Composition Property of Holomorphic Functions on the Ball

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THEOREM. *Suppose $\varphi \in A(B)$, with $\varphi(B) \subset U$, is holomorphic across its maximum modulus set. Then $g \circ \varphi \in \bigcap_{0<p<\infty} H^p(B)$ for every Bloch function g on U. If, in addition, $\{\varphi^m\}_{m=0}^{\infty}$ forms an orthogonal set in $H^2(B)$, then there exists a weight $\alpha = \alpha(\varphi)$ such that $h \circ \varphi \in H^p(B)$ for every $h \in A_\alpha^p(U)$ and for every p $(0<p<\infty)$.*

This result will be easily derived from a careful analysis of the behavior of such a function φ near its maximum modulus set. For a class of functions φ we obtain the best possible weights $\alpha(\varphi)$. These are nonhomogeneous (even rational) functions, unlike the previous examples of P. Ahern, P. Russo, and the author.

1. Introduction

We will write $B = B_n$ for the open unit ball of $\mathbf{C}^n$ $(n \geq 1)$ and let $S = S_n = \partial B_n$. For $n = 1$, we let $U = B_1$ and $T = S_1$; for further notation see Section 2. Throughout the paper $n \geq 2$ unless otherwise specified.

It has been known that the homogeneous polynomials

$$\varphi(z) = n^{n/2} z_1 \cdots z_n, \quad ([1])$$

$$\varphi(z) = z_1^2 + \cdots + z_n^2, \quad ([15])$$

$$\varphi(z) = b_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad ([2])$$

normalized so that $\varphi(B) = U$, have the following composition property:

If $g \in \mathfrak{B}(U)$, the Bloch space on U, then $g \circ \varphi \in$ BMOA(B).

Here BMOA(B) denotes the space of holomorphic functions in $H^2(B)$ whose boundary functions are of bounded mean oscillations with respect to the non-isotropic balls on S (see [8]). These results have been recently generalized by the author. In [5] it is shown that the same property holds for every φ belonging to a certain class of holomorphic homogeneous polynomials. It is, however, still open whether the same holds for every holomorphic homogeneous polynomial that maps B onto U. It is known that

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$\mathrm{BMOA}(B) \subset H^p(B)$ for every p $(0<p<\infty)$. Thus, before attacking this problem, one might naturally ask the following:

If $g \in \mathfrak{B}(U)$, does $g \circ \varphi \in \bigcap H^p(B)$ then follow for every holomorphic homogeneous polynomial φ normalized so that $\varphi(B) = U$?

Here $\bigcap H^p(B) = \bigcap_{0<p<\infty} H^p(B)$ for simplicity. In this paper we answer this question (and more) in the affirmative. In fact, one of the main results of this paper implies the following.

THEOREM A. *If $\varphi: B \to U$ is holomorphic on $\bar{B}$, then $g \circ \varphi \in \bigcap H^p(B)$ for every $g \in \mathfrak{B}(U)$.*

See Theorem 4.1 below. The hard part of proving Theorem A is to analyze carefully the behavior of such a holomorphic function φ near its *maximum modulus set* $M_\varphi = \varphi^{-1}(T)$ in S. For such a function φ we obtain the following estimate, which is a special case of Theorem 3.3.

THEOREM B. *Let φ be as in Theorem A. If $\|\varphi\|_\infty = 1$, then there exists a positive exponent $\alpha = \alpha(\varphi)$ such that*

$$\sigma\{\xi \in S: |\varphi(\xi)| \geq t\} = O[(1-t)^\alpha] \quad \text{as } t \uparrow 1.$$

The symbol $\sigma = \sigma_n$ denotes the unique normalized rotation-invariant Borel measure on S. Theorem B should be compared with the known result on the maximum modulus sets of functions $\varphi \in A(B)$, where $A(B)$ is (as usual) the ball algebra consisting of functions holomorphic on B and continuous on $\bar{B}$:

If $\varphi: B \to U$, $\|\varphi\|_\infty = 1$, and $\varphi \in A(B) \cap \mathrm{Lip}(\frac{1}{2})$, then $\varphi[M_\varphi] = 0$.

There is a sequence of works towards this result (see [19], [16], [13, §11], and [11, p. 157]; see also [14, §15] and [18] for related results). Observe that Theorem B shows in a more precise way how such a function achieves the conclusion of the above result.

Note that, if φ is holomorphic on $\bar{B}$ and $\|\varphi\|_\infty = 1$, then its maximum modulus set is precisely the zero set of the real-analytic function $1 - |\varphi|^2$ on S. We shall derive Theorem B from a more general estimate obtained in Section 3 on the behavior of real-analytic functions near their zeros. A classical result says that if a real-analytic function on a connected domain vanishes on a set of positive measure, then it vanishes throughout the domain. Hence the following result gives more precise information on how real-analytic functions behave near their zeros.

THEOREM C. *Suppose ψ is real-analytic near $0 \in \mathbf{R}^n$ $(n \geq 1)$. If $\psi(0) = 0$ and ψ is not identically 0, then there exist a positive exponent α and a neighborhood W of $0 \in \mathbf{R}^n$ such that*

$$m_n\{x \in W: |\psi(x)| \leq t\} = O(t^\alpha) \quad \text{as } t \downarrow 0,$$

where m_n denotes the Lebesgue measure on $\mathbf{R}^n$.

In Section 4 we apply a version of Theorem C on the spheres to obtain two composition properties of holomorphic functions. After proving Theorem 4.1 (which implies Theorem A), we also obtain the composition property in the H^p context. It is shown that the homogeneous polynomials $\varphi(z) = n^{n/2} z_1 \cdots z_n$ and $\varphi(z) = z_1^2 + \cdots + z_n^2$ have the composition property (cf., resp., [1] and [15]), not only in the BMOA context as mentioned at the beginning, but also in the following H^p context:

If h belongs to the weighted Bergman space $A^p_{(n-3)/2}(U)$, then $h \circ \varphi \in H^p(B)$ for every p $(0 < p < \infty)$.

These results are sharp in the sense that the weight $(n-3)/2$ is the best possible. In Theorem 4.3 we obtain an analogue of the above for more general functions φ. In case φ is a holomorphic homogeneous polynomial normalized so that $\varphi(B) = U$, we have the following.

THEOREM D. *To every such φ there corresponds a weight $\alpha = \alpha(\varphi) \leq n-2$ such that $h \circ \varphi \in H^p(B)$ for every $h \in A^p_\alpha(U)$ and for every p $(0 < p < \infty)$.*

Unfortunately, our proof of Theorem 4.3 does not produce the best possible weight $\alpha(\varphi)$ in Theorem D. In [6] the best possible weight $\alpha(\varphi)$ in Theorem D can be found if φ belongs to a large class of holomorphic homogeneous polynomials (see Definition 5.3 below). Finally, Section 5 is devoted to the examples of nonhomogeneous functions for which the conclusion of Theorem 4.3 holds. For these examples we also find the best possible weights. Unlike the previous functions, the examples given here are rational functions.

Most of Sections 3 and 4 is a strengthened version of part of the author's Ph.D. thesis, completed at the University of Wisconsin-Madison. The author would like to thank his advisor, Professor W. Rudin, for many helpful suggestions. The author would also like to thank Professor H. O. Kim for many conversations about the examples given in Section 5.

2. Notation and Basic Facts

We shall use the notation $f^*(\xi) = \lim_{r \uparrow 1} f(r\xi)$ for functions $f: B \to \mathbf{C}$, provided the limit exists at $\xi \in S$.

The Hardy space $H^p(B)$ $(0 < p < \infty)$ consists of holomorphic functions f on B such that

$$\|f\|_p^p = \sup_{0<r<1} \int_S |f(r\xi)|^p \, d\sigma(\xi) < \infty.$$

The space $H^p(B)$ is a Banach space with the norm $\| \;\|_p$ for $1 \leq p < \infty$, and is a complete metric space with the metric $d(f, g) = \|f-g\|_p^p$ for $0 < p < 1$. We will use the fact that every convergent sequence in $H^p(B)$ converges uniformly on compact subsets of B. Also, it is well known that if $f \in H^p(B)$ then f^* exists $[\sigma]$ a.e. and

$$\|f\|_p^p = \int_S |f^*|^p \, d\sigma. \tag{1}$$

See [13] for details.

The Bloch space $\mathfrak{B}(U)$ consists of holomorphic functions on U such that

$$\|g\|_{\mathfrak{B}} = |g(0)| + \sup_{\lambda \in U} (1-|\lambda|^2)|g'(\lambda)| < \infty.$$

We refer to [3] for a more detailed description of the space $\mathfrak{B}(U)$. In the present paper we use only the following immediate consequences of the above definition:

$$|g(\lambda)| \le 2\|g\|_{\mathfrak{B}}\left(1 + \log \frac{1}{1-|\lambda|}\right) \quad (\lambda \in U); \tag{2}$$

$$\|g_r\|_{\mathfrak{B}} \le \|g\|_{\mathfrak{B}} \quad (0<r<1). \tag{3}$$

Here g_r denotes the dilated function $\lambda \to g(r\lambda)$ for $\lambda \in U$.

For $\alpha > -1$, the weighted Bergman space $A_\alpha^p(U)$ $(0<p<\infty)$ consists of holomorphic functions h on U such that

$$\int_U |h|^p (1-|\lambda|^2)^\alpha \, dm < \infty.$$

Here m denotes the normalized area measure on $\mathbf{C}$. It is easily verified for $h \in A_\alpha^p(U)$ that

$$\lim_{r \uparrow 1} \int_U |h_r - h|^p (1-|\lambda|^2)^\alpha \, dm = 0. \tag{4}$$

Let ν_n $(n \ge 1)$ denote the normalized volume measure on $\mathbf{C}^n$. Then, for the integration with respect to the measures σ_n (for $n=1$, $\sigma_1 = d\theta/2\pi$) and ν_n, we have the following formula:

$$\int_S f \, d\sigma_n = \binom{n-1}{k} \int_{B_k} \int_{S_l} f(z, \sqrt{1-|z|^2}\,\xi) \, d\sigma_l(\xi) (1-|z|^2)^{l-1} \, d\nu_k(z) \tag{5}$$

for $f \in L^1(\sigma)$, where $n = k+l$. For $l=1$, (5) can be found in [13, §1]. This general form can be proved exactly the same way (see, e.g., [6]).

Finally, the class of holomorphic homogeneous polynomials φ (on $\mathbf{C}^n$), normalized so that $\varphi(B) = U$, will be denoted by $\mathcal{P}_n$ $(n \ge 1)$.

Any unexplained notation will be customary or as in [13].

3. Zeros of Real-Analytic Functions

Throughout this section, $n \ge 1$ except for Theorem 3.3. We first obtain the asymptotic estimate concerning the behavior of real-analytic functions near their zeros, as mentioned in Section 1. We write $x = (x'; x_n)$, where $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$ for $x \in \mathbf{R}^n$.

3.1. THEOREM. *Suppose ψ is real-analytic near $0 \in \mathbf{R}^n$. If $\psi(0)=0$ and ψ is not identically 0 near $0 \in \mathbf{R}^n$, then there exist a positive exponent α and a neighborhood W of $0 \in \mathbf{R}^n$ such that*

$$(1) \qquad m_n\{x \in W: |\psi(x)| \le t\} = O(t^\alpha) \quad \textit{as } t \downarrow 0.$$

Proof. Without loss of generality, we assume that ψ is real-valued. In case $n=1$, the theorem is trivial because zeros of real-analytic functions are isolated and have finite multiplicities. We now proceed by induction on the dimension n. Fix $n \ge 2$ and assume, as an induction hypothesis, that the theorem has been verified in all the lower-dimensional cases.

Since ψ is not identically 0, we may assume (after change of coordinate systems) that the equation

$$\psi(0; x_n) = 0$$

has isolated zero(s) at $x_n = 0$. Note that if two functions satisfy inequality (1), then so does their product (with a possibly different exponent). Hence, by the Weierstrass preparation theorem [12, Cor. 3.7], we may assume that ψ is a *distinguished* polynomial in $\mathcal{O}_{n-1}[x_n]$ ($\mathcal{O}_{n-1}$ denotes the ring of germs at $0 \in \mathbf{R}^{n-1}$ of real-valued real-analytic functions); that is,

$$\psi(x'; x_n) = x_n^d + a_1(x')x_n^{d-1} + \cdots + a_d(x') \quad \text{near } 0 \in \mathbf{R}^n$$

for some d and $a_i \in \mathcal{O}_{n-1}$ such that $a_i(0) = 0$ $(i = 1, 2, \ldots, d)$, and further that ψ is irreducible in $\mathcal{O}_{n-1}[x_n]$ (see [12, Remark 3.11]). To avoid triviality let $d \ge 2$. Choose a neighborhood V of $0 \in \mathbf{R}^{n-1}$ so that all a_i's are defined on V. By Rouche's theorem we can choose V in such a way that the complex zeros of the polynomial $\psi_{x'}$ stay bounded uniformly in $x' \in V$, where $\psi_{x'}(x_n) = \psi(x'; x_n)$. Now, given $x' \in V$, let $\Delta(x')$ be the discriminant of the polynomial $\psi_{x'}$. Being a polynomial in a_i's, Δ is a real-analytic function on V such that $\Delta(0) = 0$. Since ψ is irreducible, Δ is not identically 0. Thus, by induction hypothesis, we may assume that Δ satisfies inequality (1) with n and W replaced (respectively) by $(n-1)$ and V, after shrinking V if necessary.

Let $E(x')$ be the set of real zeros of $\psi_{x'}$ for $x' \in V$. Define, for $(x'; x_n) \in V \times \mathbf{R}$,

$$\delta(x'; x_n) = \begin{cases} \min_{r \in E(x')} |x_n - r| & \text{if } E(x') \ne \phi, \\ 1 & \text{if } E(x') = \phi. \end{cases}$$

Then there exists a constant $C \ge 1$ such that

$$\delta(x'; x_n) \le C \quad \text{for all } (x'; x_n) \in V \times [-1, 1].$$

Now fix $x' \in V$ and let $c_1, \ldots, c_k$ be the complex, but not real, zeros of the polynomial $\psi_{x'}$ (counting multiplicities). Then, for every $(x; x_n) \in V \times [-1, 1]$,

$$|\psi(x'; x_n)| = \prod_{r \in E(x')} |x_n - r|^{m(r)} \prod_{i=1}^{k} |x_n - c_i| \ge C^{-d}[\delta(x'; x_n)]^d \prod_{i=1}^{k} |\operatorname{Im} c_i|,$$

where $m(r)$ is the multiplicity of the zero at $x_n = r$. Since $\Delta(x')$ is the product of the squares of the differences of all complex zeros of $\psi_{x'}$, we have

$$\prod_{i=1}^{k} |\operatorname{Im} c_i| \geq 2^{-d} \Delta(x').$$

Hence the proof of the inequality

$$m_n\{(x'; x_n) \in V \times [-1, 1] : \delta(x'; x_n) \leq t\} = O(t) \tag{2}$$

will complete the induction. But it is easy to see that, for t sufficiently small,

$$m_1\{x_n \in [-1, 1] : \delta(x'; x_n) \leq t\} = O(t)$$

uniformly in $x' \in V$. Finally, an application of Fubini's theorem shows (2). □

Note that Theorem 3.1 is a purely local theorem. Hence we can easily prove a global version on S^n, the unit sphere of $\mathbf{R}^{n+1}$, by compactness. Let us introduce some terminology. Suppose $\psi \in C(S^n)$. Then ψ is called *real-analytic on an open subset V of S^n* if ψ has a real-analytic continuation to some open set $W \subset \mathbf{R}^{n+1}$ such that $V = W \cap S^n$. Also ψ is called *real-analytic on a compact subset K of S^n* if ψ is real-analytic on some neighborhood of K in S^n.

3.2. THEOREM. *Let $\psi \in C(S^n)$, and suppose that ψ is real-analytic on $\psi^{-1}(0)$. If $\psi^{-1}(0)$ has empty interior, then there exists a positive exponent α such that*

$$h_n\{x \in S^n : |\psi(x)| \leq t\} = O(t^\alpha) \quad \textit{as } t \downarrow 0,$$

where h_n denotes the n-dimensional Hausdorff measure on S^n.

Proof. By compactness, it is enough to show that to each $x_0 \in S$ there correspond neighborhoods $V = V(x_0)$ and $\alpha = \alpha(x_0) > 0$ such that

$$h_n\{x \in V : |\psi(x)| \leq t\} = O(t^\alpha). \tag{1}$$

Note that (1) holds trivially if $x_0 \notin \psi^{-1}(0)$. Fix $x_0 \in \psi^{-1}(0)$. We may assume, without loss of generality, that $P = (0, 0, \dots, 0, 1) \notin \psi^{-1}(0)$. Let $\phi : \mathbf{R}^n \to S^n \setminus P$ be the inverse function of the stereographic projection from $S^n \setminus P$ onto $\mathbf{R}^n$. The explicit formula of ϕ is given by

$$\phi(y) = \frac{(2y, |y|^2 - 1)}{1 + |y|^2} \quad (y \in \mathbf{R}^n).$$

Note that each component of ϕ is real-analytic everywhere on $\mathbf{R}^n$. By assumption, $\psi \circ \phi$ is therefore a real-analytic function, not identically 0, on some neighborhood of $y_0 = \phi^{-1}(x_0) \in \mathbf{R}^n$. Thus, by Theorem 3.1, we have $\alpha > 0$ and a neighborhood W of y_0 such that

$$m_n\{y \in W : |\psi \circ \phi(y)| \leq t\} = O(t^\alpha). \tag{2}$$

Since $J\phi$, the Jacobian of ϕ, is bounded on $\mathbf{R}^n$ [one may compute $J\phi(y) = (2/(1 + |y|^2))^n$], by the area formula [9, Thm. 3.2.5] we obtain (1) from (2) with $V = \phi(W)$. The proof is complete. □

The following is an immediate consequence of Theorem 3.2.

3.3. THEOREM. *Suppose* $\varphi \in A(B)$ *(recall* $n \geq 2$*) is nonconstant,* $\|\varphi\|_\infty = 1$, *and* φ *is real-analytic on* M_φ. *Then there exists a positive exponent* α *such that*

$$\phi\{\xi \in S : |\varphi(\xi)| \geq t\} = O[(1-t)^\alpha] \quad \text{as } t \uparrow 1.$$

Proof. Since $\phi \in A(B)$ is nonconstant, M_φ has empty interior by [14, Thm. 1.2]. Also, note that M_φ is the zero set of the real-analytic function $1 - |\varphi|^2$ on M_φ. Now the theorem follows from Theorem 3.2. □

3.4. REMARKS. (1) Even when $n = 1$, Theorem 3.3 holds if M_φ has empty interior.

(2) The real-analyticity assumption in Theorem 3.3 implies that φ is holomorphic on $B \cup M_\varphi$. To see this, suppose φ is real-analytic on some connected open set $W \subset \mathbf{C}^n$ containing a component of M_φ. Then $\partial\varphi/\partial\bar{z}_i = 0$ $(1 \leq i \leq n)$ on $W \cap B$, and hence $\partial\varphi/\partial\bar{z}_i = 0$ on W by real-analyticity.

4. Composition Properties

In this section we give two applications of Theorem 3.3 concerning the composition properties of holomorphic functions on the ball.

4.1. THEOREM. *Suppose* $\varphi \in A(B)$ *satisfies the hypothesis of Theorem 3.3. Then* $g \circ \varphi \in \bigcap H^p(B)$ *for every* $g \in \mathfrak{B}(U)$.

Proof. Let $0 < p < \infty$ and fix $g \in \mathfrak{B}(U)$. First of all, it is clear that $g \circ \varphi$ is holomorphic on B. By 2.(2) and 2.(3) we have

$$|g_r \circ \varphi^*| \leq 2\|g\|_{\mathfrak{B}}\left(1 + \log \frac{1}{1-|\varphi^*|}\right) \quad (0 < r < 1).$$

Note that, by Theorem 3.3, $(1-|\varphi^*|)^{-\alpha} \in L^1(\sigma)$ for some $\alpha > 0$ and that $(g_r \circ \varphi)^* = g_r \circ \varphi^*$ for $0 < r < 1$. Hence, by the dominated convergence theorem and 2.(1), there exists a function $f \in H^p(B)$ such that $\lim_{r \uparrow 1} g_r \circ \varphi = f$ in the topology of H^p-space. In particular, $g_r \circ \varphi$ converges pointwise to f on B as $r \uparrow 1$, and hence $g \circ \varphi = f$. The proof is complete. □

In what follows, a function $\varphi \in H^\infty(B)$ is called *orthogonal* if $\|\varphi\|_\infty = 1$ and $\{\varphi^m\}_{m=0}^\infty$ forms an orthogonal set in $H^2(B)$. It is not hard to see that φ is orthogonal if and only if the pullback measure $\sigma[(\varphi^*)^{-1}]$ is rotation-invariant on $\bar{U}$. Every $\varphi \in \mathcal{P}_n$ is trivially orthogonal. Using 2.(5), one may construct nonhomogeneous orthogonal functions for $n \geq 2$. In Section 5 we consider a special class of orthogonal functions obtained this way.

Our next application of Theorem 3.3 will be the composition property of orthogonal functions. Before proceeding, we recall the characterization of Carleson measures for the weighted Bergman spaces. The set $\{\lambda \in U : |1 - \lambda e^{-i\theta}| < t\}$ will be denoted by $Q(\theta, t)$.

4.2. THEOREM ([7], [10], [17]). *Suppose μ is a finite positive Borel measure on U. Let $\alpha > -1$ and $0 < p < \infty$. Then there is a constant $C_{p,\alpha} > 0$ such that*

$$\int_U |f|^p \, d\mu \le C_{p,\alpha} \int_U |f|^p (1-|\lambda|^2)^\alpha \, dm$$

for every $f \in A^p_\alpha(U)$ if and only if $\mu[Q(\theta, t)] = O(t^{2+\alpha})$ uniformly in θ.

4.3. THEOREM. *Let $\varphi \in A(B)$ be an orthogonal function. If φ is real-analytic on M_φ, then there exists $\alpha > -1$ such that $h \circ \varphi \in H^p(B)$ for every $h \in A^p_\alpha(U)$ and for every p $(0 < p < \infty)$.*

Proof. Let $\mu_\varphi = \sigma[(\varphi^*)^{-1}]$. Then, by Theorem 3.3, μ_φ is concentrated on U and

$$\mu_\varphi\{\lambda \in U: |\lambda| \ge t\} = O[(1-t)^{1+\alpha}] \tag{1}$$

for some $\alpha > -1$. Since μ_φ is rotation-invariant, we have by Fubini's theorem that

$$\mu_\varphi[Q(0,t)] = \int_0^{2\pi} \mu_\varphi[Q(\theta, t)] \frac{d\theta}{2\pi} = \int_U \sigma_1[I(\lambda, t)] \, d\mu_\varphi(\lambda), \tag{2}$$

where $I(\lambda, t) = \{e^{i\theta} \in T: |1 - \lambda e^{-i\theta}| < t\}$. Since $I(\lambda, t) = \phi$ for $|\lambda| \le 1-t$ and $\sigma_1[I(\lambda, t)] = O(t)$ uniformly in λ, we obtain from (1) and (2) that

$$\mu_\varphi[Q(\theta, t)] = \mu_\varphi[Q(0,t)] = O(t^{2+\alpha}) \tag{3}$$

for every θ. Now fix p $(0 < p < \infty)$ and $h \in A^p_\alpha(U)$. Then—by (3), 2.(4), and Theorem 4.2—there exists a function $f \in H^p(B)$ such that $\lim_{r \uparrow 1} h_r \circ \varphi = f$ in the topology of H^p-space. It follows that $h \circ \varphi \in H^p(B)$ as in the proof of Theorem 4.1. □

We conclude this section with the following remark.

4.4. REMARK. Let $\varphi \in \mathcal{P}_n$. Since homogeneity is invariant under the unitary change of variables, we may assume that $\varphi(1, 0, \ldots, 0) = 1$. Then, by 2.(5),

(1)

$$\int_S |\varphi|^{2k} \, d\sigma = (n-1) \int_U \int_{S_{n-1}} |\varphi(\lambda, \sqrt{1-|\lambda|^2}\,\xi)|^{2k} \, d\sigma_{n-1}(\xi)(1-|\lambda|^2)^{n-2} \, dm(\lambda)$$

for $k = 0, 1, 2, \ldots$. Since $|\varphi|^{2k}$ is pluri-subharmonic, the double integral in the right side of (1) is at least

$$\begin{aligned} &\int_U |\varphi(\lambda, 0, \ldots, 0)|^{2k} (1-|\lambda|^2)^{n-2} \, dm(\lambda) \\ &\quad = \int_U |\lambda|^{2dk} (1-|\lambda|^2)^{n-2} \, dm(\lambda) \quad [d = \deg(\varphi)] \\ &\quad = \frac{\Gamma(dk+1)\Gamma(n-1)}{\Gamma(dk+n)}. \end{aligned}$$

By Stirling's formula, the above behaves like k^{1-n} as $k\to\infty$. On the other hand, if

$$\sigma\{\xi\in S: |\varphi(\xi)|\ge t\}=O[(1-t)^{1+\alpha}] \quad (\alpha>-1), \tag{2}$$

then

$$\int_S|\varphi|^{2k}\,d\sigma=2k\int_0^1 t^{2k-1}O[(1-t)^{1+\alpha}]\,dt$$
$$=O\left[\frac{\Gamma(2k+1)\Gamma(2+\alpha)}{\Gamma(2k+2+\alpha)}\right].$$

Again, by Stirling's formula the above is dominated by $k^{-(\alpha+1)}$, and therefore the exponent α in (2) is at most $n-2$. Accordingly, the proof of Theorem 4.3 shows that the weight α associated with φ (as in Theorem 4.3) is at most $n-2$. This maximal weight $n-2$ can be achieved for functions φ of the form $\varphi(z)=a_1z_1^d+\cdots+a_nz_n^d$ $(d\ne 2)$; see [6].

5. Examples

For $\varphi\in H^\infty(B)$, we let $\mu_\varphi=\sigma[(\varphi^*)^{-1}]$. If $\mu_\varphi\ll m$, we then write $w_\varphi=d\mu_\varphi/dm$. Note that, if φ is an orthogonal function such that $\mu_\varphi\ll m$, then w_φ is radial. In this section we will consider a special class of rational orthogonal functions φ such that $\mu_\varphi\ll m$ and

$$w_\varphi(r)=c(1-r)^{\delta(\varphi)/2-1}[1+o(1)] \quad \text{as } r\uparrow 1 \tag{*}$$

for some constant $c=c(\varphi)$. Here $\delta(\varphi)=(2n-1)-$(topological dimension of M_φ); that is, $\delta(\varphi)$ is the topological co-dimension of M_φ in S. [For $\varphi\in H^\infty(B)$, we let $M_\varphi=\{\xi\in S: \sup_{0\le r<1}|\varphi(r\xi)|=\|\varphi\|_\infty\}$.] It is clear that for such φ the conclusion of Theorem 4.3 holds, with the best possible weight $\alpha=\delta(\varphi)/2-1$. It is known [6] that if $\varphi\in\mathcal{P}_n$ then $\mu_\varphi\ll m$ and $w_\varphi>0$ on U.

Fix positive integers k,l such that $k+l=n$, let $f\in\mathcal{P}_k$ and $g\in\mathcal{P}_l$ with degrees at least 2, and define

$$\varphi(z,w)=\frac{g(w)}{1-f(z)} \quad \text{for } (z,w)\in B. \tag{**}$$

Since degrees of f and g are at least 2, we have (by homogeneity)

$$|f(z)|+|g(w)|<1 \quad \text{for } (z,w)\in B,$$

and hence $\varphi\in H^\infty(B)$. Also, φ is orthogonal by 2.(5). Put $\deg(f)=2d$ and $\deg(g)=2e$. By assumption, $d,e\ge 1$. It is not difficult to verify that $\varphi\in A(B)$ if $e>1$, while φ does not extend to a continuous function on $\bar{B}$ if $e=1$. The case $f(z)=z_1^2+\cdots+z_k^2$ and $g(w)=w_1^2+\cdots+w_l^2$ appears in [4].

5.1. SOME AUXILIARY FUNCTIONS. Let $0\le s\le 1$ and $0\le t\le 1$. Define, for $0\le x\le 1$,

$$\alpha_{s,t}(x)=\frac{(1-x)^e t}{1+x^d s}, \qquad \beta_{s,t}(x)=\frac{(1-x)^e t}{1-x^d s}.$$

By elementary calculus, $\alpha_{s,t}$ and $\beta_{s,t}$ are strictly decreasing on $[0,1]$ unless $d=e=s=1$. Let $\delta_{s,t}$ and $\epsilon_{s,t}$ be the inverse functions, of (respectively) $\alpha_{s,t}$, and $\beta_{s,t}$, on $[0,t]$ except for the case $d=e=s=1$. Define

$$(1) \qquad E_{s,t}(r)=\int_{\delta_{s,t}(r)}^{\epsilon_{s,t}(r)} \frac{x^{k-1}(1-x)^{l-1}\,dx}{\sqrt{1-(r/\beta_{s,t}(x))^2}\,\sqrt{(r/\alpha_{s,t}(x))^2-1}} \qquad (0\le r\le t).$$

Now assume $k\ge 2$ and $l\ge 2$ for simplicity (the analysis that follows for the case $k=1$ or $l=1$ is a bit simpler in notation). By 2.(5), integration in polar coordinates, and change of variables, we then have, for an arbitrary nonnegative Borel function ψ on $[0,1]$,

$$\int_S \psi\circ|\varphi^*|\,d\sigma = c\int_0^1 x^{k-1}(1-x)^{l-1}\int_{S_k}\int_{S_l}\psi\left(\left|\frac{(1-x)^e g(\eta)}{1-x^d f(\xi)}\right|\right)d\sigma_l(\eta)\,d\sigma_k(\xi)\,dx.$$

Here (and in the sequel) we handle the constants $c=c(k,l,f,g)$ in the usual manner; they are not necessarily the same at any two occurrences. By definitions of w_f and w_g, the triple integral of the right side of the above is the same as

$$c\int_0^1\int_0^1\left\{\int_0^1\int_0^\pi x^{k-1}(1-x)^{l-1}\psi\left(\left|\frac{(1-x)^e t}{1-x^d s e^{i\theta}}\right|\right)d\theta\,dx\right\}t w_g(t) s w_f(s)\,dt\,ds.$$

In the double integral inside braces in the above, we make change of variables

$$r=\frac{(1-x)^e t}{\sqrt{(1-x^d s)^2+2x^d s(1-\cos\theta)}} \qquad (s,t,x \text{ fixed})$$

and interchange the order of integrations to see that the resulting double integral is equal to

$$2\int_0^t E_{s,t}(r)\psi(r)\frac{dr}{r}.$$

It follows that

$$\int_S \psi\circ|\varphi^*|\,d\sigma = c\int_0^1 \psi(r)\left\{\int_0^1\int_r^1 E_{s,t}(r)\,t w_g(t) s w_f(s)\,dt\,ds\right\}\frac{dr}{r}.$$

In other words, if $\tau_\varphi=\sigma[(|\varphi^*|)^{-1}]$ then

$$d\tau_\varphi(r)=c\int_0^1\int_r^1 E_{s,t}(r)\,t w_g(t) s w_f(s)\,dt\,ds\frac{dr}{r}.$$

But, since μ_φ is rotation-invariant, there is a simple relation between μ_φ and τ_φ:

$$d\mu_\varphi(r,\theta)=d\tau_\varphi(r)\,d\sigma_1(\theta).$$

Let us summarize these observations.

5.2. PROPOSITION. *Let φ, f, g be as in (**). Then $\mu_\varphi \ll m$, $w_\varphi>0$ on U, and*

$$w_\varphi(r)=\frac{c}{r^2}\int_0^1\int_r^1 E_{s,t}(r)\,t w_g(t) s w_f(s)\,dt\,ds.$$

5.3. DEFINITION. Let $\varphi \in \mathcal{P}_n$. We say $\varphi \in \Omega_n$ if w_φ satisfies (*). Then it turns out ([6]) that Ω_n contains a large part of $\mathcal{P}_n$. For $n=1$, we simply let $\Omega_1 = \mathcal{P}_1$. We remark in passing that it is not known whether $\Omega_n = \mathcal{P}_n$ for $n \geq 2$.

We now prove the following.

5.4. PROPOSITION. *Let φ, f, g be as in (**), and suppose that $g \in \Omega_l$.*

(1) *If $d>1$ or $e>1$, then (*) holds for φ.*

(2) *If $e=1$, $f(z)=z_1^2+\cdots+z_k^2$,* and *M_g is a disjoint union of compact manifolds, then (*) holds for φ.*

Proof.

Case (1). Since $d>1$ or $e>1$, it is easily verified that

$$M_\varphi = \{(0,\eta) : \eta \in M_g\}$$

and hence $\delta(\varphi) = 2k + \delta(g)$. If $d=1$ and $e>1$, then

$$\text{(3)}\qquad \begin{aligned} \delta_{s,t}(r) &= \frac{t-r}{e+s}[1+O(1-r)] \\ \epsilon_{s,t}(r) &= \frac{t-r}{e-s}[1+O(1-r)] \end{aligned}$$

uniformly in $t \geq r$ and s. Also, we have by (3) and the mean value theorem that

$$\text{(4)}\qquad \begin{aligned} 1-\left(\frac{r}{\beta_{s,t}(x)}\right)^2 &= 2(e-s)(\epsilon_{s,t}(r)-x)[1+O(1-r)] \\ \left(\frac{r}{\alpha_{s,t}(x)}\right)^2 - 1 &= 2(e+s)(x-\delta_{s,t}(r))[1+O(1-r)] \end{aligned}$$

uniformly in $x \in (\delta_{s,t}(r), \epsilon_{s,t}(r))$, $t \geq r$, and s.

Insert (4) into 5.1.(1) and make the substitution

$$x = (\epsilon_{s,t}(r) - \delta_{s,t}(r))y + \delta_{s,t}(r).$$

Because

$$x = \frac{t-r}{e+s}\left(\frac{2sy}{e-s}+1\right)^{-1}[1+O(1-r)]$$

by (3) and hence $x = O(1-r)$ uniformly in $t \geq r$ and s, we then have

$$\text{(5)}\qquad E_{s,t}(r) = \frac{1}{2}\left(\frac{t-r}{e+s}\right)^{k-1}\frac{[1+O(1-r)]}{\sqrt{e^2-s^2}}\int_0^1\left(\frac{2sy}{e-s}+1\right)^{1-k}\frac{dy}{\sqrt{y(1-y)}}.$$

If $d>1$ and $e \geq 1$, then a similar (and simpler) argument shows that

$$\text{(6)}\qquad E_{s,t}(r) = c(t-r)^{k-1}[1+O(\sqrt{1-r}\,)].$$

The error terms in (5) and (6) are uniform in $t \geq r$ and s. It follows from (5), (6), and Proposition 5.2 that

$$w_\varphi(r)=[c+o(1)]\int_r^1 (t-r)^{k-1}(1-t)^{\delta(g)/2-1}\,dt.$$

Now a little computation shows (*) for w_φ. □

Case (2). In this case, since $d=e=1$, we have

$$M_\varphi=\{(r\xi,\sqrt{1-r^2}\eta):0\le r<1,\ \xi\in f^{-1}(1),\ \eta\in M_g\}$$

and hence $\delta(\varphi)=\delta(g)+k$. The estimate of w_φ is a little complicated because such uniform estimate for $E_{s,t}$ as in (5) or (6) is not possible. We know, however, the precise formula of w_f [15]:

$$w_f(r)=\frac{k-1}{2}(1-r^2)^{(k-3)/2}.$$

Hence, by Proposition 5.2, $w_\varphi(r)$ is the same as $c[1+O(1-r)]$ times

(7)

$$\int_r^1\left\{\int_0^1\int_{\delta_{s,t}(r)}^{\epsilon_{s,t}(r)}\frac{x^{k-1}(1-x)^{l+1}\,dxs(1-s^2)^{(k-3)/2}\,ds}{\sqrt{(1-x)^2t^2-r^2(1-xs)^2}\,\sqrt{r^2(1+xs)^2-(1-x)^2t^2}}\right\}w_g(t)\,dt.$$

Note that $\epsilon_{s,t}(r)=(t-r)/(t-rs)$ and $\delta_{s,t}(r)=(t-r)/(t+rs)$. In the double integral inside the braces we make changes of variables successively:

$$sx=\rho\quad (x\text{ fixed}),$$

$$y=\frac{1-x}{1-\rho}\quad (\rho\text{ fixed});$$

this yields

$$\int_{t/r}^1 F_{y,t}(r)\frac{(1-y^2)^{(k-3)/2}y^{l+1}\,dy}{(yt+r)\sqrt{yt-r}}.$$

Here

$$F_{y,t}(r)=\int_{\gamma_{y,t}(r)}^1\frac{\rho(1-\rho)^{l+(k-1)/2}(\rho+(1-y)/(1+y))^{(k-3)/2}}{\sqrt{\rho-\gamma_{y,t}(r)}\,\sqrt{r+yt+\rho(r-yt)}}\,d\rho$$

and

$$\gamma_{y,t}(r)=\frac{yt-r}{yt+r}.$$

It is easy to see that

$$F_{y,t}(r)=c+o(1)\quad\text{uniformly in } yt\ge r,$$

and therefore the integral in (7) is equal to

$$[c+o(1)]\iint_{yt\ge r}\frac{(1-y)^{(k-3)/2}(1-t)^{\delta(g)/2-1}}{\sqrt{yt-r}}\,dy\,dt.$$

This implies (*) for w_φ. □

Note that the proof of Proposition 5.4, Case (1), shows that if

$$w_g(r)=c(1-r)^{\delta(g)/2-1}[1+\epsilon(1-r)],$$

where $\epsilon(t) \to 0$ as $t \downarrow 0$, then

$$w_\varphi(r) = c(1-r)^{\delta(\varphi)/2-1}[1+\epsilon(1-r)][1+O\sqrt{1-r}\,)].$$

Obviously, by the same technique there exists some other type of functions satisfying (*).

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