The Action of S_n on the Components of the Hodge Decomposition of Hochschild Homology

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1. Background

Let k be a field of characteristic 0, let A be an associative k-algebra, and let M be an A-bimodule. Define $C_n(A;M)$ to be $M\otimes A^{\otimes n}$ (all tensor products over k) and define $b_n\colon C_n(A;M)\to C_{n-1}(A;M)$ by

$$b_n(m\otimes a_1\otimes\cdots\otimes a_n)=ma_1\otimes a_2\otimes\cdots\otimes a_n+(-1)^n a_n m\otimes a_1\otimes\cdots\otimes a_{n-1}$$
$$+\sum_{i=1}^{n-1}(-1)^i m\otimes a_1\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n.$$

It is easy to check that $b_n\circ b_{n+1}=0$, so that $\operatorname{im} b_{n+1}$ is contained in $\ker b_n$. The *Hochschild homology of A with coefficients in M* is defined by

$$H_n(A;M)=\frac{\ker b_n}{\operatorname{im} b_{n+1}}.$$

The symmetric group S_n acts on $C_n(A;M)$ by

$$\sigma\cdot(m\otimes a_1\otimes\cdots\otimes a_n)=m\otimes a_{\sigma^{-1}1}\otimes\cdots\otimes a_{\sigma^{-1}n}.$$

Define a *splitting sequence* $(f_n)_{n=1}^{\infty}$ to be a sequence of elements $f_n\in k[S_n]$ such that

$$b_n f_n\alpha=f_{n-1}b_n\alpha \tag{1.1}$$

for all $\alpha\in C_n(A;M)$, all associative k-algebras A, and all A-bimodules M. Given a splitting sequence (f_n) one can define $I_n(A;M)$ and $K_n(A;M)$ to be the image and kernel of $C_n(A;M)$ under f_n. Then both $(I_*(A;M),b_*)$ and $(K_*(A;M),b_*)$ are subcomplexes of $(C_*(A;M),b_*)$ which then yield new homology theories.

One can obtain a trivial splitting sequence by letting f_n be the identity in S_n. In general this is the only splitting sequence. However, under the assumption that A is commutative and that M is a symmetric bimodule ($a\cdot m=m\cdot a$ for all $a\in A$ and $m\in M$), there do exist nontrivial splitting sequences.

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Harrison [11] made implicit use of such sequences in his definition of Harrison homology for commutative algebras. In that paper he studies the image complex $(I_*(A;M), b_*)$ for a certain splitting sequence which we will call $(e_n^{(1)})$. In that paper he did not consider the actual splitting sequence itself. Later, Barr [1] identified the sequence $(e_n^{(1)})$ and showed that each $e_n^{(1)}$ is an idempotent in $k[S_n]$. In particular, Barr showed that the subcomplex $(I_*(A;M), b_*)$ splits off as a direct summand from the Hochschild complex in characteristic 0.

Gerstenhaber and Schack [7] generalized Barr's result by proving that for each positive integer j there exists a splitting sequence $(e_n^{(j)})$, with $e_n^{(j)}=0$ for $n<j$, such that for fixed n the $e_n^{(j)}$ are a set of mutually orthogonal idempotents in $k[S_n]$. Moreover, they proved that the $(e_n^{(j)})$ are universal splitting sequences in the sense of the following theorem.

THEOREM 1.2 (Gerstenhaber–Schack [7]). *There are unique elements $e_n^{(j)} \in k[S_n]$ with $e_n^{(j)}=0$ for $n<j$ satisfying the following property:*

(*) *if $(f_n)_{n=1}^{\infty}$ is a splitting sequence then, for each n,*

$$f_n = \sum_{j=1}^{n} \sigma_j(f_j) e_n^{(j)}.$$

Here $\sigma_j(\ )$ is the linear functional on $k[S_j]$ defined by $\sigma_j(\tau)=\mathrm{sgn}(\tau)$.

Because the $e_n^{(j)}$ $(j=1,2,\ldots,n)$ are orthogonal idempotents in $k[S_n]$, the complex $(C_*(A;M), b_*)$ splits as a direct sum of subcomplexes,

$$(C_*(A;M), b_*) = \bigoplus_{j=1}^{\infty} (C_*^{(j)}(A;M), b_*),$$

where $C_n^{(j)}(A;M) = e_n^{(j)} \cdot C_n(A;M)$. Let $H_*^{(j)}(A;M)$ denote the homology of the subcomplex $C_*^{(j)}(A;M)$. Then $H_*(A;M)$ is a direct sum of the $H_*^{(j)}(A;M)$. Gerstenhaber and Schack refer to this splitting of $H_*(A;M)$ as a *Hodge-type decomposition* for commutative algebra homology. This name is justified by another result of theirs which states that this decomposition coincides with the usual Hodge decomposition for smooth compact complex varieties (see [8]).

Gerstenhaber and Schack and (independently) Loday (see [14]) suggested the following problem, which came up in their efforts to better understand the subcomplexes $(C_*^{(j)}(A;M), b_*)$.

PROBLEM. For each n and j, determine the structure of $e_n^{(j)}k[S_n]$ as a right S_n-module.

In this paper we shall give a solution to this problem in the sense that we shall write $e_n^{(j)}k[S_n]$ as a direct sum of induced characters. The induced characters that appear in this direct sum have a simple description which makes the whole expression for $e_n^{(j)}k[S_n]$ surprisingly elegant. This induced character expression allows one to determine certain things about $e_n^{(j)}k[S_n]$, such as the dimension of $e_n^{(j)}k[S_n]$, the multiplicity of certain irreducibles of S_n in $e_n^{(j)}k[S_n]$, and the exact decomposition of $e_n^{(j)}k[S_n]$ for certain values of j. It also allows one to compute the Euler characteristic of the subcomplex $(C_*^{(j)}(A;M), b_*)$ in the case where A is a graded algebra with nothing in the 0-graded piece (see §7). However, the problem of decomposing this sum of induced characters for general n and j remains open.

Interestingly enough, the representation $e_n^{(j)}k[S_n]$ has been studied for entirely different reasons by F. Bergeron, N. Bergeron, and A. Garsia. They independently proved more general versions of some of the results in this paper (see [2]).

2. The Algebra L_n

For each n and k, let $S(n;k)$ denote the set of permutations in S_n with exactly $k-1$ descents. Following Loday [14], define elements $l_n^{(k)}$ and $\psi_n^{(k)}$ in $k[S_n]$ according to the following formulas:

(2.1a) $$l_n^{(k)} = (-1)^{k-1} \sum_{\sigma \in S(n;k)} \operatorname{sgn}(\sigma)\sigma;$$

(2.1b) $$\lambda_n^{(k)} = \sum_{i=0}^{k-1} (-1)^i \binom{n+i}{i} l_n^{(k-i)}.$$

The $l_n^{(k)}$ are nonzero only for $k \in \{1, 2, \ldots, n\}$. The $\lambda_n^{(k)}$ are nonzero for all positive integers k and have the remarkable property that

$$\lambda_n^{(k)}\lambda_n^{(l)} = (-1)^{(k-1)(l-1)}\lambda_n^{(kl)}.$$

The following lemma is due to Loday (see [14]).

LEMMA 2.2. *Let the idempotents $e_n^{(j)}$ be as in Section 1. Then, for every n and k, we have*

$$(-1)^{(k-1)}\lambda_n^{(k)} = \sum_{j=1}^{n} k^j e_n^{(j)}.$$

Note that the equation in Lemma 2.2 determines the $e_n^{(j)}$'s in terms of the $\lambda_n^{(k)}$'s because the transition matrix $(k^j)_{k,j}$ is a Vandermonde and hence invertible. Also, equation (2.1b) determines the $l_n^{(k)}$'s in terms of the $\lambda_n^{(k)}$'s because the transition matrix is triangular with unit diagonal. So the algebra L_n can be thought of as the subalgebra of $k[S_n]$ with basis $\{e_n^{(j)}: j = 1, 2, \ldots, n\}$, $\{\lambda_n^{(j)}: j = 1, 2, \ldots, n\}$, or $\{l_n^{(j)}: j = 1, 2, \ldots, n\}$.

It is remarkable that the elements $l_n^{(j)}$ span a commutative subalgebra of $k[S_n]$ (neither commutativity nor closure under multiplication is clear given their definition in terms of descents). This is an example of a much more general phenomena, true for any Coxeter group, which was proved originally by Solomon (see [19]).

Define $\psi_n^{(k)}$ to be $(-1)^{k-1}\lambda_n^{(k)}$. Then $\psi_n^{(k)}\psi_n^{(l)} = \psi_n^{(kl)}$ and

$$\psi_n^{(k)}\alpha = k^j\alpha \quad \text{for } \alpha \in C_n^{(j)}(A;M).$$

So the $\psi_n^{(k)}$ are Adams operations and $H_*(A;M) = \bigoplus_j H_*^{(j)}(A;M)$ is a Hodge decomposition for the Hochschild homology of A with coefficients in M.

It is worth looking at the idempotents $e_n^{(j)}$ in at least one case. For $n=3$ we have

$$e_3^{(1)} = \tfrac{1}{6}(2(a)(b)(c) + (a,b) + (b,c) - (a,b,c) - (a,c,b) - 2(a,c)),$$
$$e_3^{(2)} = \tfrac{1}{2}((a)(b)(c) + (a,c)), \quad \text{and}$$
$$e_3^{(3)} = \tfrac{1}{6}((a)(b)(c) - (a,b) - (a,c) - (b,c) + (a,b,c) + (a,c,b)).$$

Although these idempotents commute with each other, $e_3^{(1)}$ and $e_3^{(2)}$ are not central in $k[S_3]$. At first glance it is not at all clear that they are idempotents or that they are mutually orthogonal.

3. Wreath Products and Cycle Indices

Our main result will express the right S_n-module $I_n^{(j)}$ as a sum of characters induced from wreath product groups. The proof will involve some manipulations of characters of wreath products, so we begin this section with a brief review of wreath products and linear characters of wreath products. For a complete discussion of wreath products and their representation theory, the reader should consult James and Kerber [12, Chap. 4].

Let G be a subgroup S_m of size g and let H be a subgroup of S_l of size h. The *wreath product,* $H \operatorname{wr} G$, is a subgroup of S_{ml} of size $g \cdot h^g$ which as a set consists of all $(m+1)$-tuples $(\gamma_1, \ldots, \gamma_m, \pi)$ where $\gamma_i \in H$ for all i and $\pi \in G$. We think of $H \operatorname{wr} G \le S_{ml}$ as acting on the set $\underline{l} \times \underline{m} = \{(j,i) : 1 \le j \le l,$ $1 \le i \le m\}$. The permutation action is given by

$$(\gamma_1, \ldots, \gamma_m, \pi)(j,i) = (\gamma_{\pi_i} j, \pi i).$$

Suppose that α is a linear character of H and that β is a linear character of G. Then there is a linear character $\alpha \operatorname{wr} \beta$ of $H \operatorname{wr} G$ defined in the following way. For each cycle $Y = (y_1, \ldots, y_s)$ of π, define $A(Y)$ by

$$A(Y) = \alpha(\gamma_{y_1} \cdots \gamma_{y_s}).$$

Then define

$$(\alpha \operatorname{wr} \beta)(\gamma_1, \ldots, \gamma_m, \pi) = \beta(\pi)\left\{\prod_Y A(Y)\right\}.$$

It is straightforward to check that $\alpha \operatorname{wr} \beta$ is a 1-dimensional representation of $H \operatorname{wr} G$.

EXAMPLE 3.1. There is one example of the $\alpha \operatorname{wr} \beta$ construction that will be of particular interest to us. This is the case where $H = C_l$, $G = S_m$, β is the trivial character of S_m, and α is the linear character of $H = \langle (1,2,\ldots,l) \rangle$ defined by $\alpha((1,2,\ldots,l)) = e^{2\pi i/l}$.

For each permutation $\sigma \in S_n$ define $Z(\sigma)$, the *cycle indicator of* σ, by

$$Z(\sigma)=\prod_u a_u^{j_u(\sigma)},$$

where $j_u(\sigma)$ is the number of u-cycles of σ and where $a_1, a_2, \ldots$ is an infinite family of commuting indeterminants over k. If N is a subgroup of S_n and χ is a k-valued function on N, define $Z_N(\chi)$, the *cycle index of* χ, by

$$Z_N(\chi)=\frac{1}{|N|}\sum_{\sigma\in N}\chi(\sigma)Z(\sigma).$$

There are two results about cycle indices that we will find useful in subsequent sections. The first involves the composition product on cycle indices. For $A=A(a_1,a_2,\ldots)$ and $B=B(a_1,a_2,\ldots)$ in $k[[a_1,a_2,\ldots]]$, define $A[B]$ to be

$$A[B]=A(B(a_1,a_2,\ldots),B(a_2,a_4,a_6,\ldots),B(a_3,a_6,a_9,\ldots),\ldots).$$

In other words, $A[B]$ is obtained from A by replacing each occurrence of a_u with $B(a_u,a_{2u},a_{3u},\ldots)$. The next result is well known (see, e.g., [12, formula 4.4.10, p. 160]).

LEMMA 3.2. *Let β be a linear character of G and let α be a linear character of H. Then*

$$Z_{H\,\mathrm{wr}\,G}(\alpha\,\mathrm{wr}\,\beta)=Z_G(\beta)[Z_H(\alpha)].$$

EXAMPLE 3.3. Return to the set-up in Example 3.1. It is straightforward to verify that

$$Z_H(\alpha)=\frac{1}{l}\sum_{d\mid l}\mu(d)a_d^{l/d},$$

so by Lemma 3.3 we have

$$Z_{C_l\,\mathrm{wr}\,S_m}(\alpha\,\mathrm{wr}\,\beta)=\left\{\frac{1}{m!}\sum_{\sigma\in S_m}Z(\sigma)\right\}\left[\frac{1}{l}\sum_{d\mid l}\mu(d)a_d^{l/d}\right].$$

The last fact we state about cycle indices is well known. It can be proved using the classical formula for the value of an induced character (see Feit [6]).

LEMMA 3.4. *Let N be a subgroup of S_n and let χ be a class function on N. Then*

$$Z_{S_n}(\mathrm{ind}_N^{S_n}(\chi))=Z_N(\chi),$$

where $\mathrm{ind}_N^{S_n}(\chi)$ *denotes the induction of χ from N to S_n.*

We end this section with a cycle index sum computation. For each permutation $\sigma\in S_n$ let $\Gamma(\sigma)$ denote the centralizer of σ in S_n. We will describe a linear character ζ_σ of $\Gamma(\sigma)$. First assume that σ has m_u u-cycles for each u. Thus $\Gamma(\sigma)$ is isomorphic to a direct product over u of the wreath products C_u wr S_{m_u}. The linear character ζ_σ will correspondingly be a product of linear characters $\zeta_\sigma^{(u)}$ of C_u wr S_{m_u}. The character $\zeta_\sigma^{(u)}$ is exactly the character α wr β of Example 3.1, where β is the trivial character of S_{m_u} and α is the linear character of C_u given by $\alpha((1,2,\dots,u))e^{2\pi i/u}$.

In view of Example 3.3, we have

$$Z_{\Gamma(\sigma)} = \prod_u \left\{ Z(S_{m_u})\left[\frac{1}{u}\sum_{d\mid u}\mu(d)a_d^{u/d}\right]\right\}. \tag{3.5}$$

DEFINITION 3.6. For each n and l with $1 \le l \le n$, define a character $\psi_n^{(l)}$ of S_n by

$$\psi_n^{(l)} = \bigoplus_{\substack{\mu \vdash n \\ l(\mu)=l}} \operatorname{ind}_{\Gamma(\sigma_\mu)}^{S_n}(\zeta_{\sigma_\mu}),$$

where the sum is over all partitions μ of n with exactly l parts and σ_μ is an arbitrarily chosen permutation with cycle type μ.

Define the cycle index sum $Z(\psi;\lambda)$ to be

$$Z(\psi;\lambda) = \sum_{n\ge 0}\sum_{l=1}^{n} \lambda^l Z_{S_n}(\psi_n^{(l)}).$$

Note that each character $\psi_n^{(l)}$ is determined by the power series $Z(\psi;\lambda)$. The next theorem gives an explicit expression for $Z(\psi;\lambda)$.

THEOREM 3.7. *For each s, let $E_s(\lambda) = (1/s)\sum_{d\mid s}\mu(d)\lambda^{s/d}$. Then*

$$Z(\psi;\lambda) = \prod_s (1-a_s)^{-E_s(\lambda)}.$$

Proof. By (3.5) we have

$$Z(\psi;\lambda) = \prod_u \left\{ \sum_{m_u=0}^{\infty} \lambda^{m_u} Z_{S_{m_u}}(\epsilon_{m_u})\left[\frac{1}{u}\sum_{d\mid u}\mu(d)a_d^{u/d}\right]\right\}, \tag{3.8}$$

where ϵ_{m_u} is the trivial character of S_{m_u}. It is well known that

$$\sum_{m=0}^{\infty} Z_{S_m}(\epsilon_m) = \exp\left(\sum_i \frac{a_i}{i}\right)$$

(see Harary and Palmer [10]). Substituting in (3.8) we obtain

$$\begin{aligned} Z(\psi;\lambda) &= \prod_u \exp\left(\sum_i \frac{a_i\lambda^i}{i}\right)\left[\frac{1}{u}\sum_{d\mid u}\mu(d)a_d^{u/d}\right] \\ &= \exp\left(\sum_{\substack{i,d,u \\ d\mid u}} \frac{a_{id}^{u/d}\lambda^i}{iu}\mu(d)\right). \end{aligned} \tag{3.9}$$

Letting $r = u/d$ and $s = id$, we have

$$Z(\psi;\lambda) = \exp\left(\sum_{\substack{r,s,d \\ d\mid s}} \frac{a_s^r\lambda^{s/d}\mu(d)}{rs}\right) =$$

$$= \exp\left(\sum_s \left(\sum_r \frac{-a_s^r}{r}\right)\left\{\frac{-1}{s}\sum_{d|s}\mu(d)\lambda^{s/d}\right\}\right)$$
$$= \prod_s (1-a_s)^{-E_s(\lambda)}. \qquad \square$$

4. Shuffles and Necklaces

In this section we will discuss two different sets of combinatorial objects and a bijection between them. Before doing so, it is worth explaining how these objects and this bijection come into the proof of our main theorem.

For each n and j, let $\chi_n^{(j)}$ be the character of the right S_n-module $e_n^{(j)}k[S_n]$ and let $Z(\chi;\lambda)$ be the cycle index sum for the values of $\chi_n^{(j)}$,

$$Z(\chi;\lambda) = \sum_{n\geq 0}\frac{1}{n!}\sum_{\sigma\in S_n}\left\{\sum_{j=1}^{n}\lambda^j\chi_n^{(j)}(\sigma)\right\}Z(\sigma).$$

We will show that $\chi_n^{(j)} = \psi_n^{(j)}$ for all n and j, or (equivalently) that $Z(\chi;\lambda) = Z(\psi;\lambda)$. Because so little is known about the $e_n^{(j)}$'s, it is impossible to verify this equality directly; we employ the following trick. We define an isomorphism Λ from the ring of power series in $\lambda, a_1, a_2, \ldots$ to the ring of power series in $t, a_1, a_2, \ldots$. It will turn out that $\Lambda Z(\psi;\lambda)$ is the generating function for one of the two kinds of combinatorial objects above, and that $\Lambda Z(\chi;\lambda)$ is the generating function for the other. The bijection between the two sets shows that $\Lambda Z(\psi;\lambda) = \Lambda Z(\chi;\lambda)$, from which we deduce that $Z(\psi;\lambda) = Z(\chi;\lambda)$.

Let $(i_1, \ldots, i_m)$ be an ordered m-partition of n, and for each j let s_j be the partial sum $s_j = i_1 + \cdots + i_{j-1}$. An $(i_1, \ldots, i_m)$-*shuffle* is a permutation $\pi \in S_n$ such that, for each j,

$$\pi(s_j+1) < \pi(s_j+2) < \cdots < \pi(s_j+i_j).$$

Let $\mathrm{sh}(i_1, \ldots, i_m)$ be the set of $(i_1, \ldots, i_m)$-shuffles. For $\mu \vdash n$, let $\mathrm{sh}_\mu(i_1, \ldots, i_m)$ be the subset of $\mathrm{sh}(i_1, \ldots, i_m)$ consisting of those permutations with cycle type μ.

A *necklace* is a digraph all of whose connected components are directed cycles. We will deal exclusively with unlabeled digraphs in which loops are allowed. Figure 1 shows a necklace η_0 with 14 points.

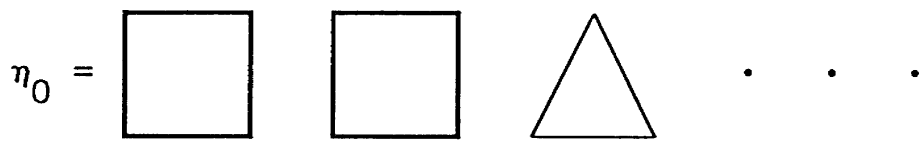

Figure 1

If η is a necklace with n points then the *cycle type of* η is the partition whose parts are the lengths of the cycles of η. For example, the necklace η_0

in Figure 1 has cycle type $4^2 3 1^3$. A *colored necklace* is one in which every point is given a color C_i chosen from an infinite set of colors $C_1, C_2, \ldots$. When we draw a colored necklace we label each point colored C_i by the number i. Figure 2 shows two colored necklaces, η_1 and η_2.

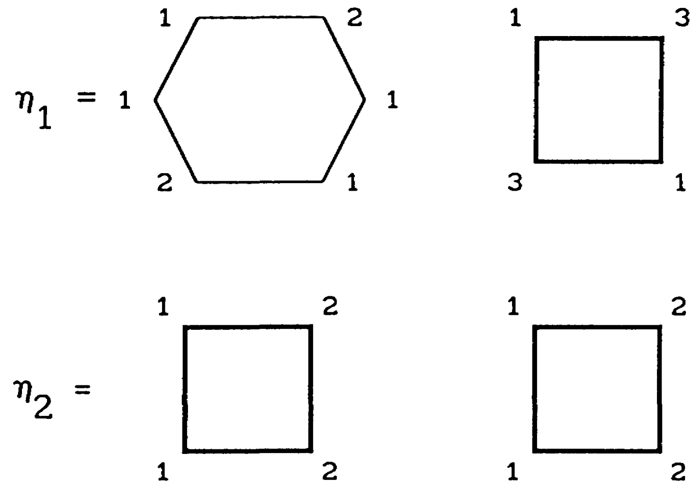

Figure 2

Let η be a colored necklace and let $Y = (y_1, \ldots, y_s)$ be a cycle of η. We say Y is *periodically colored* if there exists some $d < s$ such that y_i and y_{i+d} have the same color for all i (here $i+d$ is taken mod s). For example, both cycles in η_1 (above) are periodically colored, whereas neither cycle of η_2 is periodically colored. We say that a colored necklace η is *aperiodic* if no cycle of η is periodically colored. In Figure 2, η_2 is aperiodic but η_1 is not.

For each partition $\mu \vdash n$ and each ordered partition $(i_1, \ldots, i_m)$ of n, let $\Omega_\mu(i_1, \ldots, i_m)$ denote the set of aperiodic necklaces with cycle lengths $\mu_1, \mu_2, \ldots$ and which have exactly i_j points colored C_j.

THEOREM 4.1 (Ira Gessel). *For each $\mu \vdash n$ and each ordered partition $(i_1, \ldots, i_m)$ of n, we have*

$$|\mathrm{sh}_\mu(i_1, \ldots, i_m)| = |\Omega_\mu(i_1, \ldots, i_m)|.$$

Gessel proves this result by describing a bijection between the two sets. Although his work is unpublished, this bijection is discussed in detail in a recent paper by Désarménien and Wachs [4].

DEFINITION 4.2. Define $Z(\mathrm{sh}; t)$ and $Z(\Omega; t)$ in $Q[[t, a_1, a_2, \ldots]]$ by

$$Z(\mathrm{sh}; t) = \sum_{n \geq 0} \sum_{\mu \vdash n} \sum_{(i_1, \ldots, i_m)} |\mathrm{sh}_\mu(i_1, \ldots, i_m)| t^m a_{\mu_1} a_{\mu_2} \cdots$$

and

$$Z(\Omega; t) = \sum_{n \geq 0} \sum_{\mu \vdash n} \sum_{(i_1, \ldots, i_m)} |\Omega_\mu(i_1, \ldots, i_m)| t^m a_{\mu_1} a_{\mu_2} \cdots.$$

An immediate corollary of Theorem 4.1 is

$$Z(\text{sh}; t) = Z(\Omega; t). \tag{4.3}$$

5. The Main Result

For each n and l, let $\psi_n^{(l)}$ be the sum of induced characters given in Definition 3.6 and let $\chi_n^{(l)}$ be the character of the right S_n-module $e_n^{(l)}k[S_n]$. We can now state the main result.

THEOREM 5.1. *For every n and l, we have*

$$\chi_n^{(l)} = \text{sgn} * \psi_n^{(l)},$$

where $\text{sgn} * \psi_n^{(l)}$ *is the product of* $\psi_n^{(l)}$ *times the linear character* sgn.

Proof. For each $\sigma \in S_n$ let $e_n^{(l)}\langle\sigma\rangle$ denote the coefficient of σ in $e_n^{(l)}$.

LEMMA 5.2. *For every partition $\mu \vdash n$, let $\mathcal{C}(\mu)$ denote the conjugacy class consisting of all permutations with cycle type μ. Then*

$$\frac{1}{n!} \sum_{\sigma \in \mathcal{C}(\mu)} \chi_n^{(l)}(\sigma) = \sum_{\sigma \in \mathcal{C}(\mu)} e_n^{(l)}\langle\sigma\rangle.$$

Note: It is not true that $(1/n!)\chi_n^{(l)}(\sigma) = e_n^{(l)}\langle\sigma\rangle$ for each $\sigma \in S_n$. This is clear because $\chi_n^{(l)}(\sigma)$ is a character and hence is constant on conjugacy classes. However, for $n \geq 3$ and $l < n$, the functions $e_n^{(l)}$ are not constant on conjugacy classes.

Proof of lemma. Write $k[S_n] = \bigoplus_{\mu \vdash n} M_\mu$, where M_μ is the matrix ring corresponding to the irreducible character ω^μ of S_n. Then $e_n^{(l)}k[S_n]$ splits as a direct sum of right submodules

$$e_n^{(l)}k[S_n] = \bigoplus_{\mu} e_n^{(l)}M_\mu.$$

Moreover, $e_n^{(l)}M_\mu$ as a right S_n-module is isomorphic to some number of copies of the irreducible character ω^μ. Thus the multiplicity of the irreducible character ω^μ in $\chi_n^{(l)}$ is

$$\text{mult}_{\omega^\mu}(e_n^{(l)}k[S_n]) = \frac{\dim(e_n^{(l)}M_\mu)}{\deg(\omega^\mu)}. \tag{5.3}$$

Let D_μ denote the identity matrix in M_μ; that is, let

$$D_\mu = \frac{\deg(\omega^\mu)}{n!} \sum_{\sigma \in S_n} \omega^\mu(\sigma)\sigma. \tag{5.4}$$

Let $E_\mu^{(l)} = e_n^{(l)}D_\mu$. We have

$$E_\mu^{(l)}M_\eta = \begin{cases} e_n^{(l)}M_\mu & \text{if } \eta = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

By (5.3), the multiplicity of S^μ in $e_n^{(l)}k[S_n]$ is $\operatorname{rank}(E_\mu^{(l)})/\deg(\omega^\mu)$, where $\operatorname{rank}(E_\mu^{(l)})$ is the rank of $E_\mu^{(l)}$ as a linear transformation of $k[S_n]$.

Also observe that $E_\mu^{(l)}$ is an idempotent in $k[S_n]$, since D_μ is a central idempotent. So (as a linear transformation of $k[S_n]$) $E_\mu^{(l)}$ is diagonalizable and all its eigenvalues are 0 or 1. It follows that

$$\operatorname{rank}(E_\mu^{(l)}) = \operatorname{trace}(E_\mu^{(l)}). \tag{5.5}$$

Also, $\operatorname{trace}(E_\mu^{(l)})$ is $n!$ times the coefficient of the identity. Thus by (5.4) we have

$$\operatorname{rank}(E_\mu^{(l)}) = \dim S^\mu \sum_\sigma e_n^{(l)}\langle\sigma\rangle\omega^\mu(\sigma)$$

(here we use that $\omega^\mu(\sigma^{-1}) = \omega^\mu(\sigma)$). So the multiplicity of S^μ in $e_n^{(l)}k[S_n]$ is

$$\sum_\sigma e_n^{(l)}\langle\sigma\rangle\omega^\mu(\sigma).$$

On the other hand, the multiplicity of ω^μ in $e_n^{(l)}k[S_n]$ is equal to

$$\frac{1}{n!}\sum_\sigma \chi_n^{(l)}(\sigma)\omega^\mu(\sigma).$$

Since the $\omega^\mu(\sigma)$ span the space of linear functions on $k[S_n]$ that are constant on conjugacy classes, it follows that

$$\frac{1}{n!}\sum_{\sigma\in\mathcal{C}_\eta} \chi_n^{(l)}(\sigma) = \sum_{\sigma\in\mathcal{C}_\eta} e_n^{(l)}\langle\sigma\rangle \tag{5.6}$$

for all η, which proves Lemma 5.2.

Define $Z(E;\lambda)$ by

$$Z(E;\lambda) = \sum_n \sum_{\sigma\in S_n}\left\{\sum_{l=1}^n e_n^{(l)}\langle\sigma\rangle\lambda^l\right\}Z(\sigma).$$

By (5.6) we have $Z(E;\lambda) = Z(\chi;\lambda)$; thus, to prove Theorem 5.1, it suffices to show that $Z(E;\lambda) = Z(\psi;\lambda)$.

For each m and l, let $S_{m,l}$ denote the number of surjections from an l-element set onto an m-element set. Also, for each m and each $\mu\vdash n$, define $\mathrm{SH}_\mu^{(m)}$ by

$$\mathrm{SH}_\mu^{(m)} = \sum_{(i_1,\dots,i_m)} \sum_{\sigma\in\mathrm{sh}_\mu(i_1,\dots,i_m)} \operatorname{sgn}(\sigma)\sigma; \tag{5.7}$$

define $\mathrm{SH}_n^{(m)}$ to be the sum of the $\mathrm{SH}_\mu^{(m)}$. As observed by J. L. Loday, equations (2.1a, b) imply that

$$\mathrm{SH}_n^{(m)} = (-1)^{m-1}\sum_{k=1}^m \binom{m}{k}\lambda_n^{(k)}. \tag{5.8}$$

Combining (5.8) with Lemma 2.2, we have

$$\mathrm{SH}_n^{(m)} = (-1)^{m-1}\sum_{k=1}^m\sum_{l=1}^n \binom{m}{k}(-1)^{k-1}k^l e_n^{(l)} = \tag{5.9}$$

$$= \sum_{l=1}^{n} e_n^{(l)} \left\{ (-1)^{m-1} \sum_k \binom{m}{k} (-1)^{k-1} k^l \right\}$$
$$= \sum_{l=1}^{n} S_{m,l} e_n^{(l)}.$$

Define $\Lambda: k[[\lambda, a_1, a_2, \ldots]] \to k[[t, a_1, a_2, \ldots]]$ by

(5.10) $$\Lambda(\lambda^l a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n}) = \sum_{m \le l} S_{m,l} t^m a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n}.$$

It is clear that Λ is an isomorphism. From (5.9) we have that

(5.11) $$\Lambda(Z(E;\lambda)) = Z(\mathrm{sh}; t, a_1, -a_2, a_3, -a_4, \ldots).$$

Note that the substitution of $-a_{2p}$ for a_{2p} accounts for the factor $\mathrm{sgn}(\sigma)$ which appears on the right-hand side of (5.7).

We now consider what happens to $Z(\psi;\lambda)$ under the mapping Λ.

LEMMA 5.12. $\Lambda Z(\psi;\lambda) = Z(\Omega; t)$.

This lemma follows from standard Polya theory arguments (see, e.g., Kerber and Thurlings [13]). We will not repeat the proof here.

Now let $Z(\mathrm{sgn} * \psi; \lambda)$ be

$$Z(\mathrm{sgn} * \psi; \lambda) = \sum_{n \ge 0} \frac{1}{n!} \sum_{\sigma \in S_n} \left\{ \sum_{l=1}^{n} \mathrm{sgn}(\sigma) \psi_n^{(l)}(\sigma) \lambda^l \right\} Z(\sigma).$$

Then

$$\begin{aligned} Z(\mathrm{sgn} * \psi; \lambda) &= Z(\psi; \lambda, a_1, -a_2, a_3, -a_4, \ldots) \\ &= \Lambda^{-1} Z(\Omega; t, a_1, -a_2, a_3, -a_4, \ldots) \\ &= \Lambda^{-1} Z(\mathrm{sh}; t, a_1, -a_2, a_3, -a_4, \ldots) && \text{by (4.3)} \\ &= Z(E;\lambda) && \text{by (5.11)} \\ &= Z(\chi;\lambda). \end{aligned}$$

It follows that $\chi_n^{(l)} = \mathrm{sgn} * \psi_n^{(l)}$ for all n and l. □

COROLLARY 5.13. *For each n and l, the dimension of $e_n^{(l)} k[S_n]$ equals the number of permutations in S_n with exactly l cycles. Equivalently, for fixed n we have*

$$\sum_{l=1}^{n} \dim(e_n^{(l)} k[S_n]) q^l = q(q+1)(q+2)\cdots(q+n-1).$$

6. Decomposing $\chi_n^{(l)}$

In this section we consider what can be said about the decomposition of the characters $\chi_n^{(l)}$ into irreducibles. By Theorem 5.1 we have that

(6.1) $$Z(\chi;\lambda) = \prod_l (1 + (-1)^l a_l)^{-(1/l) \sum_{d|l} \mu(d) \lambda^{l/d}}.$$

This equation can be used to efficiently compute the values of the characters $\chi_n^{(l)}$ which, in turn, gives a simple means to decompose $\chi_n^{(l)}$ for every n and l when the character table of S_n is available. We begin with Table 1, listing the decompositions of $\chi^{(l)}$ for small values of n.

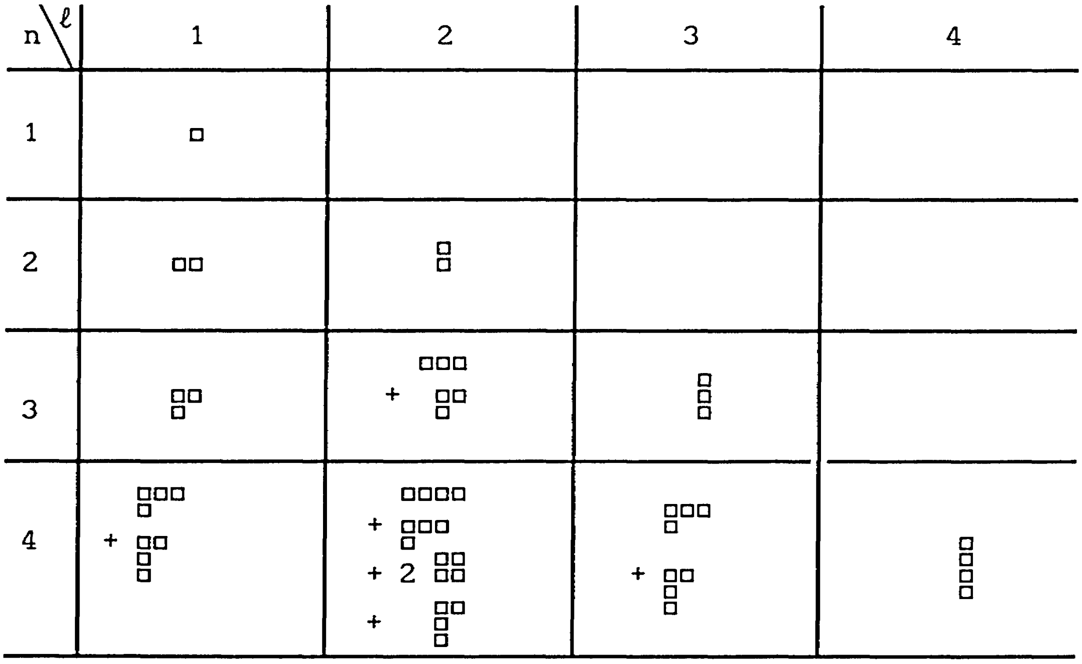

Table 1

For fixed n, the sum of the characters $\chi_n^{(l)}$ is the character of the regular representation of S_n. So, for $\mu \vdash n$, the multiplicity of the irreducible character ω^μ in $\sum_l \chi_n^{(l)}$ is the number of standard Young tableau of shape μ. We would like a combinatorial rule that assigns to each standard Young tableau T a number $l(T)$ so that the multiplicity of ω^μ in $\chi_n^{(l_0)}$ is the number of standard Young tableau T of shape μ with $l(T) = l_0$. At present, the author knows of no such rule, although certain things can be said about the decomposition of $\chi_n^{(l)}$ for special values of l, and about the multiplicity of ω^μ in $\chi_n^{(l)}$ for special partitions μ. In what follows we list some of these facts.

PROPOSITION 6.2. $\chi_n^{(n)} = \omega^{1^n}$.

This fact can be derived from the original definition of the $e_n^{(j)}$'s and hence was known to Gerstenhaber and Schack and to Loday. It can also be derived from (6.1) by observing that

$$Z(\chi;\lambda)[a_l \leftarrow (-1)^{l-1}x^l] = \sum_{n,l} \langle \chi_n^{(l)}, \omega^{1^n} \rangle x^n \lambda^l.$$

In view of this observation and (6.1), we see that Proposition 6.2 is equivalent to the identity

$$\prod_l (1+x^l)^{-(1/l)\sum_{d|l}\mu(d)\lambda^{l/d}} = \frac{1}{1-x\lambda}, \tag{6.2a}$$

a well-known result that comes up in the enumeration of necklaces [13] and the theory of Witt vectors [5].

The next result is due to Gerstenhaber, who proved it by different methods.

PROPOSITION 6.3. *The multiplicity of the trivial character ϵ_n in $\chi_n^{(l)}$ is* 1 *if $l=\lfloor(n+1)/2\rfloor$ and* 0 *if $l\neq\lfloor(n+1)/2\rfloor$.*

This is an interesting result because of the case $A=M=tk[t]/(t^2)$. In this case A is the 1-dimensional algebra with zero multiplication, so $H_n(A;A)$ is 1-dimensional (with representative $t\otimes t\otimes\cdots\otimes t$) for all $n\geq 0$. Clearly the action of S_n on $H_n(A;A)$ is the trivial representation, so by Proposition 6.3 we have that $H_n(A;A)\subseteq H_n^{(l)}(A;A)$, where $l=\lfloor(n+1)/2\rfloor$. We will return to this example in Section 7.

At present we are unaware of how Gerstenhaber proved this result. We give a separate proof based on equation (6.1).

Proof. Let $M_{l,n}^{(n)}$ denote the multiplicity of ϵ_n in $\chi_n^{(l)}$. Then

$$\begin{aligned}
\sum_n\Big(\sum_l M_{l,n}^{(n)}\lambda^l\Big)x^n &= \sum_n \frac{1}{n!}\sum_{\sigma\in S_n}\Big\{\sum_l \chi_n^{(l)}(\sigma)\lambda^l\Big\}x^n\\
&= \prod_l (1+(-1)^l x^l)^{-(1/l)\sum_{d\mid l}\mu(d)\lambda^{l/d}}\\
&= \exp\Big(\sum_l\Big(\sum_s \frac{(-1)^{(l-1)s}x^{ls}}{sl}\sum_{d\mid l}\mu(d)\lambda^{l/d}\Big)\\
&= \exp\Big(-\sum_m \frac{(-1)^{m-1}x^m}{m}\sum_{l\mid m}(-1)^{(m/l)-1}\sum_{d\mid l}\mu(d)\lambda^{l/d}\Big)\\
&= \exp\Big(-\sum_m \frac{(-1)^{m-1}x^m}{m}\sum_{t\mid m}\lambda^t\sum_{d\mid(m/t)}\mu(d)(-1)^{m/td}\Big).
\end{aligned}$$

For each N let $F_N=\sum_{d\mid N}\mu(d)(-1)^{N/d}$. It is straightforward to check that

$$F_N=\begin{cases}-1 & \text{if } N=1,\\ 2 & \text{if } N=2,\\ 0 & \text{if } N>2.\end{cases}$$

Thus

$$\begin{aligned}
\sum_{n,l} M_{l,n}^{(n)}\lambda^l x^n &= \exp\Big(+\sum_{m\text{ odd}}\frac{(-1)^{m-1}x^m\lambda^m}{m}+\sum_{m=2p}\frac{(-1)^{m-1}x^m(\lambda^m-2\lambda^p)}{m}\Big)\\
&= \exp\Big(\sum_m \frac{(-1)^{m-1}(x\lambda)^m}{m}\Big)\exp\Big(-\sum_p\frac{-(x^2\lambda)^p}{p}\Big)\\
&= \frac{1+x\lambda}{1-x^2\lambda}=1+\sum_r \lambda^r(x^{2r-1}+x^{2r}),
\end{aligned}$$

which proves the result. □

In general, for μ a partition of n let $M_{l,n}^{(\mu)}$ denote the multiplicity of ω^μ in $\chi_n^{(l)}$. The method employed in the previous proof can be pushed further.

PROPOSITION 6.4. *For $n \geq 2$, the multiplicity $M_{l,n}^{(n-1,1)}$ is given by*

$$M_{l,n}^{(n-1,1)} = \begin{cases} 1 & \text{if } l=1,\ l=\lfloor (n+1)/2 \rfloor, \text{ or } l=n/2+1 \quad (n \text{ even}), \\ 2 & \text{if } 1<l<\lfloor (n+1)/2 \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is well known that $\omega^{(n-1,1)}(\sigma)$ is the number of fixed points of σ minus 1. Thus

$$\begin{aligned} \sum_n \sum_l (M_{l,n}^{(n)} + M_{l,n}^{(n-1,1)})\lambda^l x^n &= \sum_n \frac{1}{n!} \sum_{\sigma \in S_n} \left(\sum_l \chi_n^{(l)}(\sigma)\lambda^l \right) \left(\frac{a_1 \partial}{\partial a_1} Z(\sigma) \right)_{a_l \leftarrow x^l} \\ &= \left(\frac{a_1 \partial}{\partial a_1} \left\{ \prod_l (1+(-1)^l a_l)^{-(1/l)\sum_{d \mid l} \mu(d)\lambda^{l/d}} \right\} \right)_{a_l \leftarrow x^l} \\ &= \left(\frac{+\lambda x}{1-x} \right) \left(\frac{1+x\lambda}{1-x^2\lambda} \right). \end{aligned}$$

Therefore,

$$\sum_{n,l} M_{l,n}^{(n-1,1)} \lambda^l x^n = \left(\frac{\lambda x}{1-x} \right) \left(\frac{1+x\lambda}{1-x^2\lambda} \right) - \left(\frac{1+x\lambda}{1-x^2\lambda} \right).$$

It is a tedious (though straightforward) exercise to verify Proposition 6.4 from this equation. □

In a similar manner, one can show that

$$M_{l,n}^{(2,1^{n-2})} = \begin{cases} 1 & \text{if } l<n, \\ 0 & \text{otherwise.} \end{cases}$$

We now consider the characters $\chi_n^{(l)}$ for particular values of l. We already know that $\chi_n^{(n)} = \omega^{(1^n)}$. It turns out that the other extreme $\chi_n^{(l)}$ has an explicit decomposition. Let T be a standard Young tableau. An *ascent of T* is a number i such that $i+1$ appears to the right of i in T. We let $a(T)$ denote the sum of the ascents of T. The following theorem of Josefiak and Weyman gives the decomposition of $\chi_n^{(1)}$ in combinatorial terms.

THEOREM 6.5 (Josefiak and Weyman). *Let μ be a partition of n. Then the multiplicity of ω^μ in $\chi_n^{(1)}$ is the number of standard Young tableaux T of shape μ with $a(T) \equiv 1 \pmod{n}$.*

For example, $\chi_4^{(1)} = \omega^{31} + \omega^{21^2}$ because the only standard Young tableau T with $a(T) \equiv 1 \pmod 4$ are

134		12
2	and	3
		4
$a(T)=5$		$a(T)=1.$

Using the description of $\chi_n^{(l)}$ as a sum of induced characters, it is possible to compute an explicit decomposition for some other values of l. For example, $\chi_n^{(n-1)}$ can be written as

$$\begin{aligned}\chi_n^{(n-1)} &= \omega^{(2)} \otimes \omega^{(1^{n-2})} \\ &= \omega^{(2,1^{n-2})} \oplus \omega^{(3,1^{n-3})}.\end{aligned}$$

Similarly,

$$\chi_n^{(n-2)} = (\chi_3^{(1)} \otimes \omega^{(1^{n-3})}) \oplus (\mathrm{ind}_{\Gamma(\sigma)}^{S_4}(\zeta_\sigma) \otimes \omega^{(1^{n-4})}),$$

where the σ on the right has cycle type 2^2. By explicit computation we have

$$\mathrm{ind}_{\Gamma(\sigma)}^{S_4}(\zeta_\sigma) = \omega^{(2^2)} + \omega^{(4)}.$$

Thus

$$\begin{aligned}\chi_n^{(n-2)} = &\{\omega^{(21^{n-2})} \oplus \omega^{(2^2 1^{n-4})} \oplus \omega^{(31^{n-3})} \oplus \omega^{(321^{n-5})}\} \\ &\oplus \{\omega^{(2^2 1^{n-4})} \oplus \omega^{(321^{n-5})} \oplus \omega^{(3^2 1^{n-6})}\} \oplus \{\omega^{(4,1^{n-4})} \oplus \omega^{(5,1^{n-5})}\}.\end{aligned}$$

At the other extreme, we have for n odd that

$$\chi_n^{(2)} = \sum_{r=1}^{\lfloor n/2 \rfloor} \chi_r^{(1)} \otimes \chi_{n-r}^{(1)}. \tag{6.6}$$

Using Theorem 6.5 and the Littlewood-Richardson rule, we can obtain a combinatorial decomposition of sorts for $\chi^{(2)}$ from (6.6).

7. Euler Characteristics

In this section we will assume that A is an **N**-graded k-algebra, that is, as a vector space A with a direct sum decomposition

$$A = \bigoplus_{r \geq 0} A_r$$

(with A_r finite-dimensional over k) such that $A_r A_s \subseteq A_{r+s}$. We also assume that M is a graded A-bimodule, that is,

$$M = \bigoplus_{r \geq 0} M_r$$

(with M_r finite-dimensional over k) and $M_r A_s = A_s M_r \subseteq M_{r+s}$. In addition, we will assume that $A_0 = 0$.

We define a grading ω on $C_n(A;M)$ by

$$\omega(M_{r_0} \otimes A_{r_1} \otimes \cdots \otimes A_{r_n}) = (r_0 + \cdots + r_n),$$

and we let $C_{n,\omega_0}(A;M)$ denote the span of all $\alpha = m \otimes a_1 \otimes \cdots \otimes a_n$ with $\omega(\alpha) = \omega_0$. It is easy to see that $b_n(C_{n,\omega_0}(A;M)) \subseteq (C_{n-1,\omega_0}(A;M))$, and so $(C_*(A;M), b_*)$ decomposes as a direct sum of finite-dimensional subcomplexes

$$(C_*(A;M), b_*) = \bigoplus_{\omega_0} (C_{*,\omega_0}(A;M), b_*). \tag{7.1}$$

Note that the finite-dimensionality of the subcomplexes on the right comes from the fact that $A_0=0$.

Let $\sigma \in S_n$. It is easy to see that

$$\sigma \cdot C_{n,\omega_0}(A;M) \subseteq C_{n,\omega_0}(A;M),$$

so that the direct sum decomposition (7.1) has the refinement

$$(C_*(A;M), b_*) = \bigoplus_{\omega_0, j} (C^{(j)}_{*,\omega_0}(A;M), b_*), \tag{7.2}$$

where $C^{(j)}_{n,\omega_0}(A;M) = C_{n,\omega_0}(A;M) \cap C^{(j)}_n(A;M)$.

DEFINITION 7.3. Define the Euler characteristic $\pi^{(j)}_{\omega_0}(A;M)$ by

$$\begin{aligned}\pi^{(j)}_{\omega_0}(A;M) &= \sum_n (-1)^n \dim(C^{(j)}_{n,\omega_0}(A;M)) \\ &= \sum_n (-1)^n \dim(H^{(j)}_{n,\omega_0}(A;M)).\end{aligned}$$

The latter sum is over the homology groups $H^{(j)}_{n,\omega}(A;M)$ of the complex $C^{(j)}_{n,\omega_0}(A;M)$.

Let $\Pi(A,M;\lambda,z)$ be the generating function for these Euler characteristics,

$$\Pi(A,M;\lambda,z) = \sum_{j,\omega_0} \pi^{(j)}_{\omega_0}(A;M)\lambda^j z^{\omega_0}. \tag{7.3}$$

Also, let $P_A(z)$ and $P_M(z)$ be the Poincaré series for A and M, that is,

$$P_A(z) = \sum_r (\dim A_r) z^r, \qquad P_M(z) = \sum_r (\dim M_r) z^r.$$

THEOREM 7.4. *Let A be a graded k-algebra with $A_0=0$, and let M be a symmetric, graded A-bimodule. Then*

$$\Pi(A,M;\lambda,z) = P_M(z)\left\{\prod_l (1+P_A(z^l))^{-(1/l)\sum_{d|l}\mu(d)\lambda^{l/d}}\right\}.$$

Proof. By orthogonality of the idempotents $e^{(j)}_n$ we have

$$\Pi(A,M;\lambda,z) = \sum_{j,\omega_0,n} \frac{(-1)^n}{n!} \sum_{\sigma\in S_n} \chi^{(j)}_n(\sigma)\,\mathrm{tr}(\sigma\colon C_{n,\omega_0}(A;M))\lambda^j z^{\omega_0},$$

where $\mathrm{tr}(\sigma\colon C_{n,\omega_0}(A;M))$ denotes the trace of σ as a linear transformation of $C_{n,\omega_0}(A;M)$. Recall that S_n acts on $C_{n,*}(A;M) = M\otimes A^{\otimes n}$ by permutation of the tensor positions in $A^{\otimes n}$. It follows easily that

$$(-1)^n \sum_{\omega_0} \mathrm{tr}(\sigma\colon C_{n,\omega_0}(A;M)) z^{\omega_0} = P_M(z)\{Z(\sigma)[a_l \leftarrow (-1)^l P_A(z^l)]\}.$$

Thus

$$\Pi(A,M;\lambda,z) = P_M(z)\{Z(\chi;\lambda)[a_l \leftarrow (-1)^l P_A(z^l)]\}.$$

Theorem 7.4 follows immediately from the last equation by using the expression for $Z(\chi;\lambda)$ given in (6.1). □

Consider the following application of Theorem 7.4 to a case of particular interest to this author, namely when A is the truncated polynomial ring without constants, $A = tk[t]/(t^{u+1})$. This is a graded ring whose ith graded piece is the span of t^i. Note that $A_0 = 0$ as required by Theorem 7.4.

First consider the case $M = k$ (trivial A bimodule). One can check (see, e.g., [9, p. 139]) that $H_n(A;M)$ is 1-dimensional for every value of n. The weight of the unique homology class of degree n is $(u+1)e$ if $n = 2e$ and $(u+1)e+1$ if $n = 2e+1$. Using Theorem 7.4, we will show that the homology classes of degrees $2e$ and $2e+1$ are in $H^{(e)}_{**}(A;M)$ and $H^{(e+1)}_{**}(A;M)$, respectively. To do so we first compute $\Pi(A,M;\lambda,z)$. $P_M(z) = 1$ and $P_A(z) = (z - z^{u+1})/(1-z)$, so

$$\begin{aligned}\Pi(A,M;\lambda,z) &= \prod_l \left(1 + \frac{z^l - z^{l(u+1)}}{1-z^l}\right)^{-(1/l)\sum_{d|l}\mu(d)\lambda^{l/d}}\\ &= \prod_l \left(\frac{1 - z^{l(u+1)}}{1-z^l}\right)^{-(1/l)\sum_{d|l}\mu(d)\lambda^{l/d}}\\ &= \frac{1 - z\lambda}{1 - z^{(u+1)}\lambda} = \sum_e \lambda^e\left(-z^{(u+1)(e-1)+1} + z^{(u+1)e}\right).\end{aligned}$$

As this is an Euler characteristic equation, it follows that

$$\dim(H^{(e)}_{*,(u+1)(e-1)+1}(A;M)) \geq 1 \quad \text{for all } e$$

and

$$\dim(H^{(e)}_{*,(u+1)e}(A;M)) \geq 1 \quad \text{for all } e.$$

It follows that

$$\dim(H^{(e)}_{n,\omega_0}(A;M)) = \begin{cases} 1 & \text{if } n = 2e-1,\ \omega_0 = (u+1)(e-1)+1,\\ 1 & \text{if } n = 2e,\ \omega_0 = (u+1)e,\\ 0 & \text{otherwise.}\end{cases}$$

Next consider the case $M = k[t]/(t^{u+1})$, so that $H_{**}(A;M) = H_{**}(M;M)$ (note that M is a ring and that A is just M/k). Again one can check that $H_{**}(A;M)$ has dimension u in every degree except dimension $(u+1)$ in degree 0. Moreover, the u homology classes of degree n have weights

$$(u+1)\lfloor n/2 \rfloor + s \quad (s = 1, 2, \ldots, u),$$

with an extra homology class of weight 0 in degree 0. Let $N_{n,s}$ denote the homology class of degree n and weight $(u+1)\lfloor n/2\rfloor + s$. We will show that

$$N_{n,s} \in H^{(\lfloor n/2 \rfloor)}_{n,(u+1)\lfloor n/2\rfloor + s} \quad \text{for all } n, s. \tag{7.5}$$

To see this we again compute $\Pi(A,M;\lambda,z)$. We have

$$P_M(z) = \frac{1 - z^{(u+1)}}{1-z} \quad \text{and} \quad P_A(z) = \frac{z - z^{(u+1)}}{1-z}.$$

Hence

$$\Pi(A,M;\lambda,z)=\frac{1-z^{(u+1)}}{1-z}\cdot\frac{1-z\lambda}{1-z^{(u+1)}\lambda}$$

$$\begin{aligned}(7.6)\qquad &=1+z+\cdots+z^u+\sum_s \lambda^s\{-z^{(u+1)(s-1)+1}-\cdots-z^{(u+1)(s-1)+u}\\ &\qquad\qquad +z^{(u+1)s+1}+\cdots+z^{(u+1)s+u}\}\\ &=1+z+\cdots+z^u=\sum_s\sum_{r=1}^{u} z^{(u+1)s+r}\{-\lambda^{s+1}+\lambda^s\}.\end{aligned}$$

From (7.6) it follows that

$$\dim(H^{(s+1)}_{\text{odd},(u+1)s+r}(A;M))\geq 1 \quad \text{for all } s,r$$

and

$$\dim(H^{(s)}_{\text{even},(u+1)s+r}(A;M))\geq 1 \quad \text{for all } s,r.$$

From this, (7.5) follows immediately. It should be noted that Loday found an earlier proof of (7.5) using different techniques.

At this point it is worth saying a few words about the restriction $A_0=0$. Without this restriction the alternating sum $\sum_n \dim(C^{(j)}_{n,\omega_0}(A;M))$ does not converge for all values of j and ω_0. So the Euler characteristic, defined as the alternating sum of the dimensions of the complex, makes no sense. An unfortunate consequence of the restriction $A_0=0$ is that it does not allow A to have an identity. However, this is not really a problem. If A is a graded ring with identity e and if $A_0=\langle e\rangle$, let $\bar{A}=A/A_0$. Then $\bar{A}$ is a graded ring with $\bar{A}_0=0$. It is well known (see, e.g., Quillen [18]) that $H(A;M)=H(\bar{A};M)$. So the Euler characteristic $\Pi(A;M;\lambda,z)$ defined as

$$\Pi(A,M;\lambda,z)=\sum_{j,\omega_0}\left\{\sum_n(-1)^n\dim(H^{(j)}_{n,\omega_0}(A;M))\right\}\lambda^j z^{\omega_0}$$

does converge and is equal to $\Pi(\bar{A},M;\lambda,z)$. The latter Euler characteristic can be evaluated using Theorem 7.4.

We end this section with the analogue of Theorem 7.4 for cyclic homology. Assume that A is a graded, commutative k-algebra (with $A_0=0$) and that $M=A$. Using the double complex definition of cyclic homology (see [16]), Loday showed that the Hodge decomposition $\bigoplus_j H^{(j)}(A;A)$ just obtained for Hochschild homology gives rise to a similar Hodge decomposition $\bigoplus_j HC^{(j)}(A;A)$ of cyclic homology. He also showed that for any $j\geq 1$ and any ω_0, there is a long exact sequence

$$\begin{aligned}(7.7)\qquad &\cdots\to H^{(j)}_{n,\omega_0}(A;A)\to HC^{(j)}_{n,\omega_0}(A)\to HC^{(j-1)}_{n-2,\omega_0}(A)\to H^{(j)}_{n-1,\omega_0}(A;A)\to\cdots\\ &\cdots\to H^{(j)}_{1,\omega_0}(A;A)\to HC^{(j)}_{1,\omega_0}(A)\to 0.\end{aligned}$$

The corresponding sequence for $j=0$ is

$$(7.8)\qquad 0\to H^{(0)}_{0,\omega_0}(A;A)\to HC^{(0)}_{0,\omega_0}(A)\to 0$$

(see [14]).

Define $\Pi C(A;\lambda,z)$ by

$$\Pi C(A;\lambda,z)=\sum_{j,\omega_0}\left\{\sum_n \dim(HC_{n,\omega_0}^{(j)}(A))(-1)^n\right\}\lambda^j z^{\omega_0}.$$

The following result is an immediate corollary to Theorem 7.4 and the sequences (7.7) and (7.8).

COROLLARY 7.8. *Let A be a graded, commutative k-algebra with $A_0=0$. Then*

$$\Pi C(A;\lambda,z)=\frac{P_A(z)}{1-\lambda}\left\{\prod_l (1+P_A(z^l))^{-(1/l)\sum_{d|l}\mu(d)\lambda^{l/d}}-\lambda\right\}.$$

8. Open Problems

There are a number of open problems suggested by the work in this paper. The most interesting of these is to find an explicit combinatorial decomposition of the modules $e_n^{(l)}k[S_n]$, should one exist. Another problem is to find a more conceptual proof of our main result. The proof given here obscures any homological connection that might exist between the Hodge decomposition of the Hochschild complex and the induced characters $\mathrm{ind}_{\Gamma(\sigma)}^{S_n}(\zeta_\sigma)$.

Lastly, there is a bizarre connection, at the level of character values, between the characters of S_n studied in this paper and certain topologically defined characters of S_n. For each n, let Δ_n denote the thick diagonal in $\mathbf{C}^n$, that is,

$$\Delta_n=\{(v_1,\dots,v_n)\in\mathbf{C}^n\colon v_i=v_j \text{ for some } i\neq j\}.$$

Let $\mathfrak{M}_n$ denote the complement of Δ_n in $\mathbf{C}^n$ and let $H^j(\mathfrak{M}_m)$ denote the jth cohomology group (in the topological sense) of $\mathfrak{M}_n$. Note that S_n acts on $\mathfrak{M}_n$, which gives an action of S_n on $H^j(\mathfrak{M}_n)$. Let $\beta_n^{(j)}$ denote the character of this action of S_n on $H^j(\mathfrak{M}_n)$, and let $Z(\beta,\lambda)$ be defined by

$$Z(\beta;\lambda)=\sum_n\frac{1}{n!}\sum_{\sigma\in S_n}\left\{\sum_j \beta_n^{(j)}(\sigma)\lambda^j\right\}Z(\sigma).$$

Combining Corollary 5.7 of Orlick and Solomon [17, p. 186] with Corollary 4.4 of Calderbank, Hanlon, and Robinson [3, p. 293], we have

$$Z(\beta;\lambda)=\prod_l (1+(-1)^l a_l)^{-(1/l)\sum_{d|l}\mu(d)\lambda^{l/d}} \tag{8.1}$$

Note the striking similarity between equations (6.1) and (8.1). The author knows of no direct connection between the characters $\chi_n^{(j)}$ and the characters $\beta_n^{(j)}$; however, the possible existence of some connection merits further investigation.

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References

1. M. Barr, *Harrison homology, Hochschild homology and triples,* J. Algebra 8 (1968), 314–323.
2. F. Bergeron, N. Bergeron, and A. Garsia, *Idempotents for the free Lie algebra and q-enumeration,* preprint.
3. A. R. Calderbank, P. Hanlon, and R. W. Robinson, *Partitions into even and odd block size and some unusual characters of the symmetric groups,* Proc. London Math. Soc. (3) 53 (1986), 228–320.
4. J. Désarménien and M. Wachs, *Descentes des dérangements et mots circulaires,* Adv. in Math. 70 (1988), 87–132.
5. A. Dress and C. Siebeneiches, *The Burnside ring of the infinite cyclic group and its relations to the necklace algebra, λ-rings and the universal ring of Witt vectors,* preprint.
6. W. Feit, *Characters of finite groups,* Yale University Press, New Haven, Conn., 1965.
7. M. Gerstenhaber and S. D. Schack, *A Hodge-type decomposition for commutative algebra cohomology,* J. Pure Appl. Algebra 48 (1987), 229–247.
8. ———, *Algebraic cohomology and deformation theory,* Deformation Theory of Algebras and Structures and Applications (M. Hazewinkel, M. Gersterhaber, eds.), Kluwer, Dordrecht, 1988.
9. P. Hanlon, *Cyclic homology and the Macdonald conjectures,* Invent. Math. 86 (1986), 131–159.
10. F. Harary and E. Palmer, *Graphical enumeration,* Academic Press, New York, 1973.
11. D. K. Harrison, *Commutative algebras and cohomology,* Trans. Amer. Math. Soc. 104 (1962), 191–204.
12. G. James and A. Kerber, *The representation theory of the symmetric group,* Encyclopedia of Mathematics, v. 16 (G.-C. Rota, ed.), Addison-Wesley, Reading, Mass., 1981.
13. A. Kerber and K.-J. Thürlings, *Symmetry classes of functions and their enumeration theory II,* Bayreuth. Math. Schr. 15 (1983).
14. J.-L. Loday, *Partition eulériene et opérations en homologie cyclique,* C. R. Acad. Sci. Paris Sér. I Math. 307 (1988), 283–286.
15. ———, *Opérations sur l'homologie cyclique des algèbres commutatives,* preprint.
16. J.-L. Loday and D. Quillen, *Cyclic homology and the Lie algebra homology of matrices,* Comment. Math. Helv. 59 (1984), 565–591.
17. P. Orlick and L. Solomon, *Combinatorics and topology of complements of hyperplanes,* Invent. Math. 56 (1980), 167–189.
18. D. Quillen, *On the (co-)homology of commutative rings,* Applications of Categorical Algebra (A. Heller, ed.), Amer. Math. Soc., Providence, R.I., 1970.
19. L. Solomon, *A Mackey formula in the group ring of a Coxeter group,* J. Algebra 41 (1976), 255–268.

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