

# 3-Dimensional Bordism

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In [10], C. Rourke gave an elementary proof of the following.

**THEOREM 1** (Rohlin [9], Thom [11]).  $\Omega_3^{\text{SO}} = 0$  (every closed oriented 3-manifold  $M$  is the oriented boundary of a compact oriented 4-manifold).

Rourke's proof is by induction on Heegaard genus. He shows that if the genus of  $M$  is nonzero, then  $M$  is bordant to a disjoint union of manifolds of lower genus. The bordism is achieved by surgery on a carefully chosen link lying in the splitting surface of a Heegaard decomposition of  $M$ .

In this paper we show how to generalize Rourke's argument to give elementary proofs of the following two theorems.

**THEOREM 2** (Rohlin [9], Thom [11]).  $\Omega_3^{\text{O}} = 0$  (every closed 3-manifold is the boundary of a compact 4-manifold).

**THEOREM 3** (Milnor [6; 7]).  $\Omega_3^{\text{Spin}} = 0$  (every closed spin 3-manifold is the spin boundary of a compact spin 4-manifold).

Compare Lickorish [5] for Theorem 2 and Kaplan [4] for Theorem 3.

The proof of Theorem 3 (or in particular Assertion 3, which is the chief contribution of this paper) gives an explicit construction of a family of simply-connected spin 4-manifolds with a given spin 3-manifold boundary. It is hoped that this family will be useful in the study of smooth closed simply-connected 4-manifolds and of invariants of 3-manifolds.

## 0. Preliminaries

We shall work in the smooth category. A *framing*  $\mathbf{t}$  of a trivial vector bundle  $\epsilon$  is a homotopy class of trivializations of  $\epsilon$ , that is (up to homotopy) a list  $t_1, \dots, t_r$  of  $r = \text{rank}(\epsilon)$  linearly independent nonvanishing sections of  $\epsilon$ . Write  $\mathbf{t} = [t_1, \dots, t_r]$ .

Let  $M$  be an  $m$ -manifold. A framing of the restriction of the tangent bundle of  $M$  to a subset  $Z$  will be called a *tangential framing* of  $Z$  (in  $M$ ). In particular, a *spin structure* on  $M$  is (for  $m > 2$ ) a tangential framing of

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$M^{(2)}$ , the 2-skeleton of some triangulation of  $M$ . See Milnor [7, p. 202].  $M$  together with a spin structure  $s = [s_1, \dots, s_m]$  is called a *spin manifold*. We say that  $M$  is the *spin boundary* of a spin manifold  $W$  (with spin structure  $S = [S_0, S_1, \dots, S_m]$ ) if  $M = \partial W$  and  $S|_M = s$  (i.e.,  $(S_1, \dots, S_m)|_{M^{(2)}}$  and  $(s_1, \dots, s_m)$  are homotopic). Spin manifolds  $M$  and  $N$  are *spin bordant* if the disjoint union  $M \cup (-N)$  is a spin boundary. (Note that a spin manifold  $M$  has a natural orientation—i.e., tangential framing of  $M^{(0)}$  which extends to  $M^{(1)}$ .)

Let  $Z$  be a submanifold of  $M$  with trivial normal bundle  $\nu$ . A framing of  $\nu$  will be called a *normal framing* of  $Z$  (in  $M$ ). If in addition  $Z$  is embedded with trivial normal bundle in a submanifold  $F$  of  $M$ , and  $F$  has trivial normal bundle in  $M$ , then any framing of  $\nu$  obtained by juxtaposing normal framings of  $Z$  in  $F$  and  $F$  in  $M$  will be called a *natural framing* of  $Z$  in  $M$  (relative to  $F$ ).

Now let  $Z$  be a normally framed link of spheres in  $M$ . Denote by  $M(Z)$  the manifold obtained from  $M$  by surgery along  $Z$ . If  $M$  lies in  $\partial W$ , then write  $W_Z$  for  $W$  with handles attached along  $Z$ . Observe that  $\partial(W_Z) = (\partial W)(Z)$ . In particular,  $M$  and  $M(Z)$  are bordant, since  $M \cup (-M(Z)) = \partial(M \times I_{Z \times 1})$ .

If  $M$  is spin and  $\dim(Z) > 1$ , then  $M(Z)$  can be given a spin structure for which  $M$  and  $M(Z)$  are spin bordant. For example, if  $M$  is a connected sum  $P \# Q$  with splitting sphere  $Z$ , then  $M(Z) = P \cup Q$  inherits a spin structure by restriction (since  $(P \cup Q)^{(2)}$  lies in  $(P \# Q)^{(2)}$ ), and it is easily verified that  $P \# Q$  and  $P \cup Q$  are spin bordant.

For  $Z$  1-dimensional,  $M(Z)$  need not have a spin structure. There is, however, the following presumably well-known result. We state it for 3-manifolds (for use in our proof of Theorem 3) although it holds in higher dimensions as well. A proof is given in the appendix.

**DEFINITION.** A normally framed circle  $z$  in a spin manifold  $M$  is *spin* if its framing  $\mathbf{n}$  is incompatible with the spin structure  $s$  on  $M$  (i.e., if  $\mathbf{t} \neq s|_z$ , where  $\mathbf{t}$  is obtained from  $\mathbf{n}$  by adding the oriented tangent to  $z$ ).

**LEMMA 0.** *Let  $Z$  be a normally framed link of circles in a spin 3-manifold  $M$ . If each component of  $Z$  is spin, then  $M(Z)$  has a spin structure for which  $M$  and  $M(Z)$  are spin bordant.*

## 1. Rourke's Proof

Let  $F$  be a closed oriented surface and let  $x$  be a *curve* (i.e., a smoothly embedded circle) in  $F$ . Write  $[x]$  for the class of  $x$  in  $H_1(F; \mathbf{Z}_2)$ . We say that  $x$  is *essential* if  $[x] \neq 0$  (equivalently, if  $x$  is nonseparating). Two curves in  $F$  are *dual* if they intersect transversely in exactly one point. Note that curves which have duals are essential.

A union  $X$  of disjoint essential curves in  $F$  is called a *complete system of curves* in  $F$  if  $F(X)$  is a sphere (for either normal framing of  $X$ ). Two transverse complete systems are *dual* if they have a pair of dual components.

REMARK 1. *Every essential curve in a closed oriented surface is contained in a complete system of curves (by the classification of surfaces).*

A *Heegaard diagram* of genus  $g$  is a triple  $(F, X, Y)$ , where  $X$  and  $Y$  are transverse complete systems of curves in a closed oriented surface  $F$  of genus  $g$ . Associated with the diagram is an oriented 3-manifold  $F_{X,Y}$ , obtained from  $F \times I_{X \times 0 \cup Y \times 1}$  by capping off the boundary 2-spheres ( $X$  and  $Y$  are complete) with 3-balls. (Observe that  $F$  sits naturally in  $F_{X,Y}$  as  $F \times \{1/2\}$ .) Every closed oriented 3-manifold  $M$  arises in this way, and the minimal genus of a diagram for  $M$  is called the *genus* of  $M$ . Note that if  $X$  and  $Y$  are dual, then the genus of  $F_{X,Y}$  is less than the genus of  $F$ , by a standard handle cancellation argument.

Theorem 1 follows from the following.

LEMMA 1 (Rourke). *Let  $F$  be a closed oriented surface, and let  $X$  and  $Y$  be transverse complete systems of curves in  $F$ .*

(a) *For any complete system of curves  $Z$  in  $F$  (transverse to  $X$  and  $Y$  and naturally framed in  $F_{X,Y}$ ),*

$$F_{X,Y}(Z) = F_{X,Z} \# F_{Z,Y}.$$

*In particular,  $F_{X,Y}$  and  $F_{X,Z} \cup F_{Z,Y}$  are bordant.*

(b) *There exists a sequence  $Z_0, \dots, Z_n$  of complete systems of curves in  $F$ , each dual to the next, with  $Z_0 = X$  and  $Z_n = Y$ .*

Lemma 1(a) is Rourke's Lemma 1. The proof is easy: The surgery on  $F_{X,Y}$  replaces each framed circle  $z = S^1 \times B^2$  of  $Z$  by  $B^2 \times S^1$ . But  $B^2 \times S^1$  can be viewed as the union of two 2-handles  $B^2 \times S^1_{\pm}$ , one attached to  $F \times [0, 1/2]_{X \times 0}$  and the other to  $F \times [1/2, 1]_{Y \times 1}$  along  $z \times \{1/2\}$ . This yields the desired connected sum with  $F(Z)$  as the splitting 2-sphere. The last statement follows from observations in Section 0. Lemma 1(b) is immediate from the following assertion and Remark 1.

ASSERTION 1. *Let  $F$  be a closed oriented surface. If  $x$  and  $y$  are transverse essential curves in  $F$ , then there exists a sequence  $z_0, \dots, z_n$  of essential curves, each dual to the next, with  $z_0 = x$  and  $z_n = y$ .*

This is in essence Rourke's Lemma 2. We prove a more general statement in Assertion 2 below.

*Proof of Theorem 1.* Let  $M$  be a closed oriented 3-manifold, and let  $(F, X, Y)$  be a Heegaard diagram for  $M$  of minimal genus  $g$ . If  $g = 0$ , then  $M = S^3 = \partial B^4$ , so assume  $g > 0$ . Choose a sequence  $Z_0, \dots, Z_n$  of complete systems as in Lemma 1(b). Applying Lemma 1(a) inductively, we see that  $M = F_{X,Y}$  and  $N = F_{Z_0, Z_1} \cup \dots \cup F_{Z_{n-1}, Z_n}$  are bordant. But each  $F_{Z_{i-1}, Z_i}$  has genus less than  $g$  (since  $Z_{i-1}$  and  $Z_i$  are dual) and so, by induction, bounds a 4-manifold  $W_i$ . Thus  $N = \partial(W_1 \cup \dots \cup W_n)$  and so  $M$  bounds.

REMARK. An isotopy class of a complete system of curves is called a *cut system*. In [2], Hatcher and Thurston introduced a 2-complex  $X_g$ , whose

vertices are the cut systems on a surface of genus  $g$ , and showed (using Cerf theory) that  $X_g$  is connected and simply connected. From this they derived a finite presentation for the mapping class group of the surface. Assertion 1 yields an elementary proof, by induction on  $g$ , of the connectedness of  $X_g$ . This, in turn, yields an even more direct proof of Theorem 1.

## 2. The Unoriented Case

To extend the result of Section 1 to the unoriented case, we must consider Heegaard diagrams  $(F, X, Y)$  of genus  $g$  for nonorientable 3-manifolds (see, e.g., Hempel [3]).  $F$  is taken to be a closed *nonorientable* surface of genus  $g$  (i.e., a connected sum of  $g \geq 1$  Klein bottles), and  $X$  and  $Y$  are transverse *complete systems of curves* on  $F$ , defined as in Section 1 with the appropriate modified definition of essential curves: An *essential curve* in  $F$  is a nonseparating 2-sided curve  $x$  with  $[x] \neq \omega$  when  $g > 1$ , where  $\omega \in H_1(F; \mathbf{Z}_2)$  is the dual of the first Stiefel–Whitney class of  $F$ . (The condition  $[x] \neq \omega$  is equivalent to the nonorientability of  $F - x$ .) By the classification of surfaces, once again, one has the following.

REMARK 2. *Remark 1 holds for arbitrary closed surfaces.*

The definition of  $F_{X,Y}$  is exactly as in Section 1. One readily verifies that every closed nonorientable 3-manifold is diffeomorphic to some  $F_{X,Y}$ . With these definitions, Lemma 1 goes through in the nonorientable case.

LEMMA 2. *Lemma 1 holds for arbitrary closed surfaces.*

The proof of 1(a) is unchanged, and 1(b) is a consequence of the following Assertion and Remark 2. (It is here that the condition  $[x] \neq \omega$  for curves in  $X$  and  $Y$  is used.)

ASSERTION 2. *Assertion 1 holds for arbitrary closed surfaces.* [This result is used in a forthcoming paper by D. Gabai and the second author on the classification of maps of nonorientable surfaces.]

*Proof.* Let  $x \cdot y$  denote the number of points of  $x \cap y$ . If  $x \cdot y = 0$ , then write  $x \# y$  for any *band sum* of  $x$  and  $y$  along an arc disjoint from  $x \cup y$ . In general, write  $x + y$  for the *double point sum* of  $x$  and  $y$  (with respect to any choice of orientation), obtained from  $x \cup y$  by smoothing the double points (see Figure 1). Note that  $[x \# y] = [x + y] = [x] + [y]$ . Thus if  $x \# y$  or

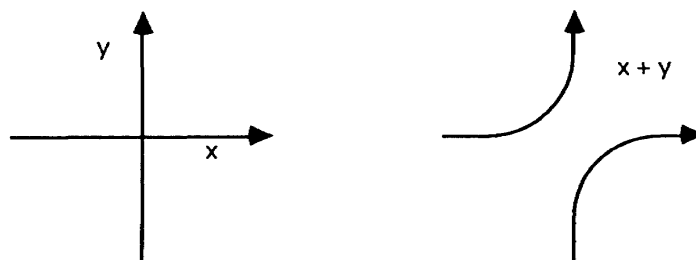


Figure 1

$x + y$  is an essential curve and  $x$  and  $y$  are 2-sided, then  $x$  or  $y$  is essential as well.

If  $x \cdot y = 0$ , then we show how to find an essential curve  $z$  dual to both  $x$  and  $y$ , provided  $g > 1$ . (If  $g = 1$  then  $x = y$ .) It is easy to find some curve  $z$  dual to both, as neither separates  $F$ . (In particular,  $[z] \neq 0$ .) If  $z$  is 1-sided, then a regular neighborhood of  $x \cup y \cup z$  is a twice-punctured Klein bottle (Figure 2). It follows that  $F - (x \cup y \cup z)$  is nonorientable, since both  $F - x$

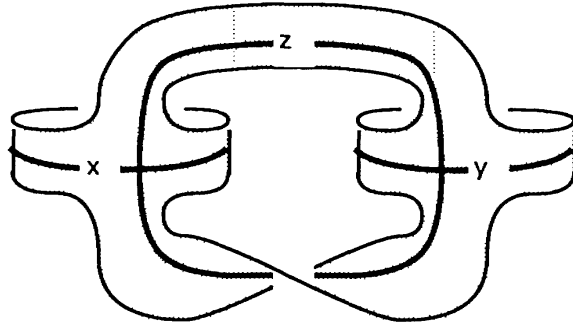


Figure 2

and  $F - y$  are nonorientable. Thus  $z$  may be modified (by band summing with a 1-sided curve in  $F - (x \cup y \cup z)$ ) to become 2-sided. Finally, arrange that  $[z] \neq \omega$  by using  $x + z$  in place of  $z$  if necessary.

If  $x \cdot y = 1$ , then  $x$  and  $y$  are dual and there is nothing to show.

Now assume that  $x \cdot y > 1$ . A curve  $z$  (transverse to  $x$  and  $y$ ) with  $x \cdot z$  and  $y \cdot z$  both less than  $x \cdot y$  will be called *admissible*. It suffices by induction to find an admissible essential curve.

A component of  $y - x$  will be called a *singular arc* of  $y$  if its endpoints are oppositely oriented, in the following sense: A choice of orientations on  $x$  and  $y$  yields orientations on the endpoints of the arc, and these can be compared within a regular neighborhood of  $x$ , which is orientable since  $x$  is 2-sided. Similarly define the singular arcs of  $x$ .

Now let  $y_1$  be a component of  $y - x$  and  $N$  be a regular neighborhood of  $x \cup y_1$ . There are evidently four possibilities for  $N$ , shown in Figure 3(a)–(d). These can be classified by the orientability of  $N$  (distinguishing (a) and (b) from (c) and (d)) and the singularity of  $y_1$  (distinguishing (a) and (c) from (b) and (d)).

For (a) and (b), consider the admissible 2-sided curves  $z$  and  $z'$  shown, with  $x = z \# z'$  in (a) and  $z + z'$  in (b). Since  $x$  is essential, either  $z$  or  $z'$  is as well. For (c) we have  $[z] = [x]$ , and so  $z$  is 2-sided and essential (and evidently admissible). Finally, case (d) reduces to (a) or (c) by swapping roles of  $x$  and  $y$ , as it is evident that some arc of  $x$  is singular. The proof of the assertion is complete.  $\square$

Theorem 2 now follows as in Section 1, except that (in the nonorientable case) the induction begins with genus 1 when  $M = S^1 \times S^2 = \partial(S^1 \times B^3)$  (cf. Hempel [3]).

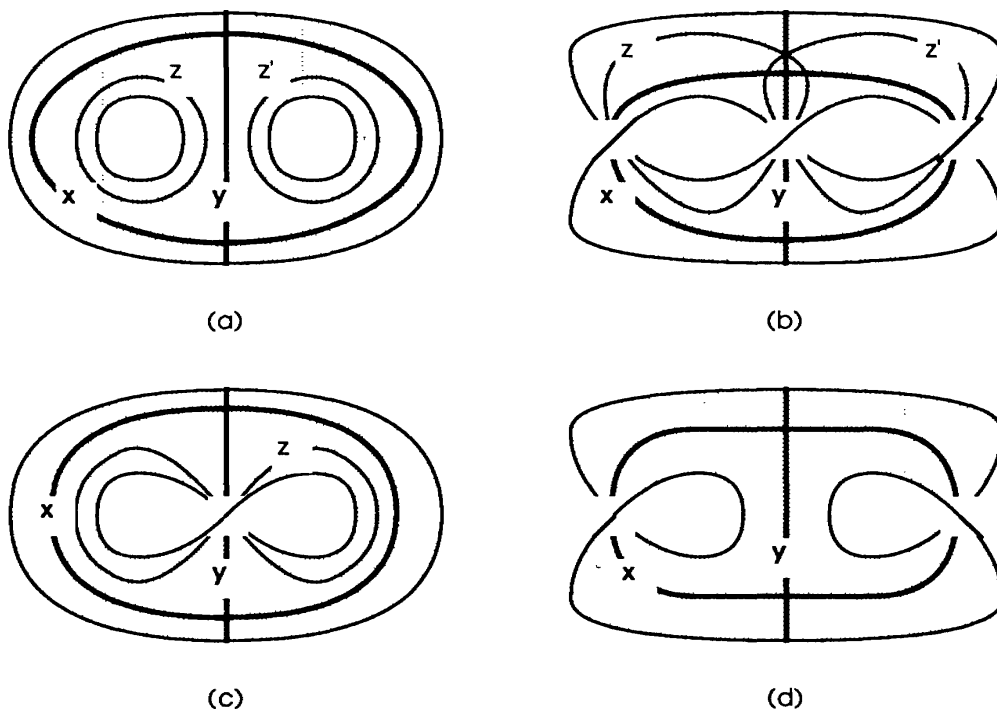


Figure 3

### 3. The Spin Case

Let  $F$  be a closed oriented surface in a spin 3-manifold  $M$ , and let  $x$  be a curve in  $F$  naturally framed in  $M$  (relative to  $F$ ). It turns out that  $x$  is spin if and only if  $[x]$  lies in the kernel of a certain *quadratic form* on  $H_1(F; \mathbf{Z}_2)$ . (See, e.g., [1, pp. 54–56] for a general discussion of  $\mathbf{Z}_2$ -valued quadratic forms.)

To be precise, set  $f(x) = 1$  if  $x$  is spin, and  $f(x) = 0$  otherwise (for each curve  $x$  in  $F$ ). Extend  $f$  to a  $\mathbf{Z}_2$ -valued function on the set of all embedded 1-manifolds in  $F$  by summing over components. Now define

$$q(x) = f(x) + |x| \pmod{2},$$

where  $|x|$  denotes the number of components of  $x$ . Thus a curve  $x$  is spin if and only if  $q(x) = 0$ , by definition.

The following result is essentially due to Pontryagin [8, pp. 103–106].

**PROPOSITION.**  $q$  defines a quadratic form on  $H_1(F; \mathbf{Z}_2)$ . That is,  $q(x) = q(y)$  if  $[x] = [y]$ , and (for  $x$  and  $y$  transverse)

$$(*) \quad q(x+y) = q(x) + q(y) + x \cdot y \pmod{2}.$$

In particular (for  $x$  and  $y$  disjoint),  $q(x \# y) = q(x) + q(y)$ .

Note: The  $+$  on the left side of the equality in  $(*)$  denotes double point sum, and  $\cdot$  denotes geometric intersection number (see the proof of Assertion 2).

*Proof.* Evidently,

$$(1) \quad f(x+y) = f(x) + f(y).$$

Furthermore, it is not hard to show that

$$(2) \quad f(x) = |x| \pmod{2} \quad \text{if } [x] = 0$$

and

$$(3) \quad |x+y| = |x| + |y| + x \cdot y \pmod{2}.$$

For (2), let  $S$  be a surface in  $F$  with  $\partial S = x$ . Note that the mod 2 Euler characteristic  $\chi(S)$  of  $S$  is the obstruction to extending the natural stable framing of  $\partial S$  to a stable framing of  $S$ . Thus  $f(x) = \chi(S)$ . But  $\chi(S) = |\partial S| \pmod{2}$ , and (2) follows.

For (3), embed  $x \cup y$  in  $F \times I$  with  $x$  in  $F \times 0$  and  $y$  in  $F \times 1$ . Now alter this embedding to an embedding of  $x + y$  in  $F \times I$  by a sequence of  $x \cdot y$  ambient surgeries, one for each double point of  $x$  and  $y$  in  $F$ . Each surgery changes the number of components by one, and so (3) follows.

It is now evident that  $q$  is well defined on  $H_1(F; \mathbf{Z}_2)$ . For if  $[x] = [y]$ , then  $[x+y] = 0$  and  $x \cdot y = 0$ , and so

$$\begin{aligned} f(x) + f(y) &= f(x+y) && \text{by (1)} \\ &= |x+y| && \text{by (2)} \\ &= |x| + |y| + x \cdot y && \text{by (3)} \\ &= |x| + |y|. \end{aligned}$$

Thus  $q(x) = f(x) + |x| = f(y) + |y| = q(y) \pmod{2}$ .

A similar argument using (1) and (3) establishes the quadratic identity (\*). The last statement follows since  $[x \# y] = [x + y]$ .  $\square$

We call  $q$  the form *induced* on  $F$  by the spin structure on  $M$ . Observe that if  $M = F_{X,Y}$  (as in §1) then  $q$  vanishes on the components of  $X$  and  $Y$ , since each bounds a disc away from  $F$  (cf. the proof of (2) in the Proposition). It follows that  $q$  has Arf invariant 0 [1, p. 56]. This motivates the following definition.

**DEFINITION.** A *quadratic surface*  $(F, q)$  is a closed oriented surface  $F$  together with a quadratic form  $q$  on  $H_1(F; \mathbf{Z}_2)$  (as in the Proposition) of Arf invariant 0. A curve  $x$  in  $(F, q)$  is *spin* if  $q(x) = 0$ , and *essential* if it is spin and nonseparating. A *complete system of curves*  $X$  in  $(F, q)$  is a union of disjoint essential curves in  $(F, q)$  with  $F(X) = S^2$ .

One then has the following analogue of Remark 1 for quadratic surfaces.

**REMARK 3.** Any essential curve  $x$  in a quadratic surface  $(F, q)$  lies in a complete system of curves in  $(F, q)$ .

*Proof.* By Remark 1,  $x$  lies in a complete system  $x_1, \dots, x_g$  in  $F$  (forgetting  $q$ ). Suppose that  $q(x_i) = 1$  for some  $i$ . Choose another complete system  $y_1, \dots, y_g$  in  $F$  such that  $x_i \cdot y_j = \delta_{ij}$  (i.e., the  $x$ 's and  $y$ 's form a symplectic basis). If  $q(y_i) = 0$  then replace  $x_i$  by  $y_i$ . If  $q(y_i) = 1$  then, since the Arf invariant of  $q$  is 0, there is some  $j \neq i$  with  $q(x_j) = q(y_j) = 1$  ([1, p. 54]). In this case, replace  $x_i$  by  $x_i \# x_j$  and  $x_j$  by  $y_i \# y_j$  (embedded disjointly). Continue until all the curves in the system are spin.  $\square$

Theorem 3 now follows, exactly as in Section 1, from the following.

LEMMA 3. *Lemma 1 holds for quadratic surfaces  $(F, q)$ , with the following addendum to 1(a): For any spin structure on  $F_{X,Y}$  which induces  $q$ , there are spin structures on  $F_{X,Z}$  and  $F_{Z,Y}$  such that  $F_{X,Y}$  and  $F_{X,Z} \cup F_{Z,Y}$  are spin bordant.*

The proof of 1(a) is unchanged. For the addendum, note that Lemma 0 provides a spin bordism between  $F_{X,Y}$  and  $F_{X,Y}(Z) = F_{X,Z} \# F_{Z,Y}$  (with an appropriate spin structure). But, as observed in Section 0, the latter is spin bordant to  $F_{X,Z} \cup F_{Z,Y}$  (with the inherited spin structure). Finally, 1(b) is a consequence of the following assertion and Remark 3.

ASSERTION 3. *Assertion 1 holds for quadratic surfaces  $(F, q)$ .*

*Proof.* (Note that the assumption that  $q$  has Arf invariant 0 is not used in the proof.) We adopt the notation and terminology of the proof of Assertion 2. Observe that if  $z$  and  $z'$  are dual and  $z + z'$  is essential, then  $z$  or  $z'$  is essential.

If  $x \cdot y = 0$ , then, as in the proof of Assertion 2, there is a curve  $z$  dual to  $x$  and  $y$ . But then  $z' = z + x$  is also dual to  $x$  and  $y$ , and  $z$  and  $z'$  are dual to each other. Since  $x = z + z'$  (appropriately oriented) is essential, one of  $z$  or  $z'$  is essential.

If  $x \cdot y = 1$ , then there is nothing to show. If  $x \cdot y = 2$ , then one can easily find a curve  $z$  dual to  $x$  and  $y$  by considering the two possibilities for a regular neighborhood of  $x \cup y$  and using the fact that neither  $x$  nor  $y$  separates. This case now proceeds as in the case when  $x \cdot y = 0$ .

Finally, assume that  $x \cdot y > 2$ . It suffices by induction to find an admissible essential curve. Recall that an arc (component) of  $x \Delta y = (x \cup y) - (x \cap y)$  is *singular* if its endpoints are oppositely oriented with respect to some orientation on  $x$  and  $y$ . There are two cases.

*Case 1.* Some arc in  $x \Delta y$  (say  $y_1 \subset y$ ) is nonsingular.

Then a regular neighborhood of  $x \cup y_1$  looks like Figure 3(b). The two admissible curves  $z$  and  $z'$  shown are dual and  $x = z + z'$ . Thus one is essential.

*Case 2.* All arcs in  $x \Delta y$  are singular.

In particular, a neighborhood of  $x \cup y_1$ , for any arc  $y_1$  in  $y - x$ , is a pair of pants with waist parallel to  $x$  [see Figure 3(a)]. The other two components of  $\partial P$  are called *cuffs of  $x$*  [ $z$  and  $z'$  in Figure 3(a)], and the two cuffs of  $P$  are said to be *twins*. Similarly define cuffs of  $y$ . All cuffs are evidently admissible.

Suppose that some cuff  $z$  (say, of  $x$ ) is spin. Then its twin  $z'$  is spin as well. Indeed,  $x = z \# z'$  and so  $0 = q(x) = q(z) + q(z')$ . But one of the twins is nonseparating, since  $x$  is nonseparating, and so one is essential. Thus we may assume that every cuff is nonspin.

Let  $y_1$  and  $y_2$  be adjacent arcs of  $y - x$ , and let  $N$  be a regular neighborhood of  $x \cup y_1 \cup y_2$ . Since  $y_1$  and  $y_2$  are singular,  $N$  is a disc with three holes as shown in Figure 4. Observe that all the boundary components  $z_1, z_2, z_3$ , and  $z_4$  of  $N$  are cuffs. Now both of the admissible curves  $u = z_1 \# z_3$  and  $v = z_2 \# z_3$



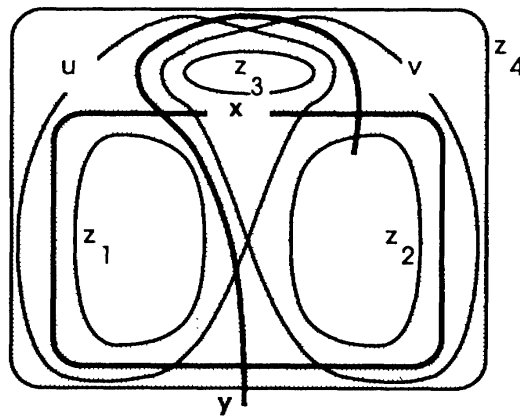


Figure 4

shown in Figure 4 are spin, since the  $z_i$  are nonspin. But at least one of  $u$  or  $v$  is nonseparating because  $[u] + [v] = [x] \neq 0$ , and so one is essential.  $\square$

REMARK. As in the remark at the end of Section 1, Assertion 2 shows that a certain *spin subcomplex* of the Hatcher–Thurston complex  $X_g$  is connected, and this yields an even more direct proof of Theorem 3. A similar remark can be made in the unoriented case.

**Appendix. Proof of Lemma 0**

Choose a component  $z$  of  $Z$ , and set  $W = (M \times I)_{z \times 1}$ ;  $W$  is obtained by attaching a 2-handle  $H = B^2 \times B^2$  to  $M \times I$ , identifying  $H_- = S^1 \times B^2$  with  $N \times 1$  (where  $N$  is a tubular neighborhood of  $z$ , identified with  $S^1 \times B^2$  by the normal framing  $\mathbf{n}$ ). Set  $H_+ = B^2 \times S^1$ . Note that (after smoothing corners)  $F = H_+ \cap H_-$  is a smooth torus which is the common boundary of  $H_+$  and  $H_-$  (see Figure 5).

Extend the spin structure  $s$  on  $M$  to a framing of  $M = M \times 0$ , and then (by adding the outward normal to  $M \times 0$  and extending trivially) to a tangential framing  $\mathbf{T}$  of  $M \times I$  in  $W$ . Similarly, extend the framing  $\mathbf{n}$  on  $z$  (after adding the oriented tangent to  $z$  and the outward normal to  $M \times 0$ ) to a tangential framing  $\mathbf{T}'$  of  $N \times I$  in  $W$ .

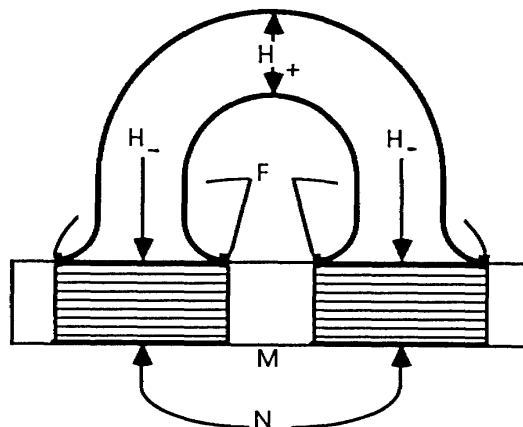


Figure 5

Now, dropping the normals,  $\mathbf{T}$  and  $\mathbf{T}'$  induce distinct tangential framings  $\mathbf{t}$  and  $\mathbf{t}'$  on  $F$  in  $H_{\pm}$  (since  $\mathbf{n}$  and  $\mathbf{s}$  are incompatible) which both extend to  $H_{-}$ . But there are only two such framings on  $F$ , and one also extends to  $H_{+}$  (since  $\pi_1 \text{SO}(3) = \mathbf{Z}_2$ ). Evidently  $\mathbf{t}'$  does not extend to  $H_{+}$ . Indeed, using polar coordinates  $(r, \theta, \varphi)$  on  $H_{+} = B^2 \times S^1$ ,  $\mathbf{t}' | (S^2 \times 0) = [\partial/\partial\theta, \partial/\partial r, \partial/\partial\varphi]$  does not even extend to  $B^2 \times 0$ . Thus  $\mathbf{t}$  extends to  $H_{+}$ , giving an extension of  $\mathbf{T}$  to  $(M \times I) \cup H_{+} \approx W$ -point, and thus (by restriction) a spin structure  $\mathbf{S}$  on  $W$  with  $\mathbf{S} | M = \mathbf{s}$ . Hence  $M$  and  $M(z)$  are spin bordant.

The lemma follows by induction on the number of components of  $Z$ .  $\square$

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### References

1. W. Browder, *Surgery on simply-connected manifolds*, Ergeb. Math. Grenzgeb., Springer-Verlag, New York, 1972.
2. A. Hatcher and W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, Topology 19 (1980), 221-237.
3. J. Hempel, *3-manifolds*, Ann. of Math. Studies, 86, Princeton Univ. Press, Princeton, 1976.
4. S. J. Kaplan, *Constructing framed 4-manifolds with given almost framed boundaries*, Trans. Amer. Math. Soc. 254 (1979), 237-263.
5. W. B. R. Lickorish, *Homeomorphisms of non-orientable two-manifolds*, Proc. Cambridge Philos. Soc. 59 (1963), 307-317.
6. J. Milnor, *Differential manifolds which are homotopy spheres*, Princeton Univ. mimeographed notes, 1959.
7. ———, *Spin structures on manifolds*, L'Enseign. Math. (2) 9 (1963), 198-203.
8. L. S. Pontryagin, *Smooth manifolds and their applications in homotopy theory*, Amer. Math. Soc. Transl. 11 (1959), 1-114.
9. V. A. Rohlin, *A three-dimensional manifold is the boundary of a four-dimensional one*, Dokl. Akad. Nauk SSSR 81 (1951), 355-357.
10. C. Rourke, *A new proof that  $\Omega_3$  is zero*, J. London Math. Soc. (2) 31 (1985), 373-376.
11. R. Thom, *Quelques propriétés des variétés-bords*, Colloque de topologie de Strasbourg V (1951).

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