The Irreducibility of the 3-Sphere

W. B. R. LICKORISH

1. Introduction

In the theory of 3-dimensional manifolds constant use is necessarily made of the fact that S^3, the 3-dimensional sphere, is irreducible. This fact is usually required in its piecewise linear interpretation, for that seems to be the commonly chosen framework for elementary work with 3-manifolds. The required result is then the following "Schönflies theorem."

THEOREM. *If S^2 is embedded piecewise linearly in S^3, then $S^3 - S^2$ has two components, the closure of each being a piecewise linear ball.*

This theorem was proved by Alexander [1], and a version of his proof is given in [8]. That proof is not, however, readily understood in the context of the standard modern theory of piecewise linear n-manifolds, and the theorem is omitted from the main expositions of that theory ([3], [6], [9], [10]). It is likewise omitted from works on 3-manifolds (e.g., [5], [7]). The purpose of this paper is to give a version of the proof based on handlebody theory. It is hoped that this proof will fill a gap in the literature and that it will bring out the 3-dimensional nature of the proof (an innermost circle argument). That itself is of interest in that the Schönflies problem for S^3 embedded in S^4 is still unsolved in the piecewise linear or smooth sense; a discussion appears in Chapter 3 of [9]. (For locally flat embeddings of S^{n-1} in S^n the result is known to be true in the topological sense for all n [2], and, using the solution to the n-dimensional Poincaré conjecture, in the piecewise linear sense for $n \geq 5$.)

2. Piecewise Linear Preliminaries

A few easily accessible results of piecewise linear topology that will be needed are listed below.

(1) *An S^1, piecewise linearly embedded in S^2, separates S^2 into two piecewise linear discs.*

Received July 14, 1987. Revision received July 22, 1988.
Michigan Math. J. 36 (1989).

The proof of this is a fairly easy exercise using induction on the number of 2-simplexes in one of the components of $S^2 - S^1$ in some triangulation, and a simple 2-dimensional version of (5). This is shown in detail in Chapter 3 of [8].

(2) *A piecewise linear manifold M has a neighbourhood of its boundary that is a piecewise linear collar, a copy of* $\partial M \times [0,1]$.

For details see Lemma 1.23 of [6], or Corollary 2.2.6 of [9].

(3) *If S^2 is embedded piecewise linearly in S^3, then $S^3 - S^2$ has two components. The closure of each component is a piecewise linear manifold.*

The first statement follows from duality in homology theory or, using (2), from a simple Mayer–Vietoris homology exact sequence argument (see (18.6) of [4]). The second follows at once, using (1), from a consideration of links of vertices in a triangulation.

(4) *If B^n is an n-ball piecewise linearly embedded in S^n, then the closure of $S^n - B^n$ is a piecewise linear n-ball.*

This weak version of the Schönflies theorem is true in all dimensions and is a fundamental result of piecewise linear theory. See Theorem 1.26 of [6], or Corollary 3.13 of [9].

(5) *Let M and B be a piecewise linear n-manifold and n-ball respectively such that $B \cap \partial M$ is a piecewise linear $(n-1)$-ball contained in ∂B.* If (i) *$B \cap M = B \cap \partial M$ then there is a piecewise linear homeomorphism $M \cup B \to M$; if* (ii) *$M \supset B$ then there is a piecewise linear homeomorphism* $\mathrm{Cl}(M - B) \to M$, *where* Cl *denotes closure. In either case, the homeomorphism may be taken to be the identity outside a small neighbourhood of B.*

The proof of this uses (2), (4), and coning constructions; when $n = 3$, (1) can be used in place of (4). See Corollary 1.29 of [6] or Lemma 3.25 of [9].

(6) *Any simplicial complex that is a piecewise linear ball has a subdivision that is simplicially collapsible.*

For this basic technical result see, for example, Theorem 2.4 of [6] or Theorem III.6 of [3]. "Collapsible" means that there is an ordering $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{2n}$ of the simplexes such that σ_0 is a vertex and, for each j, σ_{2j} is a face of σ_k for $k < 2j$ if and only if $k = 2j - 1$.

The standard language of handle theory will be used; it is explained at length in [6] and [9]. In particular, if the attaching sphere of an $(i+1)$-handle intersects the belt sphere of an i-handle transversally in precisely one point, then the two handles are said to cancel; these two handles do not change the manifold to which they are added up to (piecewise linear) homeomorphism. If an i-handle of an n-manifold is parametrised as $B^i \times B^{n-i}$, a sub-i-handle is a subset of the form $B^i \times B^{m-i}$, where B^{m-i} is piecewise linearly embedded as a standard subball of B^{n-i}. This terminology is useful for consideration of an m-submanifold of an n-manifold.

3. Proof of the Theorem

Suppose that S^3 is triangulated by a simplicial complex K that has a subcomplex L triangulating the embedded S^2. By (6) it may be assumed that K

contains a 3-simplex σ, disjoint from L, such that $K-\sigma$ collapses simplicially to a vertex v that does not belong to L. If K' and K'' denote the first and second derived barycentric subdivisions of K, the star neighbourhoods in K'' of the vertices of K' form a handle decomposition of S^3. The existence of the simplicial collapsing sequence means that this is a cancelling handle decomposition. Thus the handles cancel in pairs except for the first 0-handle (corresponding to v) and the final 3-handle (corresponding to σ). Of course, the same construction restricted to L gives a handle decomposition of S^2. Each i-handle of S^2 is a subcone (from the appropriate vertex of L') of a corresponding i-handle of S^3, and the base of this cone-pair is an (S^2, S^1)-pair. Thus, using (1), each i-handle of S^2 is a standard subhandle of an i-handle of S^3.

If S^2 has at least two 1-handles, a piecewise linear 3-ball B can be constructed by adding to the first 0-handle of S^3 some of the cancelling pairs of handles of S^3, so that B contains *some,* but *not all,* of the 1-handles of S^2. Suppose inductively that the theorem is true for any S^2 contained as a subhandle structure of a cancelling handle decomposition of S^3, in which the S^2 has either less than n 1-handles, or precisely n 1-handles but for which a ball B can be constructed as above with $\partial B \cap S^2$ having fewer than r components. (Note that $\partial B \cap S^2$ is the boundary of $B \cap S^2$, a collection of simple closed curves.) The start of this induction, when S^2 has no 1-handle or just one 1-handle, will be considered later. Suppose now that there are n 1-handles, $n \geq 2$, and that B can be constructed in the above manner with $\partial B \cap S^2$ having r components. Notice that $r > 0$ as B contains a proper subset of S^2.

Let C be a component of $\partial B \cap S^2$, innermost in the sense that (by (1)) there is a piecewise linear disc D in ∂B with $C = \partial D = D \cap S^2$. By (1), C separates S^2 into two piecewise linear discs X and Y, where X (say) contains a neighbourhood of C in $B \cap S^2$. The handle decomposition of S^3 can be changed (intuitively by infiltrating a new cancelling pair of handles to accommodate D as a subhandle) in two useful ways that will be described below. The description necessitates some notation.

Let B^* denote the closure of $S^3 - B$, and let D^+ be a piecewise linear disc in ∂B containing D in its interior such that $D^+ \cap S^2 = C$. There is, by (2), a neighbourhood of ∂B in S^3 piecewise linearly parametrised as $\partial B \times [-1, 1]$, with $\partial B \times [-1, 0]$ and $\partial B \times [0, 1]$ being collar neighbourhoods of ∂B in B and B^* respectively. It can be arranged (via a simple coning argument within collars) that $(D \times [-1, 1]) \cap S^2 = C$, that $(D^+ \times [0, 1]) \cap X = C$, and that $(D^+ \times [-1, 0]) \cap Y = C$. There are, by (5), piecewise linear homeomorphisms $\Phi: B \to \text{Closure}(B - D \times [-1, 0])$, fixed on $B \cap S^2$ and $\partial B - D$, and $\Psi: B^* \to \text{Closure}(B^* - D^+ \times [0, 1])$, fixed on $B^* \cap X$ and $\partial B^* - D^+$. The images under Φ of the original handles give a handle decomposition of ΦB; to this add the 2-handle $(D \times [-\frac{1}{2}, 0]) \cup (D^+ \times [0, 1])$ which has D as a sub-2-handle. This cancels with $D \times [-1, -\frac{1}{2}]$ viewed as a 3-handle (with no subhandle). The images under Ψ of the original handles contained in B^* complete a new cancelling handle decomposition of S^3. This contains the 2-sphere $X \cup D$ as a subhandle structure, the handles of X being exactly the same as before.

Similarly, by inserting a cancelling 0-handle and 1-handle, a new cancelling handle decomposition can be created in which $Y \cup D$ is a subhandle structure and the handles of Y are the same as before. (For this take $D \times [-\frac{1}{2}, 1]$ as the 0-handle with D as a subhandle, take the closure of $(D^+ \times [-1, 0]) - (D \times [-\frac{1}{2}, 0])$ as the cancelling 1-handle and, as before, distort the original handles slightly.)

Now, if each of $X \cup D$ and $Y \cup D$ contains a 1-handle, then each contains fewer than n 1-handles as there are only n available. So the induction hypothesis, applied to the new handle structures on S^3, implies that each of $X \cup D$ and $Y \cup D$ separates S^3 into a pair of piecewise linear balls; by (5), S^2 does likewise. If $Y \cup D$ has no 1-handle, the induction hypothesis still implies that $Y \cup D$ separates S^3 into balls, so, by (5), $X \cup D$ separates S^3 (up to piecewise linear homeomorphism) in exactly the same way as does the original S^2. However, $X \cup D$ meets the boundary of the ball consisting of ΦB and the new 2-handle and 3-handle in fewer than r components, so the result follows by induction on r. If $X \cup D$ has no 1-handle then $Y \cup D$ separates S^3 exactly as did the original S^2, and $Y \cup D$ meets in fewer than r components the boundary of the ball arising from the second new handle decomposition immediately prior to the addition of the new 0-handle and 1-handle. Again, the induction on r finishes the proof.

The start of the induction argument now follows. Consideration of the Euler characteristic shows there are only three possible handle decompositions of S^2 having at most one 1-handle; these will be considered in turn, the discussion to be understood to be entirely within the piecewise linear category.

(i) Suppose S^2 decomposes as $h^0 \cup h^2$. In this case the first 0-handle of S^3 and all the cancelling 0- and 1-handle pairs of S^3 form a 3-ball that intersects S^2 in a standard disc. For the 0-handle of S^2 is contained as a standard disc in a 0-handle of S^3 and the other cancelling 0- and 1-handle pairs are just balls added to this (as in (5)) away from the disc. Via duality, the remaining handles can be viewed similarly. That the union of these two standard pairs is standard follows from the usual piecewise linear coning constructions.

(ii) Suppose S^2 decomposes as $h^0 \cup h^0 \cup h^1 \cup h^2$. If, in the given cancelling procedure, h^1 is a subhandle of a 1-handle of S^3 that cancels a 0-handle, then one of the h^0's is in that 0-handle and the two handle pairs cancel pairwise (such cancellation can be viewed as an amalgamation of standard (ball, disc)-pairs and described entirely by pairwise coning constructions). The argument concludes as in (i). Otherwise h^1 is a subhandle of a 1-handle α of S^3 that is cancelled by a 2-handle β of S^3 (with no subhandle). Let B be the 3-ball formed by assembling all the handles of S^3 just prior to α and β. As $B \cap S^2$ is a pair of standard discs properly contained in B, the pair $(B, B \cap S^2)$ may be regarded as two 0-handle pairs joined by a 1-handle γ. Now, as before, α added to the two 0-handle pairs produces a standard (ball, disc)-pair. That ball is, then, separated by the disc into two 3-balls, and by connectivity β and γ can meet only one of these balls. Thus the ball $B \cup \beta \cup \gamma$ intersects

S^2 in a disc that separates it with a 3-ball on one side and so with a 3-ball also on the other by (5). This is then also a standard pair and the result follows as before.

(iii) Suppose S^2 decomposes as $h^0 \cup h^1 \cup h^2 \cup h^2$. If the h^1 and one of the h^2's are in a cancelling pair of 1- and 2-handles of S^3, then pairwise cancellation takes place and the result follows using the technique of (i). Otherwise check that in building up S^3, as in (ii), at the moment when h^1 has been included the manifold created is a 3-ball meeting S^2 in a standard annulus; then, when later the first h^2 has been included this becomes a 3-ball meeting S^2 in a standard ball. The finish is as before. (Alternatively, consideration of dual handles can be used to reduce case (iii) to case (ii).)

References

1. J. W. Alexander, *On the subdivision of 3-space by a polyhedron,* Proc. Nat. Acad. Sci. U.S.A. 10 (1924), 6–8.
2. M. Brown, *A proof of the generalised Schönflies theorem,* Bull. Amer. Math. Soc. 66 (1960), 74–76.
3. L. C. Glaser, *Geometrical combinatorial topology,* v. 1, Van Nostrand Reinhold Co., New York, 1970.
4. M. J. Greenberg and J. R. Harper, *Algebraic topology. A first course,* Benjamin/Cummings, Reading, Mass., 1981.
5. J. Hempel, *3-manifolds,* Ann. of Math. Stud., 86, Princeton Univ. Press, Princeton, N.J., 1976.
6. J. F. P. Hudson, *Piecewise linear topology,* W. A. Benjamin, New York, 1969.
7. W. Jaco, *Lecture notes on three-manifold topology.* Amer. Math. Soc. CBMS/43, Providence, R.I., 1980.
8. E. E. Moise, *Geometric topology in dimensions 2 and 3.* Graduate Texts in Math., 47, Springer, New York, 1977.
9. C. P. Rourke and B. J. Sanderson, *Introduction to piecewise linear topology,* Springer, Berlin, 1972.
10. E. C. Zeeman, Seminar on combinatorial topology. Notes from I.H.E.S. and University of Warwick, 1963–1966.

Department of Pure Mathematics
Cambridge University
Cambridge, CB2 1SB
England