The Structure of the Space of Co-Adjoint Orbits of a Completely Solvable Lie Group

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0. Introduction

Let G be a connected, simply connected, completely solvable Lie group with Lie algebra g. For G nilpotent, Pukanszky in [6] shows that there is an $\mathrm{Ad}^*(G)$-invariant Zariski open subset Ω of g^* in which all $\mathrm{Ad}^*(G)$-orbits have the same dimension and in which there is an algebraic subset Σ which is a cross-section for the orbits. Moreover, there is a subspace V of g^* and a computable, rational, nonsingular map $\Theta: \Sigma \times V \to \Omega$ such that, for each $\mathfrak{l} \in \Sigma$, $\Theta(\mathfrak{l}, \cdot)$ is a polynomial map whose graph is the orbit of $\mathfrak{l}$. In fact, Pukanszky's technique yields a layering of g^* by a collection of algebraic subsets $\{\Omega_j\}$ having a natural total ordering such that the maximal subset is Ω and such that in each Ω_j one can construct objects Σ_j, V_j, and Θ_j as described above. In this way a semi-algebraic cross-section for all the $\mathrm{Ad}^*(G)$-orbits is obtained. It should be emphasized that these constructions are quite explicit and depend only on the choice of a Jordan–Holder basis for g. The ordering of the layers and the computability of the cross-section in each layer makes this result particularly useful (see, e.g., [1]). For solvable groups, the layering $\{\Omega_j\}$ of g^* has itself been useful, but the space of co-adjoint orbits in each layer is more complex. For a given layer Ω, one cannot expect to obtain objects analogous to Σ, V, and Θ above. In this paper we show that, for G completely solvable, there is a refinement of the layering $\{\Omega_j\}$ such that in each of the refined layers one can obtain computable objects analogous to Σ, V, and Θ above. This refined layering also has a nice ordering, and the layers are algebraic sets. More specifically, we prove the following.

THEOREM. *Let G be a connected, simply connected, completely solvable Lie group with Lie algebra g, and let $g = g_n \supset g_{n-1} \supset \cdots \supset g_0 = (0)$ be a Jordan–Holder sequence of ideals in g. Choose a basis $X_1, X_2, \ldots, X_n$ for g such that $X_1, X_2, \ldots, X_j$ span g_j, and let $e_1, e_2, \ldots, e_n$ be the dual basis in g^*. Then there is a finite computable layering (i.e., partition) $\wp$ of g^* with the following properties:*

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(a) *each* $\Omega \in \wp$ *is G-invariant and algebraic;*
(b) *for a given* $\Omega \in \wp$, *the dimension of the co-adjoint orbits in* Ω *are constant, and there are indices* $\mathfrak{j} = \{j_1 < j_2 < \cdots < j_d\}$ ($d = \dim G\mathfrak{l}$, $\mathfrak{l} \in \Omega$) *such that for each* $\mathfrak{l} \in \Omega$, $\mathfrak{j} = \{j : g_j + g(\mathfrak{l}) \neq g_{j-1} + g(\mathfrak{l})\}$; *and*
(c) *there is a total ordering* $\Omega_1 < \Omega_2 < \cdots < \Omega_p$ *of* $\wp$ *such that* $\bigcup\{\Omega_{r'} : r' \le r\}$ *is Zariski-open in* g^* $(1 \le r \le p)$.

Given $\Omega \in \wp$ *with associated index sequence* $\mathfrak{j}$, *let*

$$V_{S(\mathfrak{j})} = \operatorname{span}\{e_j : j \in \mathfrak{j}\} \quad \textit{and} \quad V_{T(\mathfrak{j})} = \operatorname{span}\{e_j : j \notin \mathfrak{j}\}.$$

There is associated to Ω *a subset* $\mathfrak{k}^{\sim} = \{k_1 < k_2 < \cdots < k_t\}$ *of* $\{1, 2, \ldots, d\}$, *and for each* $k \in \mathfrak{k}^{\sim}$ *there is a computable real-valued rational function* b_k *on* g^* *such that*

(d) b_k *is nonsingular, nonvanishing, and semi-invariant on* Ω;
(e) *if* $V_{\mathfrak{k}^{\sim}} = \operatorname{span}\{e_{j_k} : k \in \mathfrak{k}^{\sim}\}$, *then the set*

$$\Sigma = \{\mathfrak{l} \in (V_{T(\mathfrak{j})} + V_{\mathfrak{k}^{\sim}}) \cap \Omega : b_k(\mathfrak{l}) = \pm 1\}$$

is a cross-section for the co-adjoint orbits in Ω.

For each $i \in \{1, -1\}^t$, *set* $\Omega_i = \{\mathfrak{l} \in \Omega : b_{k_s}(\mathfrak{l}) i_s > 0,\ 1 \le s \le t\}$, $\Sigma_i = \{\mathfrak{l} \in \Sigma : b_{k_s}(\mathfrak{l}) = i_s\}$, *and* $V_{T(\mathfrak{j}),i} = \{\mathfrak{l} \in V_{T(\mathfrak{j})} : \mathfrak{l}(X_{j_{k_s}}) i_s > 0\}$. *Then*

(f) *for each i there is an analytic diffeomorphism* $\Theta_i : \Sigma_i \times V_{T(\mathfrak{j}),i} \to \Omega_i$ *such that, for each* $\mathfrak{l} \in \Omega_i$, $\Theta_i(\mathfrak{l}, \cdot)$ *is an analytic map whose graph is the orbit of* $\mathfrak{l}$. *If* $\mathfrak{l} \in \Omega_i$ *then* $\Theta_i^{-1}(\mathfrak{l}) = (\mathfrak{l}', \mathfrak{l}'')$, *where* $\mathfrak{l}'$ *is the unique point in* $\Sigma_i \cap \mathrm{Ad}^*(G)\mathfrak{l}$, *and where* $\mathfrak{l}''$ *is defined by* $\mathfrak{l}''(X_{j_k}) = b_k(\mathfrak{l})^{-1}$ *if* $k \in \mathfrak{k}^{\sim}$ *and by* $\mathfrak{l}''(X_{j_k}) = \mathfrak{l}(X_{j_k})$ *if* $k \notin \mathfrak{k}^{\sim}$.

The notation here follows to some extent that of Theorem 1 of [1], where Pukanszky's results for the nilpotent case are summarized. The above theorem can, in fact, be regarded as a generalization of [1, Thm. 1].

In the first section of this paper we define the partition $\wp$ and prove that it has the properties described in parts (a), (b), and (c) of the above theorem. In the second section, a proposition is proved which is precisely analogous to Proposition 1.1 in Chapter II of [6]. The content of this proposition may be summed up by the statement: For each layer $\Omega = \bigcup \Omega_i$, there are computable analytic functions $P_i : V_{T(\mathfrak{j}),i} \times \Omega_i \to \Omega_i$ which are G-invariant in the second factor. The remaining parts of the theorem then follow from the proposition.

1. A Layering of g^*

Let $X_1, X_2, \ldots, X_n$ be a basis of g such that $g_i = \operatorname{span}\{X_1, X_2, \ldots, X_i\}$ is an ideal, $1 \le i \le n$. Let $e_i = X_i^*$, $1 \le i \le n$, be the dual basis in g^*, and let $V_i = \operatorname{span}\{e_{i+1}, \ldots, e_n\}$ and $W_i = \operatorname{span}\{e_1, \ldots, e_i\}$, $1 \le i \le n$. Let $\pi_i : g^* \to W_i$ the projection parallel to V_i. For $\mathfrak{l} \in g^*$ and $X \in g$, let $[X, \mathfrak{l}] = \mathrm{ad}^* X(\mathfrak{l})$; we denote the co-adjoint action of G on g^* multiplicatively. For each $i = 1, 2, \ldots, n$, let

$\gamma_i \in g^*$ be the root corresponding to e_i, V_i. Let $C_i = \ker \gamma_i$, $1 \le i \le n$. Given $\mathfrak{l} \in g^*$, and for each $i = 1, 2, \ldots, n$, consider the equivalence relation on the line $\pi_i(\mathfrak{l}) + \mathbf{R}e_i$ induced by the quotient space g^*/G. It is well known that there are essentially only three types of such relations. The line either (0) meets each orbit in at most one point, (1) is contained in a single orbit, or (2) meets exactly three orbits: one orbit at a single point and the other two in open "half-lines." For any $s \in G$, $\pi_i(s\mathfrak{l}) + \mathbf{R}e_i$ has the same equivalence relation type as $\pi_i(\mathfrak{l}) + \mathbf{R}e_i$. We wish to define G-invariant layers in which, for each i, the equivalence relation type of $\pi_i(\mathfrak{l}) + \mathbf{R}e_i$ is "constant." To this end we define, for each $i = 1, 2, \ldots, n$ and $\mathfrak{l} \in g^*$, the "type" $\tau_i(\mathfrak{l})$ of $\mathfrak{l}$ at i as follows:

$\tau_i(\mathfrak{l}) = 0$, if $\forall X \in g$ and $\forall t \in R$, $[X, \mathfrak{l} + te_i] \neq e_i \bmod(V_i)$;
$\tau_i(\mathfrak{l}) = 1$, if $\exists X \in C_i$ such that $[X, \mathfrak{l}] = e_i \bmod(V_i)$;
$\tau_i(\mathfrak{l}) = 2$, if $\exists X \in g$, $t \in R$ such that $[X, \mathfrak{l} + te_i] = e_i \bmod(V_i)$ but $t_i(\mathfrak{l}) \neq 1$.

It can be seen (in fact, it follows from the sequel) that $\tau_i(\mathfrak{l}) = 0$ (resp. 1, 2) if and only if $\pi_i(\mathfrak{l}) + \mathbf{R}e_i$ has equivalence relation type (0) (resp. (1), (2)). Note also that $\tau_i(\mathfrak{l}) = 0$ implies $\dim(\pi_i(G\mathfrak{l})) = \dim(\pi_{i-1}(G\mathfrak{l}))$, while $\tau_i(\mathfrak{l}) = 1$ implies $\dim(\pi_i(G\mathfrak{l})) = \dim(\pi_{i-1}(G\mathfrak{l})) + 1$.

For each $\mathfrak{l} \in g^*$ and $1 \le j \le n$, let $L_j(\mathfrak{l}) = \{X \in g : [X, \mathfrak{l}] = 0 \bmod(V_j)\}$, and let $\mathfrak{j}_{\mathfrak{l}} = \{j : L_j(\mathfrak{l}) \neq L_{j-1}(\mathfrak{l})\}$. It is easily seen that $\mathfrak{j}_{\mathfrak{l}} = \{j : g_j + g(\mathfrak{l}) \neq g_{j-1} + g(\mathfrak{l})\}$, where $g(\mathfrak{l}) = \log(\mathrm{Stab}_G(\mathfrak{l}))$. Let $J = \{\mathfrak{j}_{\mathfrak{l}} : \mathfrak{l} \in g^*\}$, and for each $\mathfrak{j} \in J$ let $\Omega_{\mathfrak{j}} = \{\mathfrak{l} \in g^* : \mathfrak{j}_{\mathfrak{l}} = \mathfrak{j}\}$. As is well known, each $\Omega_{\mathfrak{j}}$ is a G-invariant algebraic set; we let J have the total ordering as defined in [3], where it is shown that there are real nonnegative semi-invariant polynomials $\{Q_{\mathfrak{j}} : \mathfrak{j} \in J\}$ such that

$$\Omega_{\mathfrak{j}} = \{\mathfrak{l} \in g^* : Q_{\mathfrak{j}'}(\mathfrak{l}) = 0,\ \mathfrak{j}' < \mathfrak{j},\ Q_{\mathfrak{j}}(\mathfrak{l}) \neq 0\}.$$

We now define a partition of each $\Omega_{\mathfrak{j}}$ into G-invariant algebraic sets Ω_α.

Let $\mathfrak{j} \in J$, $\mathfrak{j} \neq \emptyset$, and write $\mathfrak{j} = \{j_1 < j_2 < \cdots < j_d\}$. Let $\mathfrak{l} \in \Omega_{\mathfrak{j}}$. Set $i_1 = j_1$, $h_1 = h_1(\mathfrak{l}) = L_{i_1}(\mathfrak{l})$, and let $j(i_1)$ be the smallest integer j such that $g_j + h_1 \neq h_1$. Then it is easily seen that $j(i_1) > i_1$ and $j(i_1) \in \mathfrak{j}$. Let $i_2 = \min(\mathfrak{j} \backslash \{i_1, j(i_1)\})$, let $h_1 = h_2(\mathfrak{l}) = \{X \in h_1 : \ker([X, \mathfrak{l}]) \supset g_{i_2} \cap h_1\}$, and let $j(i_2)$ be the smallest integer j such that $(g_j \cap h_1) + h_2 \neq h_2$. Then it is easily seen that $j(i_2) \in \mathfrak{j}$, $j(i_2) > i_2$, and $j(i_2) \notin \{i_1, i_2, j(i_1)\}$. Continuing in this way we obtain indices $\{i_1 < i_2 < \cdots < i_{d/2}\}$ and $\{j(i_1), j(i_2), \ldots, j(i_{d/2})\}$, with $i_k < j(i_k)$, $1 \le k \le d/2$, and $\mathfrak{j} = \{i_k, j(i_k) : 1 \le k \le d/2\}$. Each h_k is a subalgebra of g of codimension k, and $h_{d/2}$ is the Vergne polarization at $\mathfrak{l}$. (The detailed proofs of these facts may be found in [2], Lemma 3.2] for the nilpotent case; the proofs carry over verbatim to the completely solvable case.) We refer to this sequence of subalgebras $h_1 \supset h_2 \supset \cdots \supset h_{d/2}$ as the polarizing sequence for $\mathfrak{l}$ (relative to the Jordan–Holder sequence $\{g_j\}$). We denote by $\alpha(\mathfrak{l})$ the d-tuple $(i_1, i_2, \ldots, i_{d/2}, j(i_1), j(i_2), \ldots, j(i_{d/2}))$; it is easily seen that, for each $\mathfrak{l} \in g^*$ and $s \in G$, $\alpha(\mathfrak{l}) = \alpha(s\mathfrak{l})$. Let $A_{\mathfrak{j}} = \{\alpha(\mathfrak{l}) : \mathfrak{l} \in \Omega_{\mathfrak{j}}\}$ and, for each $\alpha \in A_{\mathfrak{j}}$, let $\Omega_\alpha = \{\mathfrak{l} \in \Omega_{\mathfrak{j}} : \alpha(\mathfrak{l}) = \alpha\}$. In [2, Prop. 3.3], it is shown that for each $\mathfrak{j}$ there is a total ordering of $A_{\mathfrak{j}}$ [in fact, it is the lexographic ordering on the $d/2$-tuples $(j(i_1), j(i_2), \ldots, j(i_{d/2}))$], and for each $\alpha \in A_{\mathfrak{j}}$ a polynomial function P_α on g^*, such that

$$\Omega_\alpha = \{\mathfrak{l} \in \Omega_{\mathfrak{j}} : P_{\alpha'}(\mathfrak{l}) = 0,\ \alpha' < \alpha,\ P_\alpha(\mathfrak{l}) \neq 0\};$$

in particular, each Ω_α is a G-invariant algebraic subset of $\Omega_{\mathfrak{j}}$. The set $\{\alpha: \alpha \in A_{\mathfrak{j}},\ \mathfrak{j} \in J\}$ then has a total ordering $-<$ defined as follows: If $\alpha \in A_{\mathfrak{j}}$ and $\alpha' \in A_{\mathfrak{j}'}$, then $\alpha -< \alpha'$ if $\mathfrak{j} < \mathfrak{j}'$, or if $\mathfrak{j} = \mathfrak{j}'$, $\alpha < \alpha'$. Note that $\bigcup\{\Omega_{\alpha'}: \alpha' -< \alpha\}$ is a Zariski-open subset of g^*. [Let $\alpha \in A_{\mathfrak{j}}$; then $\bigcup\{\Omega_{\alpha'}: \alpha' -< \alpha\} = \{\mathfrak{l}: \Sigma_{\mathfrak{j}'<\mathfrak{j}} Q_{\mathfrak{j}'}(\mathfrak{l}) + [Q_{\mathfrak{j}}(\mathfrak{l})(\Sigma_{\alpha'-<\alpha} P_{\alpha'}(\mathfrak{l}))] \neq 0\}$.]

LEMMA 1.1. *Let* $\mathfrak{j} \in J$ *and* $\alpha \in A_{\mathfrak{j}}$. *Then there are rational functions* r_j: $\Omega_\alpha \to g$, $j \in \mathfrak{j}$, *such that for each* $j \in \mathfrak{j}$ *and* $\mathfrak{l} \in \Omega_\alpha$, $[r_j(\mathfrak{l}), \mathfrak{l}] = e_j \operatorname{mod}(V_j)$.

REMARK. This result is more or less well known. The functions r_j were considered by Pukanszky in [6] in the nilpotent case. As their role is necessarily more explicit in the completely solvable case, we give a constructive proof.

Proof. First we inductively define elements $Y_k(\mathfrak{l}), X_k(\mathfrak{l})$ of g $(1 \le k \le d/2)$ for $\mathfrak{l} \in \Omega_\alpha$ in the following way. Fix $\mathfrak{l} \in \Omega_\alpha$, and let $h_1 \supset h_2 \supset \cdots \supset h_{d/2}$ be the inducing sequence for $\mathfrak{l}$. Set $Y_1(\mathfrak{l}) = X_{i_1}$ and $X_1(\mathfrak{l}) = X_{j(i_1)}$. Then $\mathfrak{l}([X_1(\mathfrak{l}), Y_1(\mathfrak{l})]) \neq 0$. Assume that $k > 1$, and that $Y_1(\mathfrak{l}), \ldots, Y_{k-1}(\mathfrak{l})$ and $X_1(\mathfrak{l}), \ldots, X_{k-1}(\mathfrak{l})$ have been defined such that, for each r $(1 \le r \le k-1)$, $Y_r(\mathfrak{l}) \in g_{i_r} \cap h_{r-1}$ and $X_r(\mathfrak{l}) \in (g_{j(i_r)} \cap h_{r-1}) \setminus h_r$, and such that

$$\mathfrak{l}([X_r(\mathfrak{l}), X_s(\mathfrak{l})]) = \mathfrak{l}([Y_r(\mathfrak{l}), Y_s(\mathfrak{l})]) = 0 \quad 1 \le r, s \le k-1$$

and

$$\mathfrak{l}([X_r(\mathfrak{l}), Y_s(\mathfrak{l})]) \neq 0 \quad \text{if and only if } r = s.$$

Define, for any $X \in g$,

$$c_r(\mathfrak{l}, X) = \mathfrak{l}([X_r(\mathfrak{l}), X])/\mathfrak{l}([X_r(\mathfrak{l}), Y_r(\mathfrak{l})]),$$

$$d_r(\mathfrak{l}, X) = \mathfrak{l}([Y_r(\mathfrak{l}), X])/\mathfrak{l}([X_r(\mathfrak{l}), Y_r(\mathfrak{l})]).$$

Set

$$Y_k(\mathfrak{l}) = X_{i_k} - \sum_{r=1}^{k-1} c_r(\mathfrak{l}, X_{i_k}) Y_r(\mathfrak{l}) + \sum_{r=1}^{k-1} d_r(\mathfrak{l}, X_{i_k}) X_r(\mathfrak{l}),$$

$$X_k(\mathfrak{l}) = X_{j(i_k)} - \sum_{r=1}^{k-1} c_r(\mathfrak{l}, X_{j(i_k)}) Y_r(\mathfrak{l}) + \sum_{r=1}^{k-1} d_r(\mathfrak{l}, X_{j(i_k)}) X_r(\mathfrak{l}).$$

It follows from the induction hypothesis that, for $1 \le r \le k-1$,

$$\mathfrak{l}([X_k(\mathfrak{l}), X_r(\mathfrak{l})]) = \mathfrak{l}([X_k(\mathfrak{l}), Y_r(\mathfrak{l})]) = 0,$$

and

$$\mathfrak{l}([Y_k(\mathfrak{l}), X_r(\mathfrak{l})]) = \mathfrak{l}([Y_k(\mathfrak{l}), Y_r(\mathfrak{l})]) = 0.$$

Hence, by construction of the inducing sequence, $X_k(\mathfrak{l})$ and $Y_k(\mathfrak{l})$ belong to h_{k-1}. Now we claim that $Y_k(\mathfrak{l}) \in g_{i_k}$. For this we need to show that, if $j(i_r) > i_k$ for some $r < k$, then $d_{rk}(\mathfrak{l}, X_{i_k}) = 0$. So suppose that $j(i_r) > i_k$ for some

$r<k$. By the hypothesis we have $g=h_{r-1}+\text{span}\{X_1(\mathfrak{l}),X_2(\mathfrak{l}),\ldots,X_{r-1}(\mathfrak{l})\}$, and $\{X_1(\mathfrak{l}),X_2(\mathfrak{l}),\ldots,X_{r-1}(\mathfrak{l})\}$ are linearly independent modulo h_{r-1}. Hence we have $W\in h_{r-1}$ and unique constants $c_1,c_2,\ldots,c_{r-1}$ such that

$$X_{i_k}=W+c_1X_1(\mathfrak{l})+c_2X_2(\mathfrak{l})+\cdots+c_{r-1}X_{r-1}(\mathfrak{l}).$$

It also follows from the definition of $j(i_1),j(i_2),\ldots,$ that $c_s=0$ if $j(i_s)>i_k$. [For suppose this is false and choose the smallest s such that $j(i_s)>i_k$ and $c_s\neq 0$. Solving for $X_s(\mathfrak{l})$ in the above equation then shows that $X_s(\mathfrak{l})=Y+W'$, where $Y\in g_{i_k}\cap h_{s-1}$ and $W'\in h_s$. $Y\notin h_s$ because $X_s(\mathfrak{l})\notin h_s$, and hence $(g_{i_k}\cap h_{s-1})+h_s\neq h_s$. But by definition, $j(i_s)$ is the smallest index j with the property that $(g_j\cap h_{s-1})+h_s\neq h_s$, a contradiction.] Thus $W\in g_{i_k}\cap h_{r-1}$. Now (by definition of $j(i_r)$ and h_r) $W\in h_r$, and by the induction hypothesis $Y_r(\mathfrak{l})\in g_{i_r}\cap h_{r-1}$. Hence $\mathfrak{l}([W,Y_r(\mathfrak{l})])=0$. But (also by the induction hypothesis) $\mathfrak{l}([X_s(\mathfrak{l}),Y_r(\mathfrak{l})])=0$, $1\le s\le r-1$, and thus we have $\mathfrak{l}([X_{i_k},Y_r(\mathfrak{l})])=0$ and $d_r(\mathfrak{l},X_{i_k})=0$. A similar argument shows that $X_k(\mathfrak{l})\in g_{j(i_k)}$. To sum up the above, we have shown that $Y_k(\mathfrak{l})\in g_{i_k}\cap h_{s-1}$ and $X_k(\mathfrak{l})\in g_{j(i_k)}\cap h_{s-1}$. Hence, by definition of $j(i_k)$ we have $\mathfrak{l}([X_k(\mathfrak{l}),Y_k(\mathfrak{l})])\neq 0$.

We now define the $r_j(\mathfrak{l})$: Set $r_j(\mathfrak{l})=-X_k(\mathfrak{l})/\mathfrak{l}([X_k(\mathfrak{l}),Y_k(\mathfrak{l})])$ if $j=i_k$ and set $r_j(\mathfrak{l})=Y_k(\mathfrak{l})/\mathfrak{l}([X_k(\mathfrak{l}),Y_k(\mathfrak{l})])$ if $j=j(i_k)$, $1\le k\le d/2$. It follows from the above that, for each $j\in\mathfrak{j}$, $\mathfrak{l}([X_i,r_j(\mathfrak{l})])=0$, $i<j$, and $\mathfrak{l}([X_j,r_j(\mathfrak{l})])=1$, and hence that $[r_j(\mathfrak{l}),\mathfrak{l}]=e_j \bmod(V_j)$, $j\in\mathfrak{j}$. The proof is finished. □

REMARK. The formulas for $X_k(\mathfrak{l})$ and $Y_k(\mathfrak{l})$ show that, for each $k=1,2,\ldots,d/2$, the function on Ω_α given by

$$\mathfrak{l}\to\mathfrak{l}([X_1(\mathfrak{l}),Y_1(\mathfrak{l})])\mathfrak{l}([X_2(\mathfrak{l}),Y_2(\mathfrak{l})])\cdots\mathfrak{l}([X_k(\mathfrak{l}),Y_k(\mathfrak{l})])$$

extends to a polynomial function on $\Omega_{\mathfrak{j}}$. If $P_{\alpha,k}$ denotes this function, then the polynomials P_α are given by

$$P_\alpha(\mathfrak{l})=P_{\alpha,1}(\mathfrak{l})P_{\alpha,2}(\mathfrak{l})\cdots P_{\alpha,d/2}(\mathfrak{l})$$

for each $\alpha\in A_{\mathfrak{j}}$.

Fix $\alpha=(i_1,i_2,\ldots,i_{d/2},j(i_1),j(i_2),\ldots,j(i_{d/2}))$. The functions τ_j may not be constant on Ω_α; note, however, that $\tau_{j(i_k)}(\mathfrak{l})=1$ (since $i_k<j(i_k)$) and that $\tau_{i_k}(\mathfrak{l})=1$ or 2, $1\le k\le d/2$. Lemma 1.2 and Proposition 1.3 will show that the subsets of Ω_α on which the functions τ_j are constant are algebraic subsets of Ω_α.

LEMMA 1.2. *Let $\mathfrak{l}\in\Omega_\alpha$, and let $h_1\supset h_2\supset\cdots\supset h_{d/2}$ be the polarizing sequence for $\mathfrak{l}$. Then $\tau_{i_k}(\mathfrak{l})=2$* if and only if

$$(g_{j(i_k)}\cap h_{k-1})+C_{i_k}\neq(g_{j(i_k)-1}\cap h_{k-1})+C_{i_k}.$$

Proof. Suppose that $(g_{j(i_k)}\cap h_{k-1})+C_{i_k}=(g_{j(i_k)-1}\cap h_{k-1})+C_{i_k}$. Then $X_k(\mathfrak{l})=X+Y$, where $X\in C_{i_k}$ and $Y\in g_{j(i_k)-1}\cap h_{k-1}$. By definition of $j(i_k)$, $Y\in h_k$, and thus $X\in h_{k-1}\backslash h_k$. Now the element

$$X'=X-\sum_{r=1}^{k-1} c_r(\mathfrak{l},X)Y_r(\mathfrak{l})$$

belongs to $L_{i_k-1}(\mathfrak{l})\backslash L_{i_k}(\mathfrak{l})$, and since $i_r<i_k$ $(1\leq r\leq k-1)$, $X'\in C_{i_k}$. Thus $\tau_{i_k}(\mathfrak{l})\neq 2$.

Suppose that $(\mathfrak{g}_{j(i_k)}\cap \mathfrak{h}_{k-1})+C_{i_k}\neq(\mathfrak{g}_{j(i_k)-1}\cap \mathfrak{h}_{k-1})+C_{i_k}$. We claim that $C_{i_k}\cap \mathfrak{h}_{k-1}=\mathfrak{h}_k$. Assume that $C_{i_k}\cap \mathfrak{h}_{k-1}\neq \mathfrak{h}_k$. Let j' be the smallest integer j such that $\mathfrak{g}_j\cap \mathfrak{h}_{k-1}\cap C_{i_k}$ is not contained in $\mathfrak{h}_k$, and let

$$X'\in(\mathfrak{g}_{j'}\cap \mathfrak{h}_{k-1}\cap C_{i_k})\backslash \mathfrak{h}_k.$$

By definition of $j(i_k)$ we have $j'\geq j(i_k)$, and by the hypothesis $j'\neq j(i_k)$. Now $[X',Y_k(\mathfrak{l})]\in \mathfrak{g}_{i_k-1}\cap \mathfrak{h}_{k-1}$; hence, by definition of i_k,

$$0=\mathfrak{l}([X_k(\mathfrak{l}),[X',Y_k(\mathfrak{l})]])=\mathfrak{l}([[X_k(\mathfrak{l}),X'],Y_k(\mathfrak{l})])+\mathfrak{l}([X',[X_k(\mathfrak{l}),Y_k(\mathfrak{l})]]).$$

But $[X_k(\mathfrak{l}),X']\in C_{i_k}\cap \mathfrak{g}_{j(i_k)}\cap \mathfrak{h}_{k-1}$, and $\mathfrak{h}_k\supset C_{i_k}\cap \mathfrak{g}_{j(i_k)}\cap \mathfrak{h}_{k-1}$ since $j'>j(i_k)$. Thus $[X_k(\mathfrak{l}),X']\in \mathfrak{h}_k$ and so $\mathfrak{l}([[X_k(\mathfrak{l}),X'],Y_k(\mathfrak{l})])=0$. Therefore (by the above) we have $\mathfrak{l}([X',[X_k(\mathfrak{l}),Y_k(\mathfrak{l})]])=0$. On the other hand, the hypothesis implies that $\gamma_{i_k}(X_k(\mathfrak{l}))\neq 0$. Since $\mathfrak{l}([X',Z])=0$ for all $Z\in \mathfrak{h}_{k-1}\cap \mathfrak{g}_{i_k-1}$ (again by definition of i_k and $\mathfrak{h}_{k-1}$), it follows that $\mathfrak{l}([X',Y_k(\mathfrak{l})])=0$. But this means that $X'\in \mathfrak{h}_k$, a contradiction. This proves the claim. Now suppose that $X\in \mathfrak{g}$ and $[X,\mathfrak{l}]=e_{i_k} \bmod V_{i_k}$. Then $X\in L_{i_k-1}(\mathfrak{l})$. But by construction of $\mathfrak{h}_{k-1}$ and the fact that $i_{k-1}\leq i_k-1$, $\mathfrak{h}_{k-1}\supset L_{i_{k-1}}(\mathfrak{l})$. Hence $X\in \mathfrak{h}_{k-1}\backslash \mathfrak{h}_k$, and (by the above) $X\notin C_{i_k}$. This proves that $\tau_{i_k}(\mathfrak{l})=2$, and the proof is finished. □

PROPOSITION 1.3. *Let* $\mathfrak{j}\in J$, *and let* $\alpha\in A_{\mathfrak{j}}$, $\alpha=(i_1,i_2,\ldots,i_{d/2},j(i_1),j(i_2),\ldots,j(i_{d/2}))$. *For each* $\mathfrak{l}\in\Omega_\alpha$, $\tau_{i_k}(\mathfrak{l})=2$ *if and only if* $\gamma_{i_k}(X_k(\mathfrak{l}))\neq 0$.

Proof. Suppose that $\gamma_{i_k}(X_k(\mathfrak{l}))\neq 0$. We claim that $C_{i_k}\supset(\mathfrak{g}_{j(i_k)-1}\cap \mathfrak{h}_{k-1})$. Let $X\in(\mathfrak{g}_{j(i_k)-1}\cap \mathfrak{h}_{k-1})$. Since $\mathfrak{h}_k\supset \mathfrak{g}_{j(i_k)-1}\cap \mathfrak{h}_{k-1}$,

$$\begin{aligned}\gamma_{i_k}(X)\mathfrak{l}([X_k(\mathfrak{l}),Y_k(\mathfrak{l})])&=\mathfrak{l}([X_k(\mathfrak{l}),[X,Y_k(\mathfrak{l})]])\\&=\mathfrak{l}([[X_k(\mathfrak{l}),X],Y_k(\mathfrak{l})])+\mathfrak{l}([X,[X_k(\mathfrak{l}),Y_k(\mathfrak{l})]])=0,\end{aligned}$$

and the claim follows. Thus $C_{i_k}\supset(\mathfrak{g}_{j(i_k)-1}\cap \mathfrak{h}_{k-1})+C_{i_k}$ and the hypothesis implies that $(\mathfrak{g}_{j(i_k)}\cap \mathfrak{h}_{k-1})+C_{i_k}\neq(\mathfrak{g}_{j(i_k)-1}\cap \mathfrak{h}_{k-1})+C_{i_k}$. Hence, by Lemma 1.2, $\tau_{i_k}(\mathfrak{l})=2$. The converse follows immediately from the fact that $X_k(\mathfrak{l})\in L_{i_k-1}(\mathfrak{l})\backslash L_{i_k}(\mathfrak{l})$. This finishes the proof. □

We now complete the definition of the layering $\wp$. Let $\mathfrak{j}\in J$, $\mathfrak{j}=\{j_1,j_2,\ldots,j_d\}$, and let $\alpha\in A_{\mathfrak{j}}$. Consider the set $\mathfrak{k}_\alpha$ of all k such that τ_{j_k} is not constant on Ω_α. For each $\mathfrak{l}\in\Omega_\alpha$, let $\mathfrak{k}(\mathfrak{l})=\{k\in\mathfrak{k}_\alpha:\tau_{j_k}(\mathfrak{l})=2\}$. Let $K_\alpha=\{\mathfrak{k}(\mathfrak{l}):\mathfrak{l}\in\Omega_\alpha\}$ and, for each $\mathfrak{k}\in K_\alpha$, let

$$\Omega_{\alpha,\mathfrak{k}}=\{\mathfrak{l}\in\Omega_\alpha:\mathfrak{k}(\mathfrak{l})=\mathfrak{k}\}.$$

For each k $(1\leq k\leq d)$, set $b_k(\mathfrak{l})=\gamma_{j_k}(r_{j_k}(\mathfrak{l}))$. By Proposition 1.3, for any $\mathfrak{l}\in\Omega_\alpha$, $\tau_{j_k}(\mathfrak{l})=2$ if and only if $b_k(\mathfrak{l})\neq 0$. Since $\mathfrak{l}\to b_k(\mathfrak{l})$ is a nonsingular rational function on Ω_α, it follows that, for each $\mathfrak{k}\in K_\alpha$, $\Omega_{\alpha,\mathfrak{k}}$ is an algebraic subset of Ω_α. Set

$$\wp=\{\Omega_{\alpha,\mathfrak{k}}:\mathfrak{k}\in K_\alpha,\ \alpha\in A_{\mathfrak{j}},\ \mathfrak{j}\in J\}.$$

Each $\Omega\in\wp$ is G-invariant, and the functions τ_j are constant on Ω. The layering $\wp$ also has the following property.

PROPOSITION 1.4. *There is an ordering $\Omega_1<\Omega_2<\cdots<\Omega_p$ of $\wp$ such that, for each q $(1\leq q\leq p)$, $\bigcup\{\Omega_r:1\leq r\leq q\}$ is a Zariski-open subset of g^*.*

Proof. Define an ordering of $\wp$ as follows. Let Ω and Ω' belong to $\wp$, with $\Omega=\Omega_{\alpha,\mathfrak{k}}$, and $\Omega'=\Omega_{\alpha',\mathfrak{k}'}$. If $\alpha-<\alpha'$ then $\Omega'<\Omega$. Suppose that $\alpha=\alpha'$. We then order K_α as follows. Let r be the number of elements in $\mathfrak{k}_\alpha$, and let the minimal element $\mathfrak{k}_1$ be the $\mathfrak{k}_\alpha$. The next r elements $\mathfrak{k}_2<\mathfrak{k}_3<\cdots<\mathfrak{k}_{r+1}$ in the ordering are the r elements of K_α having $r-1$ terms each (taken in any order); $\mathfrak{k}_{r+2}<\mathfrak{k}_{r+3}<\cdots<\mathfrak{k}_{2r+1}$ are the elements having $r-2$ terms, and so on. We say that $\Omega'<\Omega$ if $\mathfrak{k}'<\mathfrak{k}$. This completes the definition of the total order on $\wp$.

Now, for each $k\in\mathfrak{k}_\alpha$, there is a nonnegative real polynomial function t_k such that $\tau_{j_k}(\mathfrak{l})=2$ if and only if $t_k(\mathfrak{l})\neq 0$. Set

$$t_{\mathfrak{k}}(\mathfrak{l})=\prod_{k\in\mathfrak{k}} t_k(\mathfrak{l}).$$

For any $\Omega\in\wp$ and $\Omega=\Omega_{\alpha,\mathfrak{k}}$, let α' be the predecessor of α and let $Q^{\sim}, Q^{\sim\prime}$ be nonnegative real polynomial functions whose zero sets are the complements of $\Omega_\alpha,\Omega_{\alpha'}$, respectively. Then

$$\Omega=\{\mathfrak{l}\in\Omega_\alpha: t_{k'}(\mathfrak{l})=0,\ k'<k,\ t_k(\mathfrak{l})\neq 0\},$$

and hence

$$\bigcup\{\Omega':\Omega'\leq\Omega\}=\left\{\mathfrak{l}\in g^*: Q^{\sim\prime}(\mathfrak{l})+Q^{\sim}(\mathfrak{l})\left(\sum_{\mathfrak{k}'\leq\mathfrak{k}} t_{\mathfrak{k}}(\mathfrak{l})\right)\neq 0\right\}.\qquad\square$$

2. The Collective Orbit Structure

Let G be a connected, simply connected, completely solvable Lie group with Lie algebra g. Fix a Jordan–Holder sequence $g=g_n\supset g_{n-1}\supset\cdots\supset g_0=0$, and let $X_1,X_2,\ldots,X_n$ be a basis of g such that $\operatorname{span}\{X_1,X_2,\ldots,X_i\}=g_i$, $1\leq i\leq n$. Let $\wp$ be the layering of g^* constructed in Section 1. We shall describe the orbit structure in each layer Ω. The following lemma proves two crucial facts about the functions b_k.

LEMMA 2.1. *Let $\Omega=\Omega_{\alpha,\mathfrak{k}}\in\wp$, let $\mathfrak{l}\in\Omega$, and let $j=j_k\in\mathfrak{j}_{\mathfrak{l}}$ such that $k\in\mathfrak{k}$. Then, for any $A\in g$ such that $[A,\mathfrak{l}]=e_{j_k}\bmod(V_{j_k})$, we have $\gamma_{j_k}(A)=b_k(\mathfrak{l})$. Moreover, for each $\mathfrak{l}'\in G\mathfrak{l}$ such that $\pi_{j_k-1}(\mathfrak{l})=\pi_{j_k-1}(\mathfrak{l}')$,*

$$\mathfrak{l}'_{j_k}-\frac{1}{b_k(\mathfrak{l}')}=\mathfrak{l}_{j_k}-\frac{1}{b_k(\mathfrak{l})}.$$

Proof. Suppose that, for some $A \in g$, $[A, \mathfrak{l}] = e_j \bmod(V_j)$ and $\gamma_j(A) \neq \gamma_j(r_j(\mathfrak{l}))$. Set $B = (b_k(\mathfrak{l})A - \gamma_{j_k}(A) r_{j_k}(\mathfrak{l}))/(\gamma_{j_k}(A) - b_k(\mathfrak{l}))$. Then $\gamma_{j_k}(B) = 0$ and $[B, \mathfrak{l}] = e_{j_k} \bmod(V_{j_k})$, contradicting the fact that $k \in \mathfrak{k}$. This proves the first statement of the lemma.

Let $s = \mathfrak{l}'_j - \mathfrak{l}_j$, so that $\mathfrak{l}' = \mathfrak{l} + s e_{j_k} \bmod(V_{j_k})$. Now

$$\begin{aligned} e_{j_k} &= [r_{j_k}(\mathfrak{l}'), \mathfrak{l}']) = [r_{j_k}(\mathfrak{l}'), \mathfrak{l}] + s[r_{j_k}(\mathfrak{l}'), e_{j_k}] \\ &= [r_{j_k}(\mathfrak{l}'), \mathfrak{l}] + s b_k(\mathfrak{l}') e_{j_k} \bmod(V_{j_k}), \end{aligned}$$

and hence $[r_{j_k}(\mathfrak{l}')/(1 - s b_k(\mathfrak{l}')), \mathfrak{l}] = e_{j_k} \bmod(V_{j_k})$. Thus, by the first statement of this lemma, we have

$$b_k(\mathfrak{l}) = \frac{b_k(\mathfrak{l}')}{1 - s b_k(\mathfrak{l}')},$$

and so $-1/b_k(\mathfrak{l}) = s - 1/b_k(\mathfrak{l}')$, which gives the result. This completes the proof. □

COROLLARY 2.2. *Let $\Omega = \Omega_{\alpha, \mathfrak{k}} \in \wp$ and let $k \in \mathfrak{k}$. Set*

$$\mu_k(s) = \exp[\gamma_{j_k}(\log(s))], \quad s \in G;$$

then b_k is G-semi-invariant with multiplier μ_k^{-1}.

Proof. Let $\mathfrak{l} \in \Omega$ and $s \in G$; set $A = \mu_k(s)^{-1} \operatorname{Ad}(s) r_{j_k}(\mathfrak{l})$. Then $[A, s\mathfrak{l}] = \mu_k(s)^{-1} s([r_{j_k}(\mathfrak{l}), \mathfrak{l}]) = \mu_k(s)^{-1} s e_{j_k} = e_{j_k} \bmod(V_{j_k})$. Thus, by Lemma 2.1, $b_k(s\mathfrak{l}) = \gamma_{j_k}(A) = \mu_k(s)^{-1} b_k(\mathfrak{l})$. □

PROPOSITION 2.3. *There is a G-invariant partition of g^* into algebraic sets Ω such that, for each such set Ω, there are n functions $\{P_j\}$ in $d+n$ real variables $z_1, z_2, \ldots, z_d, \mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_n$ and indices $\{j_k\}$ and $\{k_s\}$, with $1 \leq j_1 < j_2 < \cdots < j_d \leq n$ and $1 \leq k_1 < k_2 < \cdots < k_t \leq n$, having the following properties:*

(1) *for each $\mathfrak{l} \in \Omega$, there is an open set $U = U_{1,\mathfrak{l}} \times U_{2,\mathfrak{l}} \times \cdots \times U_{d,\mathfrak{l}}$ in $\mathbf{R}^d$ such that*

$$G\mathfrak{l} = \{\mathfrak{f} \in g^* : \mathfrak{f} = \textstyle\sum P_j(z, \mathfrak{l}) e_j,\ z \in U\},$$

where $U_{k,\mathfrak{l}} = \mathbf{R}$ if $k \notin \{k_s\}$ and $U_{k,\mathfrak{l}} = (-\infty, 0)$ or $(0, +\infty)$ if $k \in \{k_s\}$;

(2) *$P_{j_k}(z, \mathfrak{l}) = z_k(z_1, z_2, \ldots, z_{k-1})$, and if $k \notin \{k_s\}$ then $P_{j_k}(z, \mathfrak{l}) = z_k$ for every $\mathfrak{l} \in \Omega$;*

(3) *$P_j(z, \mathfrak{l})$ depends only on those z_k for which $j_k \leq j$; and*

(4) *for each z, $P_j(z, \cdot)$ is G-invariant.*

Proof. Let $\wp$ be the layering of g^* define above; each $\Omega \in \wp$ is an algebraic subset of g^* (defined by explicit polynomials).

Now fix $\mathfrak{j} \in J$, $\mathfrak{j} \neq \emptyset$, and write $\mathfrak{j} = \{j_1, j_2, \ldots, j_d\}$. Fix $\alpha \in A_{\mathfrak{j}}$ and $\mathfrak{k} \in K_\alpha$. Let $\{k_s\}$ be the set of all k such that $\tau_{j_k}(\mathfrak{l}) = 2$ and $\mathfrak{l} \in \Omega_\alpha$; note that $\{k_s\} \supset \mathfrak{k}$. For each $\mathfrak{l} \in \Omega = \Omega_{\alpha, \mathfrak{k}}$ set $g_k(t, \mathfrak{l}) = \exp(t r_{j_k}(\mathfrak{l}))$ with $t \in \mathbf{R}$ and $1 \leq k \leq d$, and set

$$g(T,\mathfrak{l}) = g_1(t_1,\mathfrak{l})g_2(t_2,\mathfrak{l}) \cdots g_d(t_d,\mathfrak{l}), \quad T\in \mathbf{R}^d.$$

The proof of Proposition 5 in [5] shows that, for each $\mathfrak{l}\in\Omega$, $G\mathfrak{l} = \{g(T,\mathfrak{l})\mathfrak{l}: T\in\mathbf{R}^d\}$, and $g(T,\mathfrak{l})\mathfrak{l} = \sum Q_j(T,\mathfrak{l})e_j$ where the Q_j $(1\le j\le n)$ satisfy the following:

(1) $Q_j(T,\mathfrak{l}) = 0$ $(t_1,t_2,\ldots,t_k)$, where k is such that $j_k\le j<j_{k+1}$;
(2) $Q_{j_k}(T,\mathfrak{l}) = t_kF(b_k(\mathfrak{l})t_k)\exp[L_k(T,\mathfrak{l})]+S_k(T,\mathfrak{l})$, where $F(x) = (e^x-1)/x$, $L_k(T,\mathfrak{l})$ is a linear form in $t_1,t_2,\ldots,t_{k-1}$ $(L_1(T,\mathfrak{l})\equiv 0)$, and $S_k(T,\mathfrak{l})$ depends only on $t_1,t_2,\ldots,t_{k-1}$.

Moreover, $L_k(T,\mathfrak{l})$ depends only on $\pi_{j_k-1}(\mathfrak{l})$, and $S_k(T,\mathfrak{l})$ is of the form

$$S_k(T,\mathfrak{l}) = \exp[L_k(T,\mathfrak{l})]\mathfrak{l}_{j_k}+R_k(T,\mathfrak{l}),$$

where $R_k(T,\mathfrak{l})$ depends only on $\pi_{j_k-1}(\mathfrak{l})$ and $R_1(T,\mathfrak{l})\equiv 0$.

Define $z_1,z_2,\ldots,z_d$ and $P_j(z,\mathfrak{l})$, $1\le j\le n$, as follows. For each k, if $k\notin\mathfrak{k}$ set $z_k = Q_{j_k}(T,\mathfrak{l})$, and if $k\in\mathfrak{k}$ then

$$z_k = \frac{\exp[b_k(\mathfrak{l})t_k]\exp[L_k(T,\mathfrak{l})]}{b_k(\mathfrak{l})}.$$

The properties of the functions Q_j above allow us to solve for each t_k in terms of $\mathfrak{l}$ and the variables $z_1,z_2,\ldots,z_k$; we write $t_k = \varphi_k(z,\mathfrak{l})$ and substitute into Q_1 to obtain P_1, Q_2 to obtain P_2, and so on. Thus

$$P_j(z,\mathfrak{l}) = Q_j(\varphi_1(z_1,\mathfrak{l}),\varphi_2(z_1,z_2,\mathfrak{l}),\ldots,\varphi_j(z_1,z_2,\ldots,z_j,\mathfrak{l}),\mathfrak{l}),$$

and $\varphi_j(z,\mathfrak{l})$ depends only on $\pi_j(\mathfrak{l})$. In this way we obtain analytic functions $P_1,P_2,\ldots,P_n$ satisfying the first three conditions above.

It remains to show that $P_j(z,\cdot)$ is G-invariant. Fix

$$s\in G \quad\text{and}\quad z=(z_1,z_2,\ldots,z_d).$$

Let $\mathfrak{l}' = \sum P_j(z,\mathfrak{l})e_j$ and $\mathfrak{l}'' = \sum P_j(z,s\mathfrak{l})e_j$; we show that, for each j, $\pi_j(\mathfrak{l}') = \pi_j(\mathfrak{l}'')$ by induction on j. Suppose that $j=1$; we consider cases (1) $1\notin\mathfrak{j}$ and (2) $1\in\mathfrak{j}$.

Case (1). In this case $P_1(z,\mathfrak{l}) = P_1(z,s\mathfrak{l}) = \mathfrak{l}_1$ by construction; hence $\pi_1(\mathfrak{l}') = \pi_1(\mathfrak{l}'') = \mathfrak{l}_1$.

Case (2). Here $j_1=1$. If $1\notin\{k_s\}$ then $P_1(z,\mathfrak{l}) = P_1(z,s\mathfrak{l}) = z_1$; thus $\pi_1(\mathfrak{l}') = \pi_1(\mathfrak{l}'')$. Suppose that $1\in\{k_s\}$. By the form of the function $Q_1(T,\mathfrak{l})$ we have that, for each $\mathfrak{f}\in G\mathfrak{l}$,

$$P_1(z,\mathfrak{f}) = z_1+\mathfrak{f}_1-\frac{1}{b_1(\mathfrak{f})}.$$

We now apply Lemma 2.1 to obtain that $P_1(z,\mathfrak{l}) = P_1(z,s\mathfrak{l})$; hence $\pi_1(\mathfrak{l}') = \pi_1(\mathfrak{l}'')$.

Now suppose that $j>1$ and that $\pi_{j-1}(\mathfrak{l}') = \pi_{j-1}(\mathfrak{l}'')$. We regard W_{j-1} as a subspace of W_j, so that $W_j = W_{j-1}+\mathbf{R}e_j$. We again consider the two cases: (1) $j\notin\mathfrak{j}$; (2) $j\in\mathfrak{j}$.

Case (1). Since $j \notin \mathfrak{j}$, $\tau_j(\mathfrak{l}') = \tau_j(\mathfrak{l}'') = 0$. Because $\pi_j(\mathfrak{l}')$ and $\pi_j(\mathfrak{l}'')$ lie in the same G-orbit in W_j, and since $\pi_{j-1}(\mathfrak{l}') = \pi_{j-1}(\mathfrak{l}'')$, we have $\pi_j(\mathfrak{l}') = G\mathfrak{l} \cap (\pi_{j-1}(\mathfrak{l}') + \mathbf{R}e_j) = G\mathfrak{l} \cap (\pi_{j-1}(\mathfrak{l}'') + \mathbf{R}e_j) = \pi_j(\mathfrak{l}'')$.

Case (2). Let $j = j_k$; first suppose that $k \notin \{k_s\}$. Then $P_{j_k}(z, \mathfrak{l}) = P_{j_k}(z, s\mathfrak{l}) = z_k$ so that $\pi_{j_k}(\mathfrak{l}') = \pi_{j-1}(\mathfrak{l}') + z_k e_{j_k} = \pi_{j-1}(\mathfrak{l}'') + z_k e_{j_k} = \pi_{j_k}(\mathfrak{l}'')$. Suppose then that $k \in \{k_s\}$. We have $\mathfrak{l}' = g(T', \mathfrak{l})\mathfrak{l}$ and $\mathfrak{l}'' = g(T'', s\mathfrak{l})s\mathfrak{l}$, for some $T', T'' \in \mathbf{R}^d$. Now set $g_0' = g_1(t_1', \mathfrak{l}) g_2(t_2', \mathfrak{l}) \cdots g_{k-1}(t_{k-1}', \mathfrak{l})$ and $g_k' = g_k(t_k', \mathfrak{l})$, and define g_0'' and g_k'' similarly, so that $\mathfrak{l}' = g_0' g_k' \mathfrak{l}$ and $\mathfrak{l}'' = g_0'' g_k'' s\mathfrak{l}$. Recalling the definition of μ_k in Corollary 2.2, one sees by the proof of [5, Prop. 5] that $\mu_k(g_0') = \exp[L_k(T', \mathfrak{l})]$ and $\mu_k(g_0'') = \exp[L_k(T'', s\mathfrak{l})]$. Hence, by Corollary 2.2,

$$
(1)\qquad
\begin{aligned}
\frac{1}{b_k(\mathfrak{l}')} &= \frac{\exp[b_k(\mathfrak{l})t_k'] \exp[L_k(T', \mathfrak{l})]}{b_k(\mathfrak{l})} = z_k \\
&= \frac{\exp[b_k(s\mathfrak{l})t_k''] \exp[L_k(T'', s\mathfrak{l})]}{b_k(s\mathfrak{l})} = \frac{1}{b_k(\mathfrak{l}'')}.
\end{aligned}
$$

But by Lemma 2.1, $\mathfrak{l}_{j_k}' - 1/b_k(\mathfrak{l}') = \mathfrak{l}_{j_k}'' - 1/b_{j_k}(\mathfrak{l}'')$, since $\mathfrak{l}'$ and $\mathfrak{l}''$ lie in the same orbit. Thus $\mathfrak{l}_{j_k}' = \mathfrak{l}_{j_k}''$ and $\pi_{j_k}(\mathfrak{l}') = \pi_{j_k}(\mathfrak{l}'')$. This finishes the proof. □

We now describe a cross-section in each layer. Fix $\Omega \in \wp$, and fix $\Omega = \Omega_{\alpha, \mathfrak{k}}$ with $\alpha = \{i_1, i_2, \ldots, i_{d/2}, j(i_1), j(i_2), \ldots, j(i_{d/2})\}$ and with the range of $\alpha = \mathfrak{j} = \{j_1, j_2, \ldots, j_d\}$. For each j, the "type" τ_j at j is constant on Ω, and if $\tau_j = 2$ then $j = i_k$ for some k. Let $\mathfrak{k}^{\sim} = \{k_s\} = \{k : \tau_{i_k} = 2\}$. Set

$$V_{S(\mathfrak{j})} = \operatorname{span}\{e_j : j \in \mathfrak{j}\}, \quad V_{T(\mathfrak{j})} = \operatorname{span}\{e_j : j \notin \mathfrak{j}\}, \qquad V_{\mathfrak{k}^{\sim}} = \operatorname{span}\{e_{i_k} : k \in \mathfrak{k}^{\sim}\}.$$

Let t be the number of elements in $\mathfrak{k}^{\sim}$. For each $i = (i_1, i_2, \ldots, i_t) \in \{-1, 1\}^t$, let $\Omega_i = \Omega_{\alpha, \mathfrak{k}, i} = \{\mathfrak{l} \in \Omega : i_s \in U_{k_s, \mathfrak{l}}, 1 \le s \le t\}$; we have $\Omega = \bigcup \Omega_i$ and $\Omega_i \cap \Omega_{i'} = \emptyset$ if $i \ne i'$. Let $U_i = U_{1,\mathfrak{l}} \times U_{2,\mathfrak{l}} \times \cdots \times U_{d,\mathfrak{l}}$, where $\mathfrak{l} \in \Omega_i$. Define $z(i) \in \mathbf{R}^d$ by $z(i)_k = 0$ if $k \notin \mathfrak{k}^{\sim}$ and by $z(i)_k = i_s$ if $k = k_s \in \mathfrak{k}^{\sim}$. Define

$$\Sigma_i = \Sigma_{\alpha, \mathfrak{k}, i} = \{\mathfrak{l} \in \mathfrak{g}^* : \mathfrak{l} = \Sigma P_j(z(i), \mathfrak{f}) e_j,\ \mathfrak{f} \in \Omega_i\}.$$

By Proposition 2.3, Σ_i is a cross-section for the co-adjoint orbits in Ω_i. From the equations (1) above it follows that $U_{k_s, \mathfrak{l}} = \{b_{k_s}(g\mathfrak{l}) : g \in G\}$, $1 \le s \le t$. Hence $\Omega_i = \{\mathfrak{l} \in \Omega : \operatorname{sgn}(b_{k_s}(\mathfrak{l})) = i_s,\ k_s \in \mathfrak{k}^{\sim}\}$ and

$$\Sigma_i = \{\mathfrak{l} \in (V_{T(\mathfrak{j})} + V_{\mathfrak{k}^{\sim}}) \cap \Omega_\alpha : \text{for each } k_s \in \mathfrak{k}^{\sim},\ b_{k_s}(\mathfrak{l}) = i_s\};$$

thus Ω_i is an open, semi-algebraic subset of Ω, and Σ_i is an algebraic subset. The cross-section $\Sigma = \bigcup \Sigma_i$ for Ω can be described as:

$$\Sigma = \{\mathfrak{l} \in (V_{T(\mathfrak{j})} + V_{\mathfrak{k}^{\sim}}) \cap \Omega_\alpha : \text{for each } k \in \mathfrak{k}^{\sim},\ b_k(\mathfrak{l}) = \pm 1\}.$$

Finally, let $V_{S(\mathfrak{j}), i} = \{\Sigma z_k e_{j_k} : z = (z_1, z_2, \ldots, z_d) \in U_i\}$; the above results can be summarized as follows.

THEOREM 2.3. *Let G be a connected, simply connected, completely solvable Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \cdots \supset \mathfrak{g}_0 = (0)$ be a Jordan-Holder sequence of ideals in $\mathfrak{g}$. Choose a basis $X_1, X_2, \ldots, X_n$ for $\mathfrak{g}$*

such that $X_1, X_2, \ldots, X_j$ *span* $\mathfrak{g}_j$, *and let* $e_1, e_2, \ldots, e_n$ *be the dual basis in* $\mathfrak{g}^*$. *Then there is a finite computable layering* $\mathcal{P}$ *of* $\mathfrak{g}^*$ *with the following properties:*

(a) *each* $\Omega \in \mathcal{P}$ *is G-invariant;*
(b) *for a given* $\Omega \in \mathcal{P}$, *the dimension of the co-adjoint orbits in* Ω *is constant, and* $\tau_j(\mathfrak{l})$ *is constant on* Ω *for* $1 \leq j \leq n$;
(c) *there is a total ordering* $\Omega_1 < \Omega_2 < \cdots < \Omega_p$ *of* $\mathcal{P}$ *such that* $\bigcup\{\Omega_{r'} : r' \leq r\}$ *is Zariski-open in* $\mathfrak{g}^*$, $1 \leq r \leq p$.

Given $\Omega \in \mathcal{P}$ *with associated index sequence* $\{j_1 < j_2 < \cdots < j_d\} = \mathfrak{j}_{\mathfrak{l}}$ $(\mathfrak{l} \in \Omega)$, *let* $\mathfrak{k}^{\sim} = \{k_1 < k_2 < \cdots < k_t\} = \{k : \tau_{j_k} = 2\}$. *For each* $k \in \mathfrak{k}^{\sim}$ *there is a real-valued rational function* b_k *on* $\mathfrak{g}^*$ *such that*

(d) b_k *is nonsingular, nonvanishing, and semi-invariant on* Ω *with multiplier* μ_k^{-1};
(e) *if* $V_{\mathfrak{k}^{\sim}} = \operatorname{span}\{e_{j_k} : k \in \mathfrak{k}^{\sim}\}$, *then the set* $\Sigma = \{\mathfrak{l} \in (V_{T(\mathfrak{j})} + V_{\mathfrak{k}^{\sim}}) \cap \Omega : b_k(\mathfrak{l}) = \pm 1\}$ *is a cross-section for the co-adjoint orbits in* Ω.

For each $i \in \{1, -1\}^t$, *set* $\Omega_i = \{\mathfrak{l} \in \Omega : b_{k_s}(\mathfrak{l}) i_s > 0,\ 1 \leq s \leq t\}$, $\Sigma_i = \Sigma \cap \Omega_i = \{\mathfrak{l} \in \Sigma : b_{k_s}(\mathfrak{l}) = i_s,\ 1 \leq s \leq t\}$, *and* $V_{T(\mathfrak{j}), i} = \{\mathfrak{l} \in V_{T(\mathfrak{j})} : \mathfrak{l}(X_{j_{k_s}}) i_s > 0\}$. *Then*

(f) *for each i there is an analytic diffeomorphism* $\Theta_i : \Sigma_i \times V_{T(\mathfrak{j}), i} \to \Omega_i$ *such that, for each* $\mathfrak{l} \in \Omega_i$, $\Theta_i(\mathfrak{l}, \cdot)$ *is an analytic map whose graph is the orbit of* $\mathfrak{l}$. *If* $\mathfrak{l} \in \Omega_i$ *then* $\Theta_i^{-1}(\mathfrak{l}) = (\mathfrak{l}', \mathfrak{l}'')$, *where* $\mathfrak{l}'$ *is the unique point in* $\Sigma_i \cap \mathrm{Ad}^*(G)\mathfrak{l}$, *and where* $\mathfrak{l}''$ *is defined by* $\mathfrak{l}''(X_{j_k}) = b_k(\mathfrak{l})^{-1}$ *if* $k \in \mathfrak{k}^{\sim}$ *and by* $\mathfrak{l}''(X_{j_k}) = \mathfrak{l}(X_{j_k})$ *if* $k \notin \mathfrak{k}^{\sim}$.

We are currently investigating the extension of these ideas in two directions. Much of the above can be carried over to the case of exponential groups, but there are still some complications to be worked out. Once these are handled, the general solvable case should present few additional difficulties. Secondly, we are considering cross-sections of general double-coset spaces $H \backslash G / K$ (H and K closed, connected) for G solvable. Progress has been made in this direction as well. In [4, Prop. A.2] it is shown that for G nilpotent there is a layering of G into finitely many algebraic sets, and a map $c : G \to G$ which is rational on each layer and whose image is a cross-section for the H-K double cosets.

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