Corona Theorems for Subalgebras of H^∞

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Let H^∞ be the Banach algebra of all bounded analytic functions in the open unit disk $\mathbf{D}$. The famous corona theorem of Carleson [1] states that the unit disk is dense in the maximal ideal space $M(H^\infty)$ of H^∞. Equivalently, the ideal $I=(f_1,\dots,f_N)$ generated by the functions $f_i \in H^\infty$ $(i=1,\dots,N)$ equals the whole algebra H^∞ if and only if $\sum_{k=1}^N |f_k| \geq \delta > 0$ in $\mathbf{D}$.

Let L^∞ denote the space of essentially bounded, Lebesgue measurable functions on the unit circle T. It is standard to identify, via radial limits, H^∞ with a uniformly closed subalgebra of L^∞. Let B be a Douglas algebra, that is, a uniformly closed subalgebra of L^∞ containing H^∞. Associated with each Douglas algebra is the largest C^*-algebra QB contained in B, that is,

$$QB = B \cap \bar{B} = \{f \in B : \bar{f} \in B\},$$

and the C^*-algebra CB generated by the invertible inner functions in B and their conjugates.

By using the corona theorem for H^∞, Chang and Marshall [2] showed that the unit disk is dense in the maximal ideal space of $CA_B := CB \cap H^\infty$. Later Sundberg and Wolff [15] could prove by highly sophisticated methods that the corona theorem is also true in the algebra $QA_B := QB \cap H^\infty$. It is now quite surprising that the methods of Chang and Marshall [2] not only yield another proof of the corona theorem for QA_B, but that they can be used to show that *every* subalgebra A of H^∞ of the form $A = \mathcal{C} \cap H^\infty$ has the corona property, where $\mathcal{C}$ is a C^*-algebra satisfying $CB \subseteq \mathcal{C} \subseteq QB$. The proof of this result will be a major object of this paper. Incidentally, we obtain some other properties of algebras of this type. This will answer a question of Dawson [3, §6] concerning the ideal structure of subalgebras of H^∞.

The Corona Theorem for Admissible Algebras

DEFINITION 1. Let A be a closed subalgebra of H^∞. According to Metzger [12] we shall say that A has the "weak F-property" if f belongs to A whenever $uf \in A$ for some inner function $u \in A$. If we merely assume that u is an inner function in H^∞, then we say that A has the "F-property" (in the sense of Khavin).

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Most of the standard algebras arising in function theory have the weak F-property; for example, if A is the trace in H^∞ of any C^*-algebra $\mathcal{C} \subseteq L^\infty$ (that is, if $A = \mathcal{C} \cap H^\infty$) then it follows from $f = \bar{u}(uf)$ that A has the weak F-property. A nontrivial example of a closed subalgebra of H^∞ which does not have the weak F-property was recently discovered by Gorkin [6], who shows that the algebra $H^\infty \cap B_0 = \{f \in H^\infty : \lim_{|z| \to 1}(1-|z|^2)|f'(z)| = 0\}$ of bounded analytic functions belonging to the little Bloch class B_0 is such an example.

Let $C = C(T)$ be the space of continuous complex-valued functions on the unit circle T, and let $QA = \{f \in H^\infty : \bar{f} \in H^\infty + C\}$. As usual, $A(\mathbf{D}) = C \cap H^\infty$ denotes the disk algebra.

If A is a closed subalgebra between $A(\mathbf{D})$ and QA, then A has the weak F-property if and only if A is invariant under the backward shift operator; that is, if $f \in A$ then $(f(z) - f(0))/z \in A$. This follows from [3, p. 56] and the fact that every inner function in A is a finite Blaschke product.

DEFINITION 2. A closed subalgebra of H^∞ is said to satisfy the "D. J. Newman property" if the Shilov boundary ∂A of A coincides with the set

$$K_A = \{m \in M(A) : |u(m)| = 1 \text{ for every inner function } u \in A\},$$

where $M(A)$ denotes the maximal ideal space of A.

Here also, a large class of algebras has this property. Let us mention at this point the algebras QA_B and CA_B (see [13, p. 38, Thm. 5.3]). In Proposition 7 we shall encounter further examples.

DEFINITION 3. Let $A \subseteq B$ be two commutative algebras with the same identity element. Then (A, B) is called a "Wiener pair" if every element $f \in A$ which is invertible in B is also invertible in A (see [14, p. 203]).

In order to expect that the corona theorem holds in a subalgebra A of H^∞ which contains the polynomials, it is of course necessary that (A, H^∞) forms a Wiener pair.

DEFINITION 4. We shall call a closed subalgebra A of H^∞ "admissible" if it satisfies the following conditions:

$$A \text{ has the weak F-property,} \tag{1}$$

$$A \text{ has the D. J. Newman property,} \tag{2}$$

$$(A, H^\infty) \text{ forms a Wiener pair.} \tag{3}$$

Since we are dealing only with uniform algebras A, we shall identify a function $f \in A$ with its Gelfand transform. Moreover, $Z(f) = \{m \in M(A) : f(m) = 0\}$ denotes the zero set of $f \in A$, and $Z(I) = \bigcap_{f \in I} Z(f)$ will be the zero set (or hull) of an ideal $I \subseteq A$.

We are now able to show that the admissible algebras A with $CA_B \subseteq A \subseteq QA_B$ behave essentially in the same way as CA_B and QA_B. The main tool to achieve this is the following result of Chang and Marshall.

LEMMA 1 (Chang and Marshall [2]). *If $g \in H^\infty$ and if u is an inner function in CA_B with* $\operatorname{dist}(g\bar{u}, H^\infty) := \inf\{\|g\bar{u} - h\|, h \in H^\infty\} < 1$, *then there exists a unimodular function $u_1 \in g\bar{u} + H^\infty$ with $u_1 \in CB$.*

A result of the following type was first proved by Marshall [12, p. 20] for the algebra H^∞.

THEOREM 2. *Let A be an admissible algebra with $CA_B \subseteq A \subseteq QA_B$, and let I be an ideal in A whose hull does not intersect the Shilov boundary of A. Then I is generated* (*algebraically*) *by inner functions.*

Proof. In the first step we show that I contains an inner function u.

Step I. By a compactness argument there exist finitely many functions $f_1, \dots, f_N \in I$ such that

$$\bigcap_{i=1}^{N} Z(f_i) \cap \partial A = \emptyset. \tag{4}$$

Define S to be the closure of the set

$$\{f\bar{u} : f \in A,\ u \text{ inner},\ u \in CA_B\}.$$

Then S is a closed subalgebra of QB which contains CB. Since every inner function $u \in A$ is invertible in S, we see that the maximal ideal space $M(S)$ of S coincides with

$$K_S = \{m \in M(S) : |u(m)| = 1 \text{ for every inner function } u \in A\}.$$

Since A has the D. J. Newman property, it follows that the restriction mapping $\Gamma : M(S) \to \partial A$ is well defined. Moreover, by a theorem of Shilov [8, §12], Γ is surjective. It is also easy to see that A separates the points of $M(S)$; thus $M(S)$ is homeomorphic to ∂A. Hence, by (4),

$$\sum_{i=1}^{N} |f_i| \geq \delta > 0 \quad \text{on } M(S).$$

This implies the existence of functions $q_i \in S$ such that $1 = \sum_{i=1}^{N} q_i f_i$. By the definition of S we can choose functions $u_i, h_i \in A$ $(i = 1, \dots, N)$, u_i inner, such that

$$\|q_i - \bar{u}_i h_i\| \leq \frac{1}{2C} \quad (i = 1, \dots, N),$$

where $C = \sum_1^N \|f_i\|$. Thus we have

$$\left|\sum_1^N h_i \bar{u}_i f_i\right| \geq \left|\sum_1^N q_i f_i\right| - \sum_1^N \|q_i - h_i \bar{u}_i\| \|f_i\| \geq 1 - \frac{1}{2C} C = \frac{1}{2} \tag{5}$$

on $M(S)$. Let $g_i = h_i \prod_{j\neq i} u_j$ and $b = \prod_{j=1}^{N} u_j$. Then $g_i, b \in A$ and $|b| = 1$ on $M(S)$. Multiplying (5) by $|b|$, we obtain

$$\left|\sum_1^N g_i f_i\right| \geq \frac{1}{2}|b| = \frac{1}{2} \quad \text{on } M(S). \tag{6}$$

Let $f = \sum_1^N g_i f_i$. Then $f \in I$. Let $uF = f$ be the inner–outer factorization of f. Since $|f| \geq \frac{1}{2}$ on $M(S)$, f does not vanish on $M(L^\infty)$. Thus the outer function F is invertible in H^∞.

Since $\bar{F} \neq 0$ on $M(B) \subseteq M(H^\infty)$, $\bar{F}^{-1} \in B$ (note that $\bar{F} = u(\bar{u}\bar{F}) \in B$). Hence $\bar{u} = \bar{u}\bar{F}(\bar{F})^{-1} \in B$ and therefore $u \in A$. Since A has the weak F-property, $F \in A$. Because (A, H^∞) is a Wiener pair, $F^{-1} \in A$. Thus $u = (uF)F^{-1} \in I$.

Step II. This works exactly in the same manner as that of [13, p. 49]. We include the proof for the convenience of the reader. Let $g \in I$. Without loss of generality we may assume that $\|g\| \leq \frac{1}{2}$. Since $\operatorname{dist}(g\bar{u}, H^\infty) \leq \|g\bar{u}\| \leq \frac{1}{2} < 1$, the hypotheses of Lemma 1 are fulfilled. Thus there exist functions $h \in H^\infty$, $u_1 \in CB$ and u_1 unimodular (i.e., $|u_1| = 1$ a.e. on T) such that

$$u_1 = g\bar{u} + h.$$

Let $v = v_g := g + uh = uu_1$. Then v is an inner function in H^∞. Moreover $v = uu_1 \in CB$, hence $v \in A$ and $uh = v - g \in A$. Since A has the weak F-property, $h \in A$. Thus $v \in I$. The set $\{v_g : g \in I\}$ and the function u now generate I. □

REMARK. Theorem 2 generalizes Theorem 6.2 in [13, p. 48].

As a corollary we obtain the following separation property.

COROLLARY 3. *The inner functions in A separate the points of $M(A)\setminus\partial A$.*

Proof. Let $m_1 \neq m_2$ be two maximal ideals in $M(A)\setminus\partial A$. Since the m_i are generated by inner functions, there exists $u \in A$, u inner, such that $u(m_1) = 0$ but $u(m_2) \neq 0$. □

We can now prove the main result of this paper.

THEOREM 4. *Let A be an admissible algebra with $CA_B \subseteq A \subseteq QA_B$. Then the corona theorem holds in A.*

Proof. Using standard arguments and the fact that the corona theorem holds in H^∞, it is easy to see that the assertion of the theorem is equivalent to the assertion that the restriction mapping $\Gamma : M(H^\infty) \to M(A)$ is onto.

Assume that $\Gamma(M(H^\infty))$ is a proper subset of $M(A)$. Let $m \in M(A)\setminus \Gamma(M(H^\infty))$. Because by a theorem of Shilov [8, §12], every $x \in \partial A$ extends to a maximal ideal of $M(H^\infty)$, we see that $m \notin \partial A$. Choose an arbitrary $p \in \Gamma(M(H^\infty))$. Then by Corollary 3 there exists an inner function $u \in A$ such that $u(m) = 0$, but $u(p) \neq 0$. (Note that if $p \in \partial A$, then $|u(p)| = 1 \neq 0$ for every inner function $u \in A$.) Since by the continuity of Γ the set $\Gamma(M(H^\infty))$ is compact, there exist finitely many inner functions $u_i \in A$ with $u_i(m)=0$ $(i=1,\dots,n)$ and $\sum_1^n |u_i| \ge \delta > 0$ on $\Gamma(M(H^\infty))$. We shall now proceed in the same manner as Chang and Marshall [2].

Because $\mathbf{D} \subseteq \Gamma(M(H^\infty))$, $\sum_1^n |u_i|$ is bounded away from zero on $\mathbf{D}$. By the corona theorem for H^∞ there exist functions $g_i \in H^\infty$ such that $1=\sum_1^n g_i u_i$. Let $u=\prod_{i=1}^n u_i$. By Lemma 1 there exist functions $h_i \in H^\infty$, $v_i \in CB$, $|v_i| = 2\|g_i\|$ a.e. on T with $\bar{u} u_i g_i = h_i + v_i$ $(i=1,\dots,n)$. Hence

$$1=\sum_1^n u_i g_i = u \sum_1^n h_i + \sum_1^n u_i (u\bar{u}_i v_i).$$

Since $u, u\bar{u}_i v_i \in CB \cap H^\infty \subseteq A$, we obtain that $u \sum_1^n h_i \in A$. Because A has the weak F-property, $\sum_1^n h_i \in A$. Finally, we see that $1=\sum_1^n p_i u_i$, where

$$p_1 = u\bar{u}_1\left(\sum_1^n h_i + v_1\right), \qquad p_i = u\bar{u}_i v_i \quad (i=2,\dots,n)$$

are functions in A. This gives the contradiction

$$1 = m(1) = \sum_1^n p_i(m) u_i(m) = 0. \qquad \square$$

Copying the proof of Theorem 6.5 in [13, p. 50], we also obtain the following result.

THEOREM 5. *Let A be an admissible algebra with $CA_B \subseteq A \subseteq QA_B$. Then the ideals $I \subseteq A$ satisfying $Z(I) \cap \partial A = \emptyset$ are in a one-to-one correspondence with the ideals $J \subseteq H^\infty$ such that $Z(J) \cap M(B) = \emptyset$. In particular, every ideal I in A whose zero set is disjoint from the Shilov boundary of A is the trace of a unique ideal J in H^∞. Moreover $Z(I) = \Gamma(Z(J))$, where $\Gamma: M(H^\infty) \to M(A)$ is the restriction mapping.*

If we specialize to maximal ideals, we obtain the following corollary.

COROLLARY 6. *The restriction mapping $\Gamma: M(H^\infty) \to M(A)$ is a homeomorphism from $M(H^\infty)\setminus M(B)$ onto $M(A)\setminus \partial A$.*

Traces of C^*-Algebras in H^∞ and the Corona Theorem

In the second part of this paper we shall now use Theorem 4 to show that every subalgebra A of H^∞ which is the trace of a C^*-algebra between CB and QB has the corona property. This answers (at least for C^*-algebras of the above type) a question of Dawson [3, p. 95]. In view of Theorem 4, it suffices to show that every such algebra is admissible. This is done in our next proposition.

PROPOSITION 7. *Let $\mathfrak{C}$ be any C^*-algebra between CB and QB and let $A = \mathfrak{C} \cap H^\infty$. Then A is an admissible algebra which is logmodular on $\mathfrak{C}$.*

Proof. That A satisfies the weak F-property was already discussed in the introduction. Of course (A, H^∞) is also a Wiener pair. In fact, let $f \in A$ be invertible in H^∞; then f is an outer function. Because the restriction map $M(L^\infty) \to M(\mathcal{C})$ is onto, f does not vanish on $M(\mathcal{C})$; hence $f^{-1} \in \mathcal{C}$. Thus $f^{-1} \in \mathcal{C} \cap H^\infty = A$.

In order to prove that A has the D. J. Newman property, we show first that A is logmodular on $\mathcal{C}$. Let q be any function in $\mathcal{C} \subseteq QB$. Since $QB = QA_B + CB$ (see [5, p. 386]), there exist functions $f \in QA_B$ and $v \in CB$ with $q = f + v$. Hence $f = q - v \in \mathcal{C} \cap H^\infty = A$, from which we can conclude that

$$\mathcal{C} = A + CB. \tag{7}$$

Now let q be any real-valued function in $\mathcal{C}$; then $e^q \in \mathcal{C}$. By (7), there exist for every $\epsilon > 0$ functions $f \in A$ and $u \in CB$, u inner, such that $\|e^q - f\bar{u}\| < \epsilon$. The next step of the proof now works in exactly the same manner as in Chang and Marshall [2, p. 15]. We need only replace CB by $\mathcal{C}$. This leads to a function $F \in A$ which is invertible in A and has the property that $\log|F|$ approximates q. (Note that F can be taken to be the outer part of f.) Hence A is logmodular on $\mathcal{C}$.

Consequently we obtain the result that the restriction mapping

$$\Gamma : M(\mathcal{C}) \to M(A)$$

is a homeomorphism between $M(\mathcal{C})$ and ∂A [4, p. 38]. Since $\mathcal{C} = A + CB$, the closure S of the set $\{f\bar{u} : f \in A,\ u \in A,\ u \text{ inner}\}$ coincides with $\mathcal{C}$. On the other hand, it is not hard to prove that every maximal ideal $m \in K_A := \{m \in M(A) : |u(m)| = 1 \text{ for every inner function } u \in A\}$ has an extension to S (see [13, p. 39]). Now let $\Gamma : M(S) \to M(A)$ be the restriction mapping. Since $K_S = M(S) = M(\mathcal{C})$, we therefore obtain

$$K_A = \Gamma(K_S) = \Gamma(M(S)) = \Gamma(M(\mathcal{C})) = \partial A.$$

This shows that A has the D. J. Newman property. □

Combining Theorem 4 and Proposition 7, we obtain the following result.

THEOREM 8. *Let A be a subalgebra of H^∞ which is the trace of a C^*-algebra $\mathcal{C}$ between CB and QB; that is, $A = \mathcal{C} \cap H^\infty$. Then A has the corona property.*

REMARK. As a special case of Theorem 8 we obtain Sundberg and Wolff's result that the corona theorem holds in QA_B (see [15, p. 563]).

Application of the Corona Theorem to the Ideal Structure

In the above proposition we showed that $\partial A = K_A := \{m \in M(A) : |u(m)| = 1$ for every inner function $u \in A\}$. It is now a natural question to ask whether one can restrict to Blaschke products. The following theorem now gives a positive answer to this question, thus solving a problem posed in [13, p. 39].

THEOREM 9. *Let A be an admissible algebra with $CA_B \subseteq A \subseteq QA_B$ and let $m \in M(A)$. Then the following assertions are equivalent:*

(1) $m \in \partial A$;
(2) $|u(m)| = 1$ *for every inner function* $u \in A$;
(3) $|u(m)| > 0$ *for every inner function* $u \in A$;
(4) $|b(m)| = 1$ *for every Blaschke product* $b \in A$;
(5) $|b(m)| > 0$ *for every Blaschke product* $b \in A$.

Proof. (1) $\Rightarrow$ (2) is part of the definition of an admissible algebra.

(2) $\Rightarrow$ (3): trivial.

(3) $\Rightarrow$ (4): Assume there is a Blaschke product $b \in A$ with $|b(m)| < 1$. Then the function $u = (b - b(m))/(1 - \overline{b(m)}b)$ is inner and belongs to A. Here we have used the fact that (A, H^∞) is a Wiener pair. But $u(m) = 0$, which contradicts (3).

(4) $\Rightarrow$ (5): trivial.

(5) $\Rightarrow$ (1). This is the only nontrivial part. Assume that $m \notin \partial A$. Then there exists by Theorem 1 an inner function $u \in A$ with $u(m) = 0$. Using a result of Guillory and Sarason [9, p. 180], we obtain a Blaschke product $b \in H^\infty$ such that $q := u\bar{b} \in QC$. Hence $\bar{b} = q\bar{u} \in QC \cdot B \subseteq B$. Therefore $b \in CA_B \subseteq A$. Since by Theorem 4 the map $\Gamma : M(H^\infty) \to M(A)$ is onto, there exists $x \in M(H^\infty)$ with $x|_A = m$. Note that by (5) $m \notin \mathbf{D}$; hence $x \in M(H^\infty + C)$. Thus we have the following relations:

$$0 = u(m) = x(u) = x(qb) = x(q)x(b).$$

Since $|x(q)| = 1$, we see that $b(m) = x(b) = 0$, which contradicts hypothesis (5). $\square$

REMARK. Using Proposition 7 we see that Theorem 9 applies in particular to every algebra A of the form $A = \mathcal{C} \cap H^\infty$, where $\mathcal{C}$ is a C^*-algebra between CB and QB.

Let A be an admissible algebra with $CA_B \subseteq A \subseteq QA_B$. Recall that a Blaschke product $b \in H^\infty$ is said to be interpolating if $(1 - |z_n|^2)|b'(z_n)| \geq \delta > 0$ for every n, where z_n are the zeros of b in $\mathbf{D}$. Put

$$G_A = \{m \in M(A) : m \text{ contains an interpolating Blaschke product } b \in A\}$$

and let I be an ideal in A whose hull is contained in G_A. Since by Theorem 9 G_A does not intersect the Shilov boundary ∂A of A, it follows from Theorem 2 that I contains an inner function u. Can one say more? Does I contain even a finite product of interpolating Blaschke products? (Note that one cannot expect, of course, that I contains an interpolating Blaschke product b, as the example $I = (b^2)$ shows.) It is known that in the algebra H^∞ the answer is yes. This is a recent result of Tolokonnikov [16]. Using his result and the results of our previous sections, we shall now give a positive answer to this question.

THEOREM 10. *Let A be an admissible algebra with $CA_B \subseteq A \subseteq QA_B$ and let I be an ideal in A whose hull is contained in G_A. Then I contains a finite product of interpolating Blaschke products.*

REMARK. We do not know if I is generated by such Blaschke products.

Proof. By the previous discussion, we have $Z(I) \cap \partial A = \emptyset$. Therefore there exists, according to Theorem 5, a unique ideal J in H^∞ such that $J \cap A = I$. Moreover, we have $Z(J) \cap M(B) = \emptyset$. Since $Z(I) \subset G_A$, every maximal ideal $m \in Z(J)$ contains an interpolating Blaschke product; that is, $Z(J) \subseteq G_{H^\infty}$. In particular, $Z(J)$ does not contain any point m in $M(H^\infty)$ whose Gleason part is trivial (see [5, p. 413] and [10]). Theorem 2 of Tolokonnikov [16] now implies that J contains a function b of the form $b = \prod_{i=1}^N b_i$, where the b_i are interpolating Blaschke products in H^∞. The problem is that, in general, b fails to be in A. The clue of the next step is therefore to factorize each interpolating Blaschke product b_i in a product $b_i = c_i d_i$ of two Blaschke products such that

$$Z(c_i) \cap M(B) = \emptyset \quad \text{and} \quad Z(d_i) \cap Z(J) = \emptyset. \tag{8}$$

To this end, let U_1, U_2 be open sets in $M(H^\infty)$ containing $M(B)$ (resp., $Z(J)$) such that the closures of U_1 and U_2 are disjoint. This choice is clearly possible because $M(H^\infty)$ is a normal topological space. Let $V_1 = M(H^\infty) \setminus \operatorname{clos} U_2$. Let $\{z_n^{(i)}\}$ be the zero set in $\mathbf{D}$ of the b_i. Construct the (interpolating) Blaschke product d_i with zero set $\{z_n^{(i)}\} \cap V_1$. Then $b_i = c_i d_i$ for some interpolating Blaschke product c_i which satisfy (8). Here we have used the fact that every point $m \in M(H^\infty)$, with $b(m) = 0$ and b an interpolating Blaschke product, lies in the $M(H^\infty)$ closure of the zero set of b in $\mathbf{D}$ (see [5, p. 379]).

Since $Z(d_1 \cdots d_N) \cap Z(J) = \emptyset$, it is easy to see that $c := c_1 \cdots c_N$ belongs to the ideal J. Moreover $Z(c) \cap M(B) = \emptyset$. Hence the function c is invertible in the Douglas algebra B, and thus $c \in CB \subseteq A$. Therefore $c \in J \cap A = I$, which concludes the proof. □

To conclude, we state several open questions.

1. Let $\mathcal{C}$ be a C^*-algebra between C and L^∞. Does $A = \mathcal{C} \cap H^\infty$ have the corona property? (See also [3, p. 25].)
2. Let A be an admissible algebra containing the polynomials. Is the unit disk dense in the maximal ideal space of A?
3. *Generalized corona theorem:* Let A be an algebra of the above type and let $f, f_1, \dots, f_N$ be functions in A satisfying

$$|f| \leq \sum_1^N |f_i|$$

in $\mathbf{D}$. Do there exist functions $g_1, \dots, g_N \in A$ such that

$$(*) \qquad f^n = \sum_1^N g_i f_i$$

for some $n \in N$? The problem is open even when A is the disk algebra $A(\mathbf{D})$. Wolff showed that in the case $A = H^\infty$ one may take $n = 3$ (see [5, p. 329]). For $A = QA_B$ one may take $n = 5$ (see [7]).

REMARK. If A is an admissible algebra between CA_B and QA_B, we can show that (*) holds for $n = 3$ if $|f| \le \sum_1^N |f_i|$ in $\mathbf{D}$ and $\sum_1^N |f_i| \ge \delta > 0$ almost everywhere on T.

REMARK. After this paper was written we discovered the paper [17] of Volberg and Tolokonnikov, where they claim (without giving any reference) that the second author has also obtained a proof of the corona theorem for $\mathcal{C} \cap H^\infty$, where $\mathcal{C}$ is a C^*-algebra between CB and QB. However, we were not able to locate this proof anywhere in the literature.

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