# Regularity of Certain Rigid Isometric Immersions of $n$-dimensional Riemannian Manifolds into $\mathbf{R}^{n+1}$

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## 1. Introduction and Statement of Results

Let $M$ be a real analytic ($C^{\omega}$) Riemannian manifold and let $F$ be an isometry of differentiability class $C^1$ of $M$ onto another $C^{\omega}$ Riemannian manifold $\tilde{M}$. Then $F$ is $C^{\omega}$ and uniquely determined by $F(P)$ and $dF(P)$ at a point $P \in M$. The reason is that $F$ is locally a linear mapping between the normal coordinates of $M$ near $P$ and the normal coordinates of $\tilde{M}$ near $F(P)$. This uniqueness and analyticity of the isometries do not hold for the isometric immersions, as the following example shows.

EXAMPLE 1.1. Let $\gamma(s) = (\gamma^1(s), \gamma^2(s))$ be a plane curve parameterized by arclength $s$. If $\gamma$ is $C^{\infty}$ but not $C^{\omega}$ then the mapping $(s,t) \mapsto (\gamma^1(s), \gamma^2(s), t)$ is a $C^{\infty}$ isometric immersion of $\mathbf{R}^2$ into $\mathbf{R}^3$, which is not $C^{\omega}$. Furthermore, there is not uniqueness either; namely, an isometric immersion $F$ of $\mathbf{R}^2$ into $\mathbf{R}^3$ cannot be determined by $F(P)$ and $dF(P)$ at a point $P \in \mathbf{R}^2$.

The author's question is whether an isometric immersion $F$ is analytic if $F$ is locally rigid. An isometric immersion $F: M \to \mathbf{R}^N$ is said to be locally rigid at $P \in M$ if, for any open neighborhood $U$ of $P$, there exists an open set $V$ such that $P \in V \subset U$ having the following property: If $F'$ is any isometric immersion of $V$ into $\mathbf{R}^N$ then there exists an isometry of $\mathbf{R}^N$ such that $F' = \tau \circ F$. Then the question is the following: Let $M$ be a $C^{\omega}$ Riemannian manifold and let $F: M \to \mathbf{R}^N$ be an isometric immersion of class $C^k$, $k \gg 0$. Let $P \in M$. Then will $F$ be $C^{\omega}$ at $P$ if $F$ is rigid at $P$? This paper is a partial answer to this question. Our main result is the following.

THEOREM 1.1. *Suppose that $M$ is a $C^{\omega}$ Riemannian manifold of dimension $n \geq 3$ and $F: M \to \mathbf{R}^{n+1}$ is an isometric immersion of class $C^2$. Suppose that the immersed submanifold $F(M)$ has at least three nonzero principal curvatures at $F(P)$. Then $F$ is $C^{\omega}$ at $P$.*

Note that the existence of three nonzero principal curvatures (the definition is recalled in Section 2) is a sufficient condition for $F(M)$ to be locally rigid,

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by the well-known rigidity theorem (cf. [6, p. 244]). In the case $n=2$ the following is shown in [1].

THEOREM 1.2. *Suppose that $M$ is a 2-dimensional $C^\omega$ Riemannian manifold of positive curvature and $F: M \to \mathbf{R}^3$ is a $C^2$ isometric immersion. Then $F$ is $C^\omega$.*

The method of our proof is showing that $F$ satisfies a system of (nonlinear) elliptic partial differential equations of second order which are $C^\omega$ in their arguments. The proof is valid in the $C^\infty$ category as well, so that one can obtain a $C^\infty$ version of this paper by replacing every $C^\omega$ by $C^\infty$.

## 2. Proof of Theorem 1.1

Showing analyticity of a mapping is a local problem, so let $M$ be a "germ" of a $C^\omega$ manifold at a reference point $O \in M$. Let $(y^1, \ldots, y^{n+1})$ be the standard coordinates of $\mathbf{R}^{n+1}$ and write $F = (f^1, \ldots, f^{n+1})$ coordinatewise. Let $\tilde{M} = F(M)$ and $\tilde{O} = F(O)$. We may assume that $\tilde{O}$ is the origin of $\mathbf{R}^{n+1}$ and $\tilde{M}$ is tangent to the plane $y^{n+1} = 0$. Let $N$ be a unit normal vector field of $\tilde{M}$ and let $\tilde{A}$ be the second fundamental form—namely, $\tilde{A}(X, Y) \equiv \langle \nabla'_X N, Y \rangle$ for any tangent vectors $X, Y$ of $\tilde{M}$ at $\tilde{O}$, where $\nabla'$ is the covariant differentiation of $\mathbf{R}^{n+1}$.

The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the linear transformation $\mathrm{v} \mapsto \nabla'_{\mathrm{v}} N$ are called the principal curvatures at $\tilde{O}$. Let $\mathrm{v}_1, \ldots, \mathrm{v}_n$ be the orthonormal eigenvectors which correspond to the principal curvatures $\lambda_1, \ldots, \lambda_n$. Let $\{e_1, \ldots, e_n\}$ be a $C^\omega$ orthonormal frame over $M$ such that $F_* e_j = \mathrm{v}_j$ at $\tilde{O}$. We see that

$$\tilde{e}_j \equiv F_* e_j = \sum_{\nu=1}^{n+1} (e_j f^\nu) \circ F^{-1} \frac{\partial}{\partial y_\nu}.$$

We may assume further that

$$\tilde{e}_j = \frac{\partial}{\partial y_j} \quad \text{at } \tilde{O}, \quad j = 1, \ldots, n.$$

Then we have

$$e_j f^\nu(O) = \begin{cases} 0 & \text{if } j \neq \nu, \\ 1 & \text{if } j = \nu. \end{cases} \tag{1}$$

Now let $(\tilde{\eta}_1, \ldots, \tilde{\eta}_{n+1})$ be the components of $N$ and let $\eta_j = \tilde{\eta}_j \circ F$. To express $\eta_j$ in terms of partial derivatives of $(f^1, \ldots, f^{n+1})$, consider the matrix

$$M \equiv \begin{bmatrix} e_1 f^1 & \cdots & e_1 f^{n+1} \\ \vdots & & \vdots \\ e_n f^1 & \cdots & e_n f^{n+1} \\ \eta_1 & \cdots & \eta_{n+1} \end{bmatrix} \in O(n+1, \mathbf{R}).$$

We may assume that $\eta_{n+1}(O) = 1$ so that $\det M = 1$.

Choose a local coordinate system $(x^1,\dots,x^n)$ of $M$ such that

$$e_j=\frac{\partial}{\partial x^j}\ \text{at}\ O\quad \text{for each}\ j=1,\dots,n.$$

We denote by $C^\omega(x, D^\alpha f^k\colon |\alpha|\le m)$ the class of $C^\omega$ functions in the local coordinates $x$ and partial derivatives up to order $m$ of $f^1,\dots,f^{n+1}$. By $C_0^\omega(x, D^\alpha f^k\colon |\alpha|\le m)$ we denote the subset of $C^\omega(x, D^\alpha f^k\colon |\alpha|\le m)$ consisting of those which vanish at $(O, D^\alpha f^k(O))$. Since $M^{-1}=M^t$, each $\eta_j$ is equal to its cofactor in $M$. Thus we have

(2) $$\eta_j=(e_j f^{n+1})B_j+\sum_{\lambda\ne j}(e_\lambda f^{n+1})\zeta_j^\lambda,\quad j=1,\dots,n,$$

and

$$\eta_{n+1}=(e_1 f^1)\cdots(e_n f^n)+\zeta,$$

where each $B_j$ is in $C^\omega(x, D^\alpha f^k\colon |\alpha|\le 1,\ k\ne n+1)$, $B_j=1$ at $(O, D^\alpha f^k(0))$, and each $\zeta_j^\lambda$, $\zeta$ are in $C_0^\omega(x, D^\alpha f^k\colon |\alpha|\le 1,\ k\ne n+1)$. Now, let $A(x)=[A_{ij}(x)]$ be the symmetric matrix defined by

$$A_{ij}(x)=\tilde A(\tilde e_i,\tilde e_j)\circ F=\langle\nabla'_{\tilde e_i}N,\tilde e_j\rangle\circ F.$$

We express $A_{ij}(x)$ in terms of $(f^1,\dots,f^{n+1})$ and their partial derivatives:

$$\begin{aligned}\nabla'_{\tilde e_i}N&=(\tilde e_i\tilde\eta_1,\dots,\tilde e_i\tilde\eta_{n+1})\\&=(e_i\eta_1,\dots,e_i\eta_{n+1})\circ F^{-1}.\end{aligned}$$

However, by (1) and (2) we have

$$e_i\eta_k=(e_ie_kf^{n+1})B_k+\sum_{\lambda\ne k}(e_ie_\lambda f^{n+1})\zeta_k^\lambda+C_{ik},\quad k=1,\dots,n,$$

and

$$e_i\eta_{n+1}=C_{i,n+1},$$

where each $C_{ik}$ and $C_{i,n+1}$ are in $C^\omega(x, D^\alpha f^k\colon |\alpha|\le 2,\ k\ne n+1)$; thus we see that

(3) $$A_{ij}(x)=(e_ie_jf^{n+1})B_j(e_jf^j)+\sum(e_\lambda e_\mu f^\nu)\zeta_\nu^{\lambda\mu},$$

where each $\zeta_\nu^{\lambda\mu}\in C_0^\omega(x, D^\alpha f^k\colon |\alpha|\le 1)$. Since $\tilde e_j=\mathrm{v}_j$ at $\tilde O$ $(j=1,\dots,n)$, which is the eigenvector of the linear transformation $\mathrm{v}\mapsto\nabla'_{\mathrm{v}}N$, we have

(4) $$A_{ij}(O)=\begin{cases}0 & \text{if } i\ne j,\\ \lambda_j & \text{if } i=j.\end{cases}$$

Now let $K_{ij}$ be the sectional curvature of the plane section $e_i\wedge e_j$. Then

$$K_{ij}=A_{ii}A_{jj}-A_{ij}^2$$

(cf. [4]). Consider the equation

$$A_{ii}A_{jj}-A_{ij}^2-K_{ij}=0.$$

Substituting (3) for the $A_{\alpha\beta}$'s in the above, we have

$$\begin{aligned}(5)\qquad &\{(e_ie_if^{n+1})B_i(e_if^i)+\sum(e_\lambda e_\mu f^\nu)\zeta_\nu^{\lambda\mu}\}\\ &\quad\times\{(e_je_jf^{n+1})B_j(e_jf^j)+\sum(e_\lambda e_\mu f^\nu)\zeta_\nu'^{\lambda\mu}\}\\ &\quad-\{(e_ie_jf^{n+1})B_j(e_jf^j)+\sum(e_\lambda e_\mu f^\nu)\zeta_\nu''^{\lambda\mu}\}^2-K_{ij}=0,\end{aligned}$$

where each $\zeta$ is in $C_0^\omega(x, D^\alpha f^k: |\alpha|\le 1)$. Note that $K_{ij}\in C^\omega(x)$.

To derive other equations for $(f^1,\dots,f^{n+1})$, we observe that the first $n$ rows of $M$ are orthonormal and that therefore, for each $i,j=1,\dots,n$,

$$(e_if^1)(e_jf^1)+\cdots+(e_if^{n+1})(e_jf^{n+1})=\delta_{ij}\ \text{(Kronecker's delta)}.$$

Applying $e_i$ to the above, we obtain

$$\begin{aligned}(6)\qquad &(e_ie_if^1)(e_jf^1)+(e_if^1)(e_ie_jf^1)+\cdots\\ &\qquad+(e_ie_if^{n+1})(e_jf^{n+1})+(e_if^{n+1})(e_ie_jf^{n+1})=0.\end{aligned}$$

We shall show that the system of equations (6) with $i,j=1,\dots,n$ and (5) with $i<j$ is elliptic at $(f^1,\dots,f^{n+1})$. Express equations (6) and (5) in terms of coordinates $(x_1,\dots,x_n)$:

$$\begin{aligned}(7)\qquad &\frac{\partial^2 f^1}{\partial x_i^2}\frac{\partial f^1}{\partial x_j}+\frac{\partial f^1}{\partial x_i}\frac{\partial^2 f^1}{\partial x_i\partial x_j}+\cdots+\frac{\partial^2 f^{n+1}}{\partial x_i^2}\frac{\partial f^{n+1}}{\partial x_j}+\frac{\partial f^{n+1}}{\partial x_i}\frac{\partial^2 f^{n+1}}{\partial x_i\partial x_j}\\ &\qquad+\sum\frac{\partial^2 f^\nu}{\partial x_\lambda\partial x_\mu}\zeta_\nu^{\lambda\mu}+\zeta_{ij}\equiv H_{ij}(x,D^\alpha f^k)=0\end{aligned}$$

and

$$\begin{aligned}(8)\qquad &\left(\frac{\partial^2 f^{n+1}}{\partial x_i^2}B_i\frac{\partial f^i}{\partial x_i}+\sum\frac{\partial^2 f^\nu}{\partial x_\lambda\partial x_\mu}\zeta_\nu^{\lambda\mu}+\zeta_i\right)\\ &\times\left(\frac{\partial^2 f^{n+1}}{\partial x_j^2}B_j\frac{\partial f^j}{\partial x_j}+\sum\frac{\partial^2 f^\nu}{\partial x_\lambda\partial x_\mu}\zeta_\nu'^{\lambda\mu}+\zeta_j\right)\\ &-\left(\frac{\partial^2 f^{n+1}}{\partial x_i\partial x_j}B_j\frac{\partial f^j}{\partial x_j}+\sum\frac{\partial^2 f^\nu}{\partial x_\lambda\partial x_\mu}\zeta_\nu''^{\lambda\mu}+\zeta_{ij}\right)^2\\ &-K_{ij}(x)\equiv G_{ij}(x,D^\alpha f^k)=0,\end{aligned}$$

where each $\zeta$ is in $C_0^\omega(x, D^\alpha f^k: |\alpha|\le 1)$. These $\zeta$'s are different from the $\zeta$'s that previously appeared. Consider the linear partial differential operators $L_{ij}$ and $M_{ij}$ given by

$$L_{ij}w=\sum_{\substack{|\alpha|\le 2\\ k=1,\dots,n+1}}\frac{\partial H_{ij}}{\partial(D^\alpha f^k)}D^\alpha w^k,\quad i,j=1,\dots,n,$$

$$M_{ij}w=\sum_{\substack{|\alpha|\le 2\\ k=1,\dots,n+1}}\frac{\partial G_{ij}}{\partial(D^\alpha f^k)}D^\alpha w^k,\quad i<j,$$

where $w=(w^1,\dots,w^{n+1})$. Then $L_{ij}w$ and $M_{ij}w$ are of the following form:

$$(9)\qquad L_{ij}w=a_{ij}\frac{\partial^2 w^j}{\partial x_i^2}+b_{ij}\frac{\partial^2 w^i}{\partial x_i\partial x_j}+\sum\zeta_\nu^{\lambda\mu}\frac{\partial^2 w^\nu}{\partial x_\lambda\partial x_\mu}+\text{lower order terms},$$

and

(10)
$$M_{ij}w=\left(\frac{\partial^2 f^{n+1}}{\partial x_j^2}h_j+\zeta_j\right)\frac{\partial^2 w^{n+1}}{\partial x_i^2}+\left(\frac{\partial^2 f^{n+1}}{\partial x_i^2}h_i+\zeta_i\right)\frac{\partial^2 w^{n+1}}{\partial x_j^2}$$
$$+\sum \zeta_\nu'^{\lambda\mu}\frac{\partial^2 w^\nu}{\partial x_\lambda \partial x_\mu}+\text{lower order terms},$$

where $a, b, h$ are in $C^\omega(x, D^\alpha f^k: |\alpha|\leq 1)$ with values 1 at $(O, D^\alpha f^k(O))$, and all $\zeta$'s are in $C_0^\omega(x, D^\alpha f^k: |\alpha|\leq 2)$. Observe that

$$\frac{\partial^2 f^{n+1}}{\partial x_i \partial x_j}(O)=\begin{cases}\lambda_j & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases}$$

by (3) and (4).

Now consider the principal symbol $\sigma(x,\xi)$ of system (9) with $i, j=1,\ldots,n$ and of (10) with $i<j$. $\sigma(x,\xi)$ is a matrix of size $\{n^2+\frac{1}{2}n(n-1)\}\times(n+1)$. We decompose $\sigma(x,\xi)$ into $n+1$ blocks as

$$\sigma(x,\xi)=\begin{bmatrix}\sigma_1(x,\xi)\\ \vdots\\ \sigma_n(x,\xi)\\ \sigma_{n+1}(x,\xi)\end{bmatrix},$$

where $\sigma_j(x,\xi)$, $j=1,\ldots,n$, is the principal symbol matrix of system (9) with $i=1,\ldots,n$ and fixed $j$, and $\sigma_{n+1}$ is that of (10). Then, for $j=1,\ldots,n$,

$$\sigma_j(0,\xi)=\begin{bmatrix}\xi_1\xi_j & 0 & \cdots & 0 & \xi_1^2 & 0 & \cdots & & 0\\ 0 & \xi_2\xi_j & \cdots & 0 & \xi_2^2 & 0 & \cdots & & 0\\ & & & & \vdots & & & & \\ 0 & 0 & \cdots & 0 & \xi_n^2 & 0 & \cdots & \xi_n\xi_j & 0\end{bmatrix}_{n\times(n+1)}$$

↑
$j$th column

Thus we see that $\forall\xi\neq 0$ the first $n$ columns of $\sigma(O,\xi)$ are linearly independent. Now let $\sigma_{ij}$ be the principal symbol of $M_{ij}$ $(i<j)$. Then

$$\sigma_{ij}(O,\xi)=\left(0,\ldots,0,\frac{\partial^2 f^{n+1}}{\partial x_j^2}(O)\xi_i^2+\frac{\partial^2 f^{n+1}}{\partial x_i^2}(O)\xi_j^2\right)$$
$$=(0,\ldots,0,\lambda_j\xi_i^2+\lambda_i\xi_j^2).$$

Therefore, the last column of $\sigma_{n+1}(O,\xi)$ is

$$\begin{bmatrix}\lambda_2\xi_1^2 & + & \lambda_1\xi_2^2\\ \lambda_3\xi_1^2 & + & \lambda_1\xi_3^2\\ & \vdots & \\ \lambda_n\xi_{n-1}^2 & + & \lambda_{n-1}\xi_n^2\end{bmatrix},$$

which can be written as

$$A\begin{bmatrix}\xi_1^2\\ \vdots\\ \xi_n^2\end{bmatrix},$$

where

$$A=\left[\begin{array}{ccccccc}
\lambda_2 & \lambda_1 & 0 & \cdot & \cdot & \cdot & 0\\
\lambda_3 & 0 & \lambda_1 & \cdot & \cdot & \cdot & 0\\
\vdots & & & & & & \vdots\\
\lambda_n & 0 & 0 & \cdot & \cdot & \cdot & \lambda_1\\
\hline
0 & \lambda_3 & \lambda_2 & 0 & \cdot & \cdot & 0\\
0 & \lambda_4 & 0 & \lambda_2 & \cdot & \cdot & 0\\
\vdots & & & & & & \vdots\\
0 & \lambda_n & 0 & \cdot & \cdot & \cdot & \lambda_2\\
\hline
 & & \vdots & & & & \\
\hline
0 & 0 & \cdot & \cdot & \cdot & \lambda_n & \lambda_{n-1}
\end{array}\right].$$

Let us assume that $\lambda_1$, $\lambda_2$, and $\lambda_3$ are nonzero. Then we see that the last $(n-1)$ columns of the first block of $A$ are independent. Now consider the submatrix of $A$ consisting of the first block and the first row of the second block. It is easy to see that the first column cannot be a linear combination of the other columns; thus $A$ is of rank $n$. Therefore, the last column of $\sigma_{n+1}(O, \xi)$ is a nonzero vector for any $\xi \neq 0$. Now the analyticity of $F$ follows from the regularity theorem of elliptic partial differential equations (cf. [5]).

## References


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