Spectral Invariants of CR Manifolds

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The trace of the heat semigroup for the Laplacian on a Riemannian manifold has an asymptotic expansion in powers of the time t for small positive t whose coefficients are integrals of local Riemannian invariants (see [1], [8], and [13]). Parker and Rosenberg [14] and Branson and Ørsted [5] showed that in the analogous expansion for the conformal Laplacian, certain coefficients give conformal invariants—the coefficient of t^0 is a global conformal invariant and the coefficient of t^{-1} is the integral of a local conformal invariant of weight -2.

Analogues of the Riemannian result hold for the sublaplacian $\Box_b$ acting on forms of suitable degree on a compact strictly pseudoconvex CR manifold equipped with a Levi metric ([3] and [18]). In this paper we show that the analogue of the conformal result holds for the pseudoconformal Laplacian—the CR analogue of the conformal Laplacian—on a compact strictly pseudoconvex CR manifold. The resulting invariants are pseudoconformal invariants (i.e., CR invariants) of the manifold.

After giving the background in Section 1, we prove our first main theorem—that the coefficient of t^0 is a CR invariant—in Section 2. The proof follows the outline of the one in [14]. In Section 3 we show that the coefficient of t^{-1} is the integral of a local CR invariant. Again the proof follows the outline of the one in [14]. However, new complications arise because the heat kernel for the pseudoconformal Laplacian does not have a parametrix with a simple form. In Section 4 we calculate the asymptotic expansion of the trace of the heat semigroup for the pseudoconformal Laplacian (with respect to the standard metric) on spheres and show that the invariants are 0 on spheres. In Section 5 we show that the invariants vanish on all three-dimensional CR manifolds.

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1. Background

Let M be a compact strictly pseudoconvex CR manifold of dimension $2n+1$. The CR structure is defined by a complex rank n subbundle $T_{1,0}$ of the complexified tangent bundle $\mathbf{C}\otimes TM$ with the properties

(1.1) $$T_{1,0}\cap T_{0,1}=\{0\}, \quad \text{where } T_{0,1}=\overline{T_{1,0}};$$

(1.2) if Z and W are smooth sections of $T_{1,0}$, then so is $[Z, W]$.

Let θ be a real nonvanishing 1-form which annihilates $T_{1,0}$. The *Levi form* determined by θ is the Hermitian form L_θ on $T_{1,0}$ defined by

(1.3) $$L_\theta(Z, W)=-id\theta(Z, \bar{W}).$$

Because M is strictly convex, L_θ is definite; thus, by changing the sign of θ if necessary, we may assume that it is positive definite. Following Webster [20], we call (M, θ) a *pseudo-hermitian manifold*. There is a unique vector field X on M satisfying

(1.4) $$\theta(X)=1, \qquad i(X)d\theta=0.$$

The 1-form θ determines a Hermitian metric on M by

$$X\perp T_{1,0} \quad \text{and} \quad |X|=1;$$

(1.5) $$T_{1,0}\perp T_{0,1} \text{ and conjugation is an isometry;}$$

$$\langle Z, W\rangle=L_\theta(Z, W) \quad \text{for } Z, W\in T_{1,0}.$$

This metric is a Levi metric.

Let H^* denote the orthogonal complement of θ in $T^*(M)$, so that H^* is the dual of the maximal complex tangent space $H(M)$, and let π denote orthogonal projection onto H^*. Let

(1.6) $$d_b=\pi\circ d\colon C^\infty(M)\to H^*$$

and define the *sublaplacian*

(1.7) $$\Delta_b=d_b^* d_b.$$

The sublaplacian depends on both the CR structure and the choice of 1-form θ. We have the freedom to make a *pseudoconformal change* of pseudohermitian structure, that is, to multiply θ by a positive C^∞ function e^{2f}. We call e^{2f} a *pseudoconformal factor*. The sublaplacian is not invariant under pseudoconformal changes; the following lemma shows how it changes.

LEMMA 1.8. *Let $\widetilde{\Delta_b}$ be the sublaplacian for the pseudoconformal structure $\tilde{\theta}=e^{2f}\theta$. Then*

(1.9) $$\widetilde{\Delta_b}v=e^{-2f}(\Delta_b v-2n\langle d_b v, d_b f\rangle).$$

Proof. This follows by a straightforward calculation from [11, Prop. 4.10 and Lemma 5.6]. □

We introduce the operator

$$D = b_n \Delta_b + R, \tag{1.10}$$

where R is Webster's scalar curvature [20] and $b_n = 2 + 2/n$. This operator satisfies the following conformal invariance property.

PROPOSITION 1.11 [9]. *Let $\tilde{\theta} = r^{2/n}\theta$, where r is a positive function. The corresponding operator $\tilde{D}$ is given by the transformation rule*

$$\tilde{D} r^{-1} = r^{-1-2/n} D. \tag{1.12}$$

This transformation rule is analogous to the one satisfied by the conformal Laplacian in pseudo-Riemannian geometry. In fact, by [11, Prop. 6.1 and Thm. 6.2], the pseudoconformal wave operator on the Fefferman bundle over M pushes down to $2D$, so Proposition 1.11 follows from the pseudo-Riemannian result for the pseudoconformal wave operator. This proof, which is given in [9], requires the pseudo-Riemannian result and also the Moser normal form (which is used to prove [11, Thm. 6.2]). One can also verify (1.12) directly, using Lemma 1.8 and the transformation rule for R [11, Prop. 5.15]. By analogy with the Riemannian case, we call D the *pseudoconformal Laplacian.*

If $\{Z_\alpha\}$ is a local orthonormal frame for $T_{1,0}$, then

$$\Delta_b u = -\sum_{\alpha} (Z_\alpha \overline{Z_\alpha} + \overline{Z_\alpha} Z_\alpha) u + 2 \operatorname{Re} \sum_{\alpha,\beta} \omega_\beta{}^\alpha(\overline{Z_\beta}) Z_\alpha u \tag{1.13}$$

where $(\omega_\beta{}^\alpha)$ is the connection matrix for the Tanaka–Webster connection with respect to the dual basis $\{\theta^\alpha\}$ of $T^*_{1,0}$ (see [11, Prop. 4.10]). Formula (1.13) shows that D is uniform in the sense of [3, Def. 4.12] and satisfies the hypotheses of [3, Thm. 5.22]. Hence, because D is bounded below, [3, Thm. 7.30] remains true with $\Box_{b,q}$ replaced by D. Thus, if we let e^{-tD} denote the one-parameter semigroup generated by $-D$ and $p(x,y,t)$ the kernel of e^{-tD}, we have the following theorem.

THEOREM 1.14. *Let M be a compact strictly pseudoconvex CR manifold of dimension $2n+1$ equipped with a pseudo-hermitian structure θ and the corresponding metric. Then*

$$\operatorname{tr}(e^{-tD}) \sim t^{-n-1} \sum_{j=0}^{\infty} k_j t^j \quad as \ t \to 0+, \tag{1.15}$$

where

$$k_j = \int_M K_j(x)\, dV(x) \tag{1.16}$$

and the function K_j may be evaluated at x by evaluating a polynomial (depending only on n and j) in the components of the curvature and torsion of the Tanaka–Webster connection and their covariant derivatives, computed in normal coordinates at x. In addition,

$$p(x,x,t) \sim t^{-n-1} \sum_{j=0}^{\infty} K_j(x) t^j. \tag{1.17}$$

The functions $K_j(x)$ and the coefficients k_j depend on the CR structure and the pseudo-hermitian structure.

2. A Global CR Invariant

If we scale the pseudo-hermitian structure θ on M by a constant λ^2, and let $k_{j,\lambda}$ denote the constants in the asymptotic expansion (1.15) for the associated pseudoconformal Laplacian, then

$$k_{j,\lambda} = \lambda^{2n+2-2j} k_j \tag{2.1}$$

by [3, (6.38) and (6.48)]. Hence, the only coefficient in the expansion which might be a CR invariant of M is k_{n+1}. The main result of this section is that it is.

THEOREM 2.2. *Let M be a compact strictly pseudoconvex CR manifold equipped with a pseudo-hermitian structure θ. The coefficient k_{n+1} of t^0 in the asymptotic expansion of* $\operatorname{tr} e^{-tD}$ *is a pseudoconformal invariant of M; that is, it is independent of the choice of θ and depends only on the CR structure.*

The conformal analogue of this result is proved by Branson and Ørsted [5, Cor. 3.7] and Parker and Rosenberg [14, Theorem 3.1]. Our method of proof is similar to the one in [14]. The key observation is that the constructions in [3] of the kernel and parametrix for e^{-tD} depend smoothly on parameters. The corresponding smooth dependence in the Riemannian case is well known (see [16, Prop. 6.1]).

Proof. Fix a positive C^∞ function r on M and let

$$\theta^\epsilon = r^{2\epsilon/n}\theta. \tag{2.3}$$

Let D^ϵ denote the associated pseudoconformal Laplacian, and let k_j^ϵ denote the coefficient of t^{j-n-1} in the asymptotic expansion (1.15) of $\operatorname{tr}(e^{-tD^\epsilon})$. It suffices to show that

$$\left.\frac{d}{d\epsilon} k_{n+1}^\epsilon\right|_{\epsilon=0} = 0. \tag{2.4}$$

By Theorem 1.14,

$$\operatorname{tr}(e^{-tD^\epsilon}) = \sum_{j=0}^{n+1} t^{j-n-1} k_j^\epsilon + O(t) \tag{2.5}$$

as $t \to 0$. Hence for $\operatorname{Re} s > n+1$,

$$\int_0^1 t^{s-1} \operatorname{tr}(e^{-tD^\epsilon})\, dt = \sum_{j=0}^{n+1} \frac{1}{j+s-n-1} k_j^\epsilon + f^\epsilon(s), \tag{2.6}$$

where $f^\epsilon(s)$ is holomorphic for $\operatorname{Re} s > -1$. Formula (2.6) gives a meromorphic continuation of the integral on the left for $\operatorname{Re} s > -1$. Thus

$$k_{n+1}^\epsilon = \operatorname{Res}_0 \int_0^1 t^{s-1} \operatorname{tr}(e^{-tD^\epsilon})\, dt. \tag{2.7}$$

Let $p^\epsilon(x,y,t)$ denote the kernel of $\operatorname{tr}(e^{-tD^\epsilon})$ and let Q^ϵ denote the parametrix given by [3, Thm. 5.22]. The construction in [3, §5] of the leading symbol q_{-2}^ϵ of this parametrix depends smoothly on ϵ. It follows from the argument of [3, Thm. 4.5 and Appendix] that $p^\epsilon(x,y,t)$ depends smoothly on ϵ, and also that

$$p^\epsilon(x,x,t) - \sum_{j \le N} t^{j-n-1} K_j^\epsilon(x) = O(t^{N-n}) \tag{2.8}$$

uniformly in x and ϵ, where K_j^ϵ is the function in (1.16). By Theorem 1.14, K_j^ϵ depends smoothly on ϵ. Hence

$$\frac{d}{d\epsilon} \operatorname{Res}_0 \int_0^1 t^{s-1} \operatorname{tr}(e^{-tD^\epsilon})\, dt \Big|_{\epsilon=0} = \operatorname{Res}_0 \int_0^1 t^{s-1} \operatorname{tr} \frac{d}{d\epsilon}(e^{-tD^\epsilon})\Big|_{\epsilon=0} dt. \tag{2.9}$$

For $t > 0$,

$$\left(\frac{\partial}{\partial t} + D^\epsilon\right) e^{-tD^\epsilon} = 0. \tag{2.10}$$

Differentiating (2.10) with respect to ϵ and evaluating at $\epsilon = 0$ gives

$$\dot{D} e^{-tD} + \left(\frac{\partial}{\partial t} + D\right)(e^{-tD})^{\cdot} = 0, \tag{2.11}$$

where we have used the notation $^{\cdot}$ to denote the derivative evaluated at 0 and $D = D^0$. The inverse of $\partial/\partial t + D$ is convolution with the heat semigroup e^{-tD}, so

$$(e^{-tD})^{\cdot} = -\int_0^t e^{-(t-s)D} \dot{D} e^{-sD}\, ds. \tag{2.12}$$

Because $\dot{D} e^{-sD}$ and $e^{-(t-s)D}$ have smooth kernels,

$$\operatorname{tr}(e^{-(t-s)D} \dot{D} e^{-sD}) = \operatorname{tr}(\dot{D} e^{-sD} e^{-(t-s)D}). \tag{2.13}$$

By (2.12), (2.13), and the semigroup property,

$$\operatorname{tr}(e^{-tD})^{\cdot} = -t \operatorname{tr} \dot{D} e^{-tD}. \tag{2.14}$$

By (1.12),

$$D^\epsilon = r^{-\epsilon(1+2/n)} D r^\epsilon \tag{2.15}$$

so

$$\dot{D} = -\left(1 + \frac{2}{n}\right) \log r\, D + D \log r. \tag{2.16}$$

Because D is self-adjoint,

$$\operatorname{tr}(D \log r\, e^{-tD}) = \operatorname{tr}(\log r\, D e^{-tD}). \tag{2.17}$$

Using (2.10), (2.16), and (2.17) in (2.14) yields

$$\begin{aligned} \operatorname{tr} \dot{D} e^{-tD} &= -\frac{2}{n} \operatorname{tr}(\log r\, D e^{-tD}) \\ &= \frac{2}{n} \frac{d}{dt} \operatorname{tr}(\log r\, e^{-tD}). \end{aligned} \tag{2.18}$$

By (2.7), (2.9), (2.14), and (2.18),

$$\dot{k}_{n+1} = -\frac{2}{n} \operatorname{Res}_0 \int_0^1 t^s \frac{d}{dt} \operatorname{tr}(\log r\, e^{-tD})\, dt. \tag{2.19}$$

Formula (2.8) gives

$$\operatorname{tr}(\log r\, e^{-tD}) = \sum_{j \le n+1} t^{j-n-1} \int_M \log r(x) K_j(x)\, dV(x) + O(t). \tag{2.20}$$

Thus, the residue in (2.19) is 0 and $\dot{k}_{n+1} = 0$. □

3. A Local CR Invariant

The integrand in (1.16) also scales by $\lambda^{2n+2-2j}$ under the scaling of the pseudo-hermitian structure θ by a constant λ^2, and the function $K_j(x)$ satisfies

$$K_{j,\lambda}(x) = \lambda^{-2j} K_j(x) \tag{3.1}$$

[3, (6.38) and (6.48)]. In general, the functions K_j do not satisfy a simple transformation rule under pseudoconformal changes. In Theorem 3.3 we prove that K_n does. The analogous result in the conformal case was proved by Parker and Rosenberg [14, Thm. 2.1]. The key point in their proof is that the coefficient of t^{-1} in the asymptotic expansion of the heat kernel on the diagonal is, up to a constant factor, the coefficient of the leading logarithmic singularity of the Green's function for the conformal Laplacian. To show this, they use the fact that the heat kernel has a good approximation of the form

$$(4\pi t)^{-n/2} e^{-r^2/4t} \sum_{k=0}^{L} a_k(x,y) t^k, \tag{3.2}$$

where r is the Riemannian distance between x and y and n is the dimension of the manifold, and they use error estimates for the approximation. We are also able to identify K_n with the principal logarithmic singularity of the appropriate Green's function. The main new difficulty which arises in doing this is that we do not have an approximation to the kernel of e^{-tD} with a simple form analogous to (3.2).

THEOREM 3.3. *Let M be a compact strictly pseudoconvex CR manifold of dimension $2n+1$ equipped with a pseudo-hermitian structure θ. The coefficient $K_n(x)$ of t^{-1} in the asymptotic expansion* (1.17) *of $p(x,x,t)$ is a* *local pseudoconformal invariant. If the pseudo-hermitian structure is changed to* $\tilde{\theta} = r^{2/n}\theta$, *then*

$$\begin{aligned} \tilde{K}_n(x) &= r^{-2}(x)K_n(x), \\ \tilde{K}_n(x)\,d\tilde{V}(x) &= r^{2/n}K_n(x)\,dV(x), \end{aligned} \tag{3.4}$$

where $\tilde{}$ *denotes quantities with respect to the pseudo-hermitian structure* $\tilde{\theta}$.

Proof. It suffices to prove the result for $x \in U$, where U is open and $\bar{U} \subset U_1$ a coordinate patch on M. Choose a local frame $\{Z_\alpha\}$ for $T_{1,0}$ and write $Z_j = X_j - iX_{n+j}$, $j = 1, \ldots, n$, $X_0 = T$. Fix coordinates on U_1, and for $y \in U_1$ let Ψ_y denote the coordinates

$$x = (x^0, \ldots, x^{2n}) = (x^0, x') \tag{3.5}$$

on U_1 obtained by following the fixed coordinates by the affine map which makes y the origin and satisfies

$$\left.\frac{\partial}{\partial x^j}\right|_0 = X_j(0), \quad j = 0, \ldots, 2n. \tag{3.6}$$

In the terminology of [2] and [3], Ψ_y is the y coordinate map. For $x \in \mathbf{R}^{2n+1}$, $t \in \mathbf{R}$, we have the non-Euclidean dilations

$$\lambda \cdot x = (\lambda^2 x^0, \lambda x') \quad \text{and} \quad \lambda \cdot (x, t) = (\lambda^2 x^0, \lambda x', \lambda^2 t). \tag{3.7}$$

We let $\|\ \|$ denote the homogeneous norm on $\mathbf{R}^{2n+1}$:

$$\|x\| = [(x^0)^2 + x'^4]^{1/4}. \tag{3.8}$$

In U, the kernel $p(x, y, t)$ of e^{-tD} has an asymptotic expansion

$$p(x, y, t) \sim \sum_{j=0}^{\infty} k_{-2-j,x}(\Psi_x(y), t), \tag{3.9}$$

where $k_{-2-j,x} \in C^\infty(\mathbf{R}^{2n+2} \backslash \{0\})$ is homogeneous of degree $j - 2n - 2$ with respect to the dilations (3.7), and vanishes for $t < 0$ (see the proof of Theorem 4.5 in [3].) Hence, $k_{-2-j,x}(\cdot, 1) \in \mathcal{S}(\mathbf{R}^{2n+1})$. Let

$$p_m(x, y, t) = \sum_{j=0}^{m} k_{-2-j,x}(\Psi_x(y), t). \tag{3.10}$$

For any L,

$$(p - p_m)(x, y, t) = \sum_{j=m+1}^{L+2n+4} k_{-2-j,x}(\Psi_x(y), t) + \tilde{k}_L(x, y, t), \tag{3.11}$$

where $\tilde{k}_L \in C^L(M \times M \times \mathbf{R})$. By the homogeneity of $k_{-2-j,x}$, for $j \geq m+1$,

$$k_{-2-j,x}(\Psi_x(y), t) = t^{(m-2n-1)/2}\left\{t^{(j-m-1)/2} k_{-2-j,x}\left(\frac{1}{\sqrt{t}}\Psi_x(y), 1\right)\right\}. \tag{3.12}$$

Because $k_{-2-j,x}(\cdot, 1) \in \mathcal{S}(\mathbf{R}^{2n+1})$, the expression in brackets is bounded on compact sets and vanishes to infinite order off the diagonal as $t \to 0+$. Hence

$$|(p-p_m)(x,y,t)| \le t^{m/2-n-1/2} f(x,y,t), \tag{3.13}$$

where f is bounded on compact sets and vanishes to infinite order off the diagonal as $t \to 0+$.

In the remainder of this proof, we use the notation $g \sim h$ to mean that $g-h$ is bounded. By [2, Thms. 7.8 and 9.30], D has a parametrix G and

$$G(x,y) \sim \sum_{-2n \le j \le -1} G_j(x, \Psi_x(y)) + a(x) \log\|\Psi_x(y)\|, \tag{3.14}$$

where $G_j(x, \cdot)$ is homogeneous of degree j for the dilations (3.7) with λ positive. Let L^+ denote the closed subspace of $L^2(M)$ spanned by the eigenfunctions of D with positive eigenvalues, let e^{-tD^+} denote the restriction of e^{-tD} to L^+, and let p^+ denote the kernel of e^{-tD^+}. The operator D is subelliptic, self-adjoint, and bounded below, so it has only a finite number (with multiplicities) of nonpositive eigenvalues and

$$G(x,y) \sim \int_0^\infty p^+(x,y,t)\,dt. \tag{3.15}$$

By the semigroup property,

$$\begin{aligned} \int_1^\infty e^{-tD^+}\,dt &= \int_1^\infty e^{-D^+} e^{-(t-1)D^+}\,dt \\ &= e^{-D^+} \int_0^\infty e^{-tD^+}\,dt. \end{aligned} \tag{3.16}$$

The last expression is a smoothing operator, so

$$G(x,y) \sim \int_0^1 p^+(x,y,t)\,dt. \tag{3.17}$$

Now

$$\int_0^1 p^+(x,y,t)\,dt = \int_0^1 (p^+ - p_{2n})(x,y,t)\,dt + \int_0^1 p_{2n}(x,y,t)\,dt. \tag{3.18}$$

By (3.13), the first integral on the right side of (3.18) is bounded, so by (3.17) we have

$$G(x,y) \sim \int_0^1 p_{2n}(x,y,t)\,dt. \tag{3.19}$$

For $j \le 2n-1$,

$$k_{-2-j,x}(\Psi_x(y),t) = t^{j/2-n-1} k_{-2-j,x}\left(\frac{1}{\sqrt{t}}\Psi_x(y), 1\right), \tag{3.20}$$

which is integrable on $[1,\infty)$, so by (3.10) and (3.19),

$$G(x,y) \sim \int_0^\infty p_{2n-1}(x,y,t)\,dt + \int_0^1 k_{-2n-2,x}(\Psi_x(y),t)\,dt. \tag{3.21}$$

Let $z \in \mathbf{R}^{2n+1}\setminus\{0\}$ and let $w = (1/\|z\|)\cdot z$. For $j \le 2n-1$, by the homogeneity of $k_{-2-j,x}$ we have

$$\begin{aligned}\int_0^\infty k_{-2-j,x}(z,t)\,dt &= \int_0^\infty k_{-2-j,x}(z,\|z\|^2 s)\,d(\|z\|^2 s)\\ &= \|z\|^2\int_0^\infty k_{-2-j,x}(\|z\|\cdot w,\|z\|^2 s)\,ds \\ &= \|z\|^{j-2n}\int_0^\infty k_{-2-j,x}(w,s)\,ds,\end{aligned}\tag{3.22}$$

which is positively homogeneous of degree $j-2n\le -1$ with respect to the non-Euclidean dilations (3.7). Similarly, since $k_{-2n-2,x}$ is homogeneous of degree -2,

$$\begin{aligned}\int_0^1 k_{-2n-2,x}(z,t)\,dt &= \int_0^{\|z\|^{-2}} k_{-2n-2,x}(w,s)\,ds\\ &= \int_0^{\|z\|^{-2}} k_{-2n-2,x}(s^{-1/2}\cdot w,1)\frac{ds}{s}.\end{aligned}\tag{3.23}$$

The homogeneity of $k_{-2n-2,x}$ also gives

$$|k_{-2n-2,x}(s^{-1/2}\cdot w,1)|\le C(\|s^{-1/2}\cdot w\|^4+1)^{-1/2}\le 2Cs\tag{3.24}$$

for some constant C when $s\le 1$. Using (3.24) in (3.23) gives

$$\int_0^1 k_{-2n-2,x}(z,t)\,dt\sim\int_1^{\|z\|^{-2}} k_{-2n-2,x}(s^{-1/2}\cdot w,1)\frac{ds}{s}.\tag{3.25}$$

Write

$$\begin{aligned}&k_{-2n-2,x}(s^{-1/2}\cdot w,1)\\ &\quad= k_{-2n-2,x}(0,1)+[k_{-2n-2,x}(s^{-1/2}\cdot w,1)-k_{-2n-2,x}(0,1)].\end{aligned}\tag{3.26}$$

Because $k_{-2n-2,x}(\cdot,1)\in\mathcal{S}(\mathbf{R}^{2n+1})$, the expression in brackets in (3.26) is bounded by $Cs^{-1/2}$ for some constant C, so the integral of its absolute value against ds/s on $[1,\infty)$ is bounded independent of w. Using this in (3.25) along with the fact that $k_{-2n-2,x}(0,1)=K_n(x)$ yields

$$\int_0^1 k_{-2n-2,x}(z,t)\,dt\sim\int_1^{\|z\|^{-2}} K_n(x)\frac{ds}{s}=-2K_n(x)\log\|z\|.\tag{3.27}$$

Thus, combining (3.14), (3.21), (3.22), and (3.27), for $j\le 2n-1$ we have

$$G_{j-2n}(x,\Psi_x(y))=\int_0^\infty k_{-2-j,x}(\Psi_x(y),t)\,dt\tag{3.28}$$

and

$$\begin{aligned}G(x,y)-\sum_{-2n\le j\le -1} G_j(x,\Psi_x(y)) &\sim a(x)\log\|\Psi_x(y)\|\\ &\sim -2K_n(x)\log\|\Psi_x(y)\|\end{aligned}\tag{3.29}$$

so

$$K_n(x)=-a(x)/2.\tag{3.30}$$

Applying a parametrix $\tilde{G}$ for $\tilde{D}$ on the left and a parametrix G for D on the right to the transformation rule (1.12) gives

$$\tilde{G}\tilde{D}r^{-1}G = \tilde{G}r^{-1-2/n}DG; \tag{3.31}$$

hence

$$r^{-1}G \equiv \tilde{G}r^{-1-2/n}, \tag{3.32}$$

where $\equiv$ denotes equivalence modulo smoothing operators. Now,

$$d\tilde{V} = r^{2+2/n}dV, \tag{3.33}$$

so when we pass to the level of kernels in (3.32) we have

$$\tilde{G}(x,y) \equiv r^{-1}(x)G(x,y)r^{-1}(y). \tag{3.34}$$

Hence, with the notation of (3.14),

(3.35)
$$\sum_{-2n\le j\le -1} \tilde{G}_j(x, \tilde{\Psi}_x(y)) + \tilde{a}(x)\log\|\tilde{\Psi}_x(y)\|$$
$$\sim \sum_{-2n\le j\le -1} r^{-1}(x)G_j(x, \Psi_x(y))r^{-1}(y) + r^{-1}(x)a(x)\log\|\Psi_x(y)\|r^{-1}(y).$$

If we let $z = \Psi_x(y)$ and $\tilde{z} = \tilde{\Psi}_x(y)$, then

$$\tilde{z} = (\tilde{z}^0, \tilde{z}') = r^{-1/n}(x)\cdot(z^0, z' + z^0 b') \tag{3.36}$$

for some $b' \in \mathbf{R}^{2n}$. Using this and a Taylor expansion, we have

$$\sum_{-2n\le j\le -1} \tilde{G}_j(x, \tilde{\Psi}_x(y)) \sim \sum_{-2n\le j\le -1} H_j(x, \Psi_x(y)), \tag{3.37}$$

where $H_j(x, \cdot)$ is homogeneous of degree j. Also

$$\log\|\tilde{\Psi}_x(y)\| \sim \log\|\Psi_x(y)\|. \tag{3.38}$$

Using (3.37), (3.38), and the fact that $r^{-1}(y) = r^{-1}(x) + O(\Psi_x(y))$ in (3.35) gives

$$\tilde{a}(x) = r^{-2}(x)a(x). \tag{3.39}$$

The theorem now follows from (3.30), (3.33), and (3.39). □

4. Spheres

Let S^{2n+1} be the unit sphere in $\mathbf{C}^{n+1}$. Now S^{2n+1} is pseudoconformally equivalent to the hyperquadric

$$\operatorname{Im} z^{n+1} = |z'|^2, \quad z' = (z^1, \ldots, z^n), \tag{4.1}$$

which has a pseudo-hermitian structure with curvature and torsion identically 0 ([20, §4]). Hence the local invariant $K_n(x)$ is 0. In this section we show that the global invariant k_{n+1} on S^{2n+1} is also 0. In addition, we show

how to calculate the full asymptotic expansion of $\operatorname{tr}(e^{-tD})$ for a suitable pseudo-hermitian structure.

We give S^{2n+1} the pseudo-hermitian structure determined by

$$\theta = \frac{i}{2}\sum_{1}^{n+1} z^j d\bar{z}^j. \tag{4.2}$$

The resulting metric is the standard metric on $T_{1,0}$ and is 2 times the standard metric on $(T_{1,0}\oplus T_{0,1})^{\perp}$. It follows that $\Delta_b = 2\operatorname{Re}\Box_b$, where $\Box_b$ is the $\bar{\partial}_b$-Laplacian for the standard metric acting on functions. Folland calculated the eigenfunctions and eigenvalues of $\Box_b$ ([6]). The scalar curvature of the pseudo-hermitian structure is $R=2n(n+1)$ by [20, §4]. Hence

$$\begin{aligned} D &= 2(2+2/n)\operatorname{Re}\Box_b + 2n(n+1) \\ &= (2+2/n)\Delta_b + 2n(n+1) \end{aligned} \tag{4.3}$$

and we can write down the series for $\operatorname{tr}(e^{-tD})$ explicitly. We use number-theoretic techniques to study its asymptotic behavior. Corollary 4.32 and its use in the proof of Theorem 4.34 (the case of S^3) are due to Montgomery.

Let $\mathcal{H}^{p,q}$ denote the space of harmonic polynomials of bi-degree (p,q) in $\mathbf{C}^{n+1}$, that is, harmonic polynomials which are homogeneous of degree p in the z^i's and of degree q in the $\bar{z}^i$'s. The following proposition is essentially due to Folland [6, Thm. II.6].

PROPOSITION 4.4. *$\mathcal{H}^{p,q}$ is an eigenspace of Δ_b on S^{2n+1} with eigenvalue $2(2pq+pn+qn)$.*

Proof. By [6], $\mathcal{H}^{p,q}$ is an eigenspace of $\Box_b$ with eigenvalue $2q(p+n)$. The proposition follows immediately from this and the fact that $\Delta_b = 2\operatorname{Re}\Box_b$. We sketch a different proof. Let

$$T_1 = i\sum_{j=1}^{n+1}\left(z^j\frac{\partial}{\partial z^j} - \bar{z}^j\frac{\partial}{\partial \bar{z}^j}\right). \tag{4.5}$$

Then for $f\in C^\infty(S^{2n+1})$,

$$df = d_b f + 2T_1 f\theta, \quad \text{and} \quad \Delta f = d^*df = \Delta_b f - T_1^2 f, \tag{4.6}$$

where the adjoint is with respect to the standard metric, so

$$\Delta_b = \Delta + T_1^2. \tag{4.7}$$

The restriction of a harmonic polynomial of degree k to S^{2n+1} is an eigenfunction of the Laplace–Beltrami operator, with eigenvalue $k(2n+k)$ (see [4, Ch. IIIC] or [7, Lemma 2.61]). The proposition follows from this and the fact that, on $\mathcal{H}^{p,q}$, T_1 is multiplication by $i(p-q)$. □

COROLLARY 4.8. *$\mathcal{H}^{p,q}$ is an eigenspace of D with eigenvalue*

$$(2+2/n)(2p+n)(2q+n). \tag{4.9}$$

Proof. This follows immediately from (4.3) and Proposition 4.4. □

Using Corollary 4.8, we can obtain the following explicit formula for the trace of the heat semigroup e^{-tD}.

PROPOSITION 4.10. *On S^{2n+1} with the pseudo-hermitian structure θ,*

(4.11)
$$\operatorname{tr}(e^{-tD})=\sum_{p,q\geq 0}\frac{(p+n-1)!\,(q+n-1)!\,(p+q+n)}{p!\,q!\,n!\,(n-1)!}e^{-(2+2/n)(2p+n)(2q+n)t}.$$

Proof. By Corollary 4.8,

$$\operatorname{tr}(e^{-tD})=\sum d^{p,q}e^{-(2+2/n)(2p+n)(2q+n)t}, \tag{4.12}$$

where $d^{p,q}$ is the dimension of $\mathcal{H}^{p,q}$. We calculate $d^{p,q}$ by the method used to calculate the dimension of the space of harmonic polynomials of degree k ([4, Ch. IIIC], [7, Cor. 2.53]). Let $P^{p,q}$ denote the polynomials homogeneous of bi-degree (p,q) in $\mathbf{C}^{n+1}$. The dimension of $P^{p,q}$ is

$$p^{p,q}=\frac{(p+n)!\,(q+n)!}{p!\,q!\,n!^2}. \tag{4.13}$$

Since $P^{p,q}=\mathcal{H}^{p,q}+|z|^2P^{p-1,q-1}$,

$$d^{p,q}=p^{p,q}-p^{p-1,q-1}. \tag{4.14}$$

The proposition follows from (4.12)–(4.14). □

Before we calculate the asymptotic behavior of the sum in (4.12), we introduce some notation. For $k\geq 1$, let

$$\sigma_k(m)=\begin{cases}\sum_{a\mid m}a^k, & m\in\mathbf{N},\\ 0 & \text{otherwise.}\end{cases} \tag{4.15}$$

Let

$$F_k(x)=\sum_1^\infty \sigma_k(m)e^{-2\pi m/x}. \tag{4.16}$$

A crucial ingredient in determining the asymptotic behavior of $\operatorname{tr}(e^{-tD})$ is the following functional equation for F_k.

PROPOSITION 4.17. *If k is odd, then*

(4.18)
$$F_k(x)=-\frac{\zeta(-k)}{2}+\left(\frac{x}{2\pi}\right)^{k+1}k!\,\zeta(k+1)-\epsilon_k\frac{x}{4\pi}+(-1)^{(k+1)/2}x^{k+1}F_k\left(\frac{1}{x}\right),$$

where ζ is the Riemann ζ-function and ϵ_k is 1 if $k=1$ and 0 otherwise.

Proof. Write $k=2l+1$. Let $D_k(s)$ denote the Dirichlet series

$$D_k(s)=\sum_1^\infty \frac{\sigma_k(m)}{m^s}. \tag{4.19}$$

The function σ_k is multiplicative; that is, if m and r are relatively prime then $\sigma_k(mr)=\sigma_k(m)\sigma_k(r)$. Hence

$$D_k(s)=\prod_{p\text{ prime}}\left(1+\frac{\sigma_k(p)}{p^s}+\frac{\sigma_k(p^2)}{p^{2s}}+\cdots\right). \tag{4.20}$$

For p prime,

$$\sigma_k(p^j)=p^{kj}+p^{k(j-1)}+\cdots+1. \tag{4.21}$$

Using (4.21) in (4.20) yields, for $\mathrm{Re}\, s>k+1$,

$$\begin{aligned} D_k(s)&=\prod_{p\text{prime}}\left(1+\frac{p^k+1}{p^s}+\frac{p^{2k}+p^k+1}{p^{2s}}+\cdots\right)\\ &=\prod_{p\text{prime}}\left(\frac{p^k-1}{p^k-1}+\frac{p^{2k}-1}{p^s(p^k-1)}+\frac{p^{3k}-1}{p^{2s}(p^k-1)}+\cdots\right)\\ &=\prod_{p\text{prime}}\frac{1}{p^k-1}\left(\frac{p^k}{1-p^{k-s}}-\frac{1}{1-p^{-s}}\right)\\ &=\prod_{p\text{prime}}\left(\frac{1}{1-1/p^{s-k}}\right)\left(\frac{1}{1-1/p^s}\right)\\ &=\zeta(s)\zeta(s-k). \end{aligned} \tag{4.22}$$

Here ζ is the Riemann ζ-function. By Mellin's formula [15, 23.5],

$$e^{-2\pi m/x}=\frac{1}{2\pi i}\int_{k+2-i\infty}^{k+2+i\infty}\Gamma(s)\left(\frac{x}{2\pi m}\right)^s ds. \tag{4.23}$$

Combining (4.22) and (4.23) gives

$$\begin{aligned} F_k(x)&=\frac{1}{2\pi i}\int_{k+2-i\infty}^{k+2+i\infty}\Gamma(s)D_k(s)\left(\frac{x}{2\pi}\right)^s ds\\ &=\frac{1}{2\pi i}\int_{k+2-i\infty}^{k+2+i\infty}\Gamma(s)\zeta(s)\zeta(s-k)\left(\frac{x}{2\pi}\right)^s ds. \end{aligned} \tag{4.24}$$

The integrand in (4.24) has simple poles at $s=0$ and $k+1$ and (if $k=1$) at $s=1$, and no other poles. The residues are

$$-\frac{\zeta(-k)}{2},\quad \left(\frac{x}{2\pi}\right)^{k+1}k!\,\zeta(k+1)\quad\text{and}\quad -\epsilon_k\frac{x}{4\pi}. \tag{4.25}$$

Hence

$$\begin{aligned} F_k(x)=&-\frac{\zeta(-k)}{2}+\left(\frac{x}{2\pi}\right)^{k+1}k!\,\zeta(k+1)-\epsilon_k\frac{x}{4\pi}\\ &+\frac{1}{2\pi i}\int_{-2-i\infty}^{-2+i\infty}\Gamma(s)\zeta(s)\zeta(s-k)\left(\frac{x}{2\pi}\right)^s ds. \end{aligned} \tag{4.26}$$

Using Riemann's functional equation

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}=\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2} \tag{4.27}$$

for both $\zeta(s)$ and $\zeta(s-k)$ gives

$$\begin{aligned}&\zeta(s)\zeta(s-k)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-k}{2}\right)\pi^{k/2-s}\\ &\quad=\zeta(1-s)\zeta(k+1-s)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{k+1-s}{2}\right)\pi^{s-1-k/2}.\end{aligned} \tag{4.28}$$

By Legendre's duplication formula for the gamma function,

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-k}{2}\right)&=\frac{\Gamma(s/2)\Gamma((s-1)/2)}{((s-1)/2-1)\cdots((s-1)/2-l)}\\ &=\frac{(\pi)^{1/2}2^{(3+k)/2-s}\Gamma(s-1)}{(s-3)(s-5)\cdots(s-k)}\end{aligned} \tag{4.29}$$

and similarly

$$\Gamma\left(\frac{k+1-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)=\frac{\pi^{1/2}2^{s+(1-k)/2}\Gamma(k+1-s)}{(k-s)(k-2-s)\cdots(1-s)}. \tag{4.30}$$

Hence,

$$\begin{aligned}&\zeta(s)\zeta(s-k)\Gamma(s)(2\pi)^{-s}\\ &\quad=(-1)^{(k+1)/2}\zeta(1-s)\zeta(k+1-s)\Gamma(k+1-s)(2\pi)^{-(k+1-s)}.\end{aligned} \tag{4.31}$$

The proposition now follows from using (4.31) in the integrand in (4.26) and making the change of variables $w=k+1-s$ in the integral. □

Specializing to the case $k=1$, we have the following.

COROLLARY 4.32.

$$F_1(x)=\frac{x^2}{24}-\frac{x}{4\pi}+\frac{1}{24}-x^2F_1\left(\frac{1}{x}\right). \tag{4.33}$$

The next theorem gives the asymptotic behavior of $\operatorname{tr}(e^{-tD})$ on S^3.

THEOREM 4.34. *On S^3 with the pseudo-hermitian structure* (4.2),

$$\operatorname{tr}(e^{-tD})=\frac{\pi^2}{256t^2}+O\left(\frac{1}{t^2}e^{-\pi^2/4t}\right) \tag{4.35}$$

as $t\to 0+$. In particular, there is only one term in the asymptotic expansion of $\operatorname{tr}(e^{-tD})$ *as $t\to 0+$ and the CR global invariant $k_2=0$.*

Proof. By Proposition 4.10,

$$
\begin{aligned}
\operatorname{tr}(e^{-tD}) &= \sum_{p,q\geq 0} (p+q+1)e^{-4(2p+1)(2q+1)t} \\
&= \sum_{n>0} e^{-4tn} \sum_{\substack{p,q\geq 0\\ (2p+1)(2q+1)=n}} (p+q+1) \\
&= \frac{1}{2}\sum_{\substack{n>0\\ n\text{ odd}}} e^{-4nt} \sum_{\substack{a,b>0\\ ab=n}} (a+b) \\
&= \sum_{\substack{n>0\\ n\text{ odd}}} \sigma_1(n)e^{-4tn}.
\end{aligned}
\tag{4.36}
$$

The multiplicative property of σ_1 gives

$$
\sigma_1(n) - 3\sigma_1\left(\frac{n}{2}\right) + 2\sigma_1\left(\frac{n}{4}\right) = \begin{cases} \sigma_1(n), & n\text{ odd}, \\ 0, & n\text{ even}. \end{cases}
\tag{4.37}
$$

Using (4.37) and (4.16) in (4.36) gives

$$
\begin{aligned}
\operatorname{tr}(e^{-tD}) &= \sum_{n>0}\left(\sigma_1(n) - 3\sigma_1\left(\frac{n}{2}\right) + 2\sigma_1\left(\frac{n}{4}\right)\right)e^{-4tn} \\
&= F_1\left(\frac{\pi}{2t}\right) - 3F_1\left(\frac{\pi}{4t}\right) + 2F_1\left(\frac{\pi}{8t}\right).
\end{aligned}
\tag{4.38}
$$

The theorem now follows from (4.38) and Corollary 4.32, together with the observation that $F_1(1/x) = O(e^{-2\pi x})$ as $x \to \infty$. □

On S^5 we have the following.

THEOREM 4.39.

$$
\operatorname{tr}(e^{-tD}) = \frac{1}{12}\left(\frac{\pi^2}{432t^3} - \frac{1}{24t^2}\right) + O\left(\frac{1}{t^3}e^{-\pi^2/3t}\right)
\tag{4.40}
$$

as $t \to 0+$. *In particular, the global CR invariant* $k_3 = 0$.

Proof. By Proposition 4.10,

$$
\begin{aligned}
\operatorname{tr}(e^{-tD}) &= \sum_{p,q\geq 0} \frac{(p+1)(q+1)(p+q+2)}{2} e^{-12(p+1)(q+1)t} \\
&= \sum_{n>0} \frac{n}{2} e^{-12nt} \sum_{\substack{a,b>0\\ ab=n}} (a+b) \\
&= \sum_{n>0} n\sigma_1(n)e^{-12nt} =
\end{aligned}
\tag{4.41}
$$

$$= \frac{-1}{12} \frac{d}{dt} \sum_{n>0} \sigma_1(n) e^{-12nt}$$

$$= \frac{-1}{12} \frac{d}{dt} F_1\left(\frac{\pi}{6t}\right).$$

The theorem now follows from (4.41) and (4.33). □

We return to the general case. For n odd, this case can be treated by methods similar to the proof of Theorem 4.34, and for n even by methods similar to the proof of Theorem 4.39.

THEOREM 4.42. *On S^{2n+1} with the pseudo-hermitian structure* (4.2),

$$\operatorname{tr}(e^{-tD}) = t^{-(n+1)} \sum_{j=0}^{n-1} a_j t^j + O\left(\frac{1}{t^{n+1}} e^{-1/t}\right) \tag{4.43}$$

as $t \to 0+$, where a_j is an explicitly computable constant. In particular, the CR invariant $k_{n+1} = 0$.

Proof.

Case 1. Suppose now that $n = 2m$ is even. We examine the coefficient $(1/(n!\,(n-1)!) a_{p,q}$ of the exponential in a term in (4.11). Let $a = p+m$, $b = q+m$, and $k = ab$. Then, by (4.11),

$$\begin{aligned} a_{p,q} &= (p+2m-1)(p+2m-2)\cdots(p+1) \\ &\quad \times (q+2m-1)(q+2m-2)\cdots(q+1)(p+q+n) \\ &= (a+m-1)\cdots(a-m+1) \\ &\quad \times (b+m-1)\cdots(b-m+1)(a+b) \\ &= k(ar(a,k) + br(b,k)), \end{aligned} \tag{4.44}$$

where r is a polynomial of degree $2m-2$ with all its terms of even degree in the first variable. Using (4.44) in (4.11), we have

$$\begin{aligned} \operatorname{tr}(e^{-tD}) &= \sum_{k>0} e^{-4(2+2/n)kt} \sum_{\substack{a,b>0 \\ ab=k}} k(ar(a,k) + br(b,k)) \\ &= \sum_{k>0} \sum_{\substack{0<i \\ j \text{ odd} \\ i+j \le 2m}} b_{i,j} k^i \sigma_j(k) e^{-4(2+2/n)kt} \\ &= \sum_{\substack{0<i \\ j \text{ odd} \\ i+j \le 2m}} c_{i,j} \left(\frac{d}{dt}\right)^i F_j\left(\frac{\pi}{2(2+2/n)t}\right), \end{aligned} \tag{4.45}$$

where $b_{i,j}$ and $c_{i,j}$ are explicitly computable constants. The result now follows for n even from Proposition 4.17.

Case 2. Suppose $n=2m+1$ is odd. Again, we examine the coefficient $(1/(n!\,(n-1)!)a_{p,q}$ of the exponential in a term in (4.11). Let $a=2p+n$, $b=2q+n$, and $ab=k$. Then a, b, and k are odd, and by (4.11)

$$\begin{aligned} a_{p,q} &= 2^{-2n+1}(a+n-2)(a+n-4)\cdots(a-n+2) \\ &\quad\times(b+n-2)(b-n+4)\cdots(b+n+2)(a+b) \\ &= ar(a,k)+br(b,k), \end{aligned} \tag{4.46}$$

where r is a polynomial of degree $n-1$ with all its terms of even degree in the first variable. Using (4.46) in (4.11) gives

$$\begin{aligned} \operatorname{tr} e^{-tD} &= \sum_{\substack{k>0\\ k\,\text{odd}}} e^{-(2+2/n)kt} \sum_{\substack{a,b>0\\ ab=k}} (ar(a,k)+br(b,k)) \\ &= \sum_{\substack{k>0\\ k\,\text{odd}}} \sum_{\substack{j\,\text{odd}\\ i+j\le n}} b_{i,j}k^i\sigma_j(k)e^{-(2+2/n)kt}, \end{aligned} \tag{4.47}$$

where $b_{i,j}$ is an explicitly computable constant. The multiplicative property of σ_j gives

$$\sigma_j(k)-(2^j+1)\sigma_j\left(\frac{k}{2}\right)+2^j\sigma_j\left(\frac{k}{4}\right)=\begin{cases}\sigma_j(k), & k\,\text{odd},\\ 0, & k\,\text{even}.\end{cases} \tag{4.48}$$

Using (4.48) in (4.47) gives

$$\begin{aligned} \operatorname{tr}(e^{-tD}) = \sum_{\substack{j\,\text{odd}\\ i+j\le n}} \left(\frac{d}{dt}\right)^i \Bigg(c^1_{i,j}F_j\left(\frac{2\pi}{(2+2/n)t}\right)-(2^j+1)c^2_{i,j}F_j\left(\frac{\pi}{(2+2/n)t}\right) \\ +2^jc^3_{i,j}F_j\left(\frac{\pi}{2(2+2/n)t}\right)\Bigg), \end{aligned} \tag{4.49}$$

where $c^k_{i,j}$ are explicitly computable constants and $c^1_{0,j}=c^2_{0,j}=c^3_{0,j}$. The theorem now follows for n odd from (4.49) and Proposition 4.17. □

5. Three-Dimensional CR Manifolds

In this section we show that the invariants $K_1(x)$ and k_2 are 0 on three-dimensional CR manifolds. We begin with the local invariant.

THEOREM 5.1. *Let M be a compact strictly pseudoconvex three-dimensional CR manifold equipped with a pseudo-hermitian structure θ. The local invariant $K_1(x)=0$.*

Proof. By the arguments in Section 1, [3, Thm. 8.31] remains true with $\Box_b$ replaced by D. Hence

$$K_1=aR \tag{5.2}$$

for some constant a independent of (M, θ). By Theorem 3.3 and (5.2), if the pseudoconformal structure is changed to $\tilde{\theta} = e^{2f}\theta$ then

$$\tilde{K}_1 = e^{-2f} K_1, \tag{5.3}$$

so

$$a\tilde{R} = ae^{-2f} R. \tag{5.4}$$

By [11, Prop. 5.15],

$$\tilde{R} = e^{-2f}(R + 4\Delta_b f - 4\langle d_b f, d_b f\rangle). \tag{5.5}$$

Using (5.5) in (5.4) shows that $a = 0$ hence, by (5.2), $K_1 = 0$. □

Before discussing the global invariant, we introduce some notation. Let M be a strictly pseudoconvex CR manifold of dimension $2n+1$ equipped with a pseudo-hermitian structure θ. We let $\{Z_\alpha\}$ denote a local orthonormal frame for $T_{1,0}$, $\{\theta^\alpha\}$ the dual coframe for $T^*_{1,0}$. In what follows, we use the convention that repeated Greek indices are summed from one to n. The connection matrix $(\omega_\beta{}^\alpha)$ and the torsion forms τ^α of the Tanaka–Webster connection ([19], [20]) are uniquely determined by the structure equations

$$\begin{aligned} d\theta &= i\theta^\alpha \wedge \bar{\theta}^\alpha \\ d\theta^\alpha &= \theta^\beta \wedge \omega_\beta{}^\alpha + \theta \wedge \tau^\alpha \end{aligned} \tag{5.6}$$

together with the requirements

$$\begin{aligned} \omega_\beta{}^\alpha + \bar{\omega}_\alpha{}^\beta &= 0 \\ \tau^\alpha \wedge \bar{\theta}^\alpha &= 0. \end{aligned} \tag{5.7}$$

By [20, (1.41) and (2.9)], the curvature matrix

$$\Pi_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta \tag{5.8}$$

has the form

$$\Pi_\alpha{}^\beta = R_\alpha{}^\beta{}_{\rho\bar{\sigma}}\theta^\rho \wedge \bar{\theta}^\sigma + W_\alpha{}^\beta{}_\rho \theta^\rho \wedge \theta - W^\beta{}_{\alpha\bar{\sigma}}\bar{\theta}^\sigma \wedge \theta + i\bar{\theta}^\alpha \wedge \tau^\beta - i\bar{\tau}^\alpha \wedge \theta^\beta. \tag{5.9}$$

If we view the curvature tensor as a complex-valued 2-form with values in $\operatorname{Hom}(T(M) \otimes \mathbf{C})$, its projection onto $T^*_{1,0} \wedge T^*_{0,1}$ is

$$\Pi^{1,1} = R_\alpha{}^\beta{}_{\rho\bar{\sigma}}\theta^\rho \wedge \bar{\theta}^\sigma \otimes Z_\beta \otimes \theta^\alpha. \tag{5.10}$$

We denote the covariant derivative with respect to the Tanaka–Webster connection by ∇. We define the torsion tensor T by

$$T = \tau^\alpha \otimes Z_\alpha. \tag{5.11}$$

Let

$$C : T^* \otimes T \otimes T^* \to T^* \tag{5.12}$$

be contraction on the second and third indices. Here T and T^* denote the complexified tangent and cotangent bundle of M.

Our proof that $k_2=0$ if $n=1$ requires knowing the general form of the integrand K_2. We use invariance theory, as in [3, §8], to obtain the general form in any dimension.

PROPOSITION 5.13. *There are constants a, b, c, d, e, and f depending only on n such that*

$$K_2=aR^2+b|Ric|^2+c|\Pi^{1,1}|^2+d\Delta_b R+e|T|^2+f\operatorname{Im} d_b^* C\nabla T. \tag{5.14}$$

Here Ric is Webster's Ricci tensor [20, (2.16)].

Proof. By Theorem 1.14, $K_2(x)$ is a polynomial, depending only on n, in the components of the curvature and torsion and their covariant derivatives computed in normal coordinates at x. By (3.1), the terms of this polynomial scale by λ^{-4} when the pseudo-hermitian structure is scaled by the constant λ^2. We write

$$\bar\tau^\alpha=A_{\alpha\beta}\theta^\beta \tag{5.15}$$

and let

$$A_{\overline{\alpha\beta}}=\overline{A_{\alpha\beta}}. \tag{5.16}$$

It follows from [3, (8.5) and (8.9)] and [20, (1.33) and (1.36)] that $K_2(x)$ is a monomial in

$$\begin{gathered} R_\alpha{}^\beta{}_{\rho\bar\sigma}R_\gamma{}^\mu{}_{\nu\bar\eta},\ A_{\alpha\beta}A_{\mu\nu},\ A_{\alpha\beta}A_{\overline{\mu\nu}},\ R_\alpha{}^\beta{}_{\rho\bar\sigma}A_{\mu\nu},\\ R_\alpha{}^\beta{}_{\rho\bar\sigma;0},\ R_\alpha{}^\beta{}_{\rho\bar\sigma;\mu\nu},\ R_\alpha{}^\beta{}_{\rho\bar\sigma;\mu\bar\nu},\ A_{\alpha\beta;0},\ A_{\alpha\beta;\mu\nu},\\ A_{\alpha\beta;\mu\bar\nu},\ A_{\alpha\beta;\overline{\mu\nu}},\ W_\alpha{}^\beta{}_{\rho;\mu},\ W_\alpha{}^\beta{}_{\rho;\bar\mu}, \end{gathered} \tag{5.17}$$

and their complex conjugates. Here we have written components of tensors with respect to the basis $\{X,Z_\alpha,\overline{Z_\alpha}\}$ of $T(M)\otimes\mathbf{C}$ and the dual basis $\{\theta,\theta^\alpha,\bar\theta^\alpha\}$ of $T^*(M)\otimes\mathbf{C}$. As in the proof of [3, Lemma 8.12], we use the metric to raise and lower indices and thereby eliminate conjugates. The polynomial is invariant under the action of $U(n)$ on the tensor algebra of $\mathbf{C}^n$. By classical invariant theory, it is a linear combination of monomials, each of which has the same number of indices up and down [17]. Also by classical invariant theory [17], it is given by a complete contraction. Hence, by [20, (1.23), (1.33), (1.36), (2.16) and (2.17)], it is a linear combination of

$$R^2,\ |Ric|^2,\ |\Pi^{1,1}|^2,\ |T|^2,\ XR,\ R_{;\alpha}{}^\alpha,\ R_\nu{}^\mu{}_{;\mu}{}^\nu,\ A_{\alpha\beta;}{}^{\alpha\beta},\ W_\alpha{}^\beta{}_{\rho;}{}^\mu \tag{5.18}$$

and their conjugates. We define a new CR structure $T'_{1,0}$ on M by $T'_{1,0}=T_{0,1}$, as in the proof of [3, Lemma 8.24]. Then

$$\Delta'_b=\Delta_b, \tag{5.19}$$

so

$$K_2' = K_2. \tag{5.20}$$

For a local orthonormal frame for $T_{1,0}'^{*}$, we take $\{\theta'^\alpha = \bar\theta^\alpha\}$. By (5.6)-(5.9) we have

$$\omega_\alpha'^{\ \beta} = \bar\omega_\alpha{}^\beta, \qquad \tau'^\alpha = -\bar\tau^\alpha, \tag{5.21}$$

hence

$$X' = -X,\ \Pi_\alpha'^{\ \beta} = \bar\Pi_\alpha{}^\beta,\ R_\alpha'^{\ \beta}{}_{\rho\bar\sigma} = \bar R_\alpha{}^\beta{}_{\rho\bar\sigma},\ A'_{\overline{\alpha\beta}} = -A_{\alpha\beta}. \tag{5.22}$$

By [20, (1.33)], R is real, so $R' = R$. Hence, we may assume that the coefficient of XR in K_2 is real. By (5.22), $X'R' = -XR$. Combining this with (5.20), we may assume that the coefficient of XR is 0. Similarly, we may assume that the coefficients of $\operatorname{Im} R_{;\alpha}{}^\alpha$ and $\operatorname{Re} A_{\alpha\beta;}{}^{\alpha\beta}$ are 0. By (5.6) and [20, (1.43)],

$$\begin{aligned} W_\alpha{}^\rho{}_\beta &= A_{\alpha\beta;\bar\rho} \\ &= (\nabla \bar T)_{\alpha\beta;\bar\rho}, \end{aligned} \tag{5.23}$$

so the term $W_\alpha{}^\beta{}_{\rho;}{}^\mu$ in (5.18) is redundant. By the Bianchi identities [12, Lemma 2.2],

$$R_\nu{}^\mu{}_{;\mu} = R_{;\nu} - i(n-1)A_{\alpha\nu;\bar\alpha}, \tag{5.24}$$

so $R_\nu{}^\mu{}_{;\mu}{}^\nu$ is redundant in (5.18). Thus, K_2 can be written as a linear combination of

$$R^2,\ |Ric|^2,\ |\Pi^{1,1}|^2,\ |T|^2,\ \operatorname{Re} R_{;\alpha}{}^\alpha,\ \operatorname{Im} A_{\alpha\beta;}{}^{\alpha\beta}. \tag{5.25}$$

By (1.13),

$$\operatorname{Re} R_{;\alpha}{}^\alpha = \Delta_b R. \tag{5.26}$$

By (5.11) and (5.12),

$$\operatorname{Im} A_{\alpha\beta;}{}^{\alpha\beta} = -\operatorname{Im} d_b^* C \nabla T. \tag{5.27}$$

The proposition now follows from (5.25)-(5.27). □

Proposition 5.13 is a crucial ingredient in the proof of the next proposition.

PROPOSITION 5.28. *If M is three-dimensional then there is a constant a independent of M such that*

$$k_2 = \int_M a|T|^2 \theta \wedge d\theta. \tag{5.29}$$

Proof. Because $n = 1$,

$$|\Pi^{1,1}|^2 = |Ric|^2 = R^2; \tag{5.30}$$

so, by Proposition 5.13, there are constants a, b, c, and d such that

$$K_2 = a|T|^2 + b \operatorname{Im} d_b^* C \nabla T + cR^2 + d\Delta_b R. \tag{5.31}$$

The volume element is $\theta \wedge d\theta$, so for any one form μ,

$$\int_M d_b^* \mu \theta \wedge d\theta = 0. \tag{5.32}$$

Because Δ_b is self-adjoint,

$$\int_M \Delta_b R \theta \wedge d\theta = 0. \tag{5.33}$$

For the case $M = S^3$ with the pseudo-hermitian structure (4.2), $T = 0$ and $R = 4$, so

$$k_2 = 4c \int_M \theta \wedge d\theta. \tag{5.34}$$

By Theorem 4.34, $k_2 = 0$ for S^3, so by (5.34), $c = 0$. The proposition now follows from integrating (5.31). □

Finally, we prove the following.

THEOREM 5.35. *On a compact strictly pseudoconvex three-dimensional CR manifold M, the global invariant k_2 is 0.*

Proof. We will show that there is a pseudo-hermitian structure on S^3 (with the standard underlying CR structure) for which $T \not\equiv 0$. The theorem then follows from Proposition 5.28 and Theorem 4.34. For the pseudo-hermitian structure (4.2) on S^3, we may take

$$\theta^1 = \frac{1}{\sqrt{2}}(z\,dw - w\,dz) \quad \text{and} \quad Z = \sqrt{2}\left(\bar{z}\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial z}\right). \tag{5.36}$$

Here we use (z, w) as the coordinates in $\mathbf{C}^2$. Then, by (5.6) and (5.7),

$$\omega_1{}^1 = -4i\theta \quad \text{and} \quad \tau^1 = 0. \tag{5.37}$$

For any real-valued function λ, we let $\tilde{\theta} = e^{\lambda}\theta$. By [11, Lemma 5.6], the torsion form for this new pseudo-hermitian structure is

$$\overline{\tilde{\tau}^1} = e^{-\lambda}(i\lambda_{11} - i(\lambda_1)^2)\tilde{\theta}^1. \tag{5.38}$$

We take

$$\lambda = (z + \bar{z}). \tag{5.39}$$

By (5.36)-(5.39),

$$\tilde{A}_{11} = -2i\bar{w}^2 e^{-(z+\bar{z})}. \tag{5.40}$$

Hence, $\tilde{T} \not\equiv 0$. □

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