Positive Entropy Homeomorphisms on the Pseudoarc

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Answering another of Marcy Barge's questions, we show that there is a pseudoarc homeomorphism with positive topological entropy. Since iterating a homeomorphism of positive entropy yields one of arbitrarily large entropy, it follows that the pseudoarc admits homeomorphisms of arbitrarily large entropy. Whether or not given a positive number r there is a pseudoarc homeomorphism of that entropy r is not known, and is not answered here, but is another of Marcy Barge's questions. As is often the case, in order to obtain this result we developed a tool which itself yields more information about the pseudoarc.

A *continuum* is a compact connected metric space. A continuum X is *homogeneous* if for $x, y \in X$ there is a space homeomorphism h such that $h(x) = y$. A continuum is *chainable* or *arclike* or *snakelike* if for each $\epsilon > 0$ there is a chain $C = \{C_0, ..., C_n\}$ of open sets of diameter less than ϵ that covers X. C is a *chain* if $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. A pseudoarc, which is a nonseparating plane continuum, can be characterized as a homogeneous chainable continuum. Pseudoarcs, although arclike, contain no continuous nontrivial images of arcs, and in fact every nondegenerate subcontinuum of a pseudoarc is itself a pseudoarc. Another extraordinary fact about this continuum is that most continua [in the sense that they form a dense G_δ-set in the space of all continua (Vietoris topology)] in the plane are pseudoarcs.

A compact metric space is a *compactum*. A compactum X is *indecomposable* if every proper subcontinuum of X is nowhere dense in X. It is *hereditarily indecomposable* if every subcontinuum of X is itself indecomposable. The pseudoarc is a hereditarily indecomposable continuum.

For us, P will denote a pseudoarc, $I = [0, 1]$, $\mathbf{Z}$ is the integers, and $\mathbf{N}$ is the positive integers. If X is a compact metric space, $H(X)$ denotes its group of self homeomorphisms.

A chain $C = \{C_0, \dots, C_n\}$ is *taut* whenever $C_i \cap C_j \neq \emptyset$ if and only if $\bar{C}_i \cap \bar{C}_j \neq \emptyset$. A chain covers a set A *essentially* if there is a continuum Q contained in A such that each link contains a point of Q not in the closure of any other link. An open set o in a space X is *regular* if $\operatorname{Int} \bar{o} = o$. In the

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discussion that follows, we will assume that our open chain covers are regular, taut, and essential.

If B is a collection of sets, then B^* denotes the union of the sets in B. If C is a collection of sets, then C is an *amalgamation* of B if $B^* = C^*$ and each set in C is the union of some sets in B. If the closure of each set in B is a subset of a set in C, then B is said to *closure refine* C. The chain C *properly covers* the chain D if D closure refines C and if, for every c in C, there is some d in D such that $\bar{d} \subseteq c$.

If $C = \{C_0, \ldots, C_n\}$ is an open chain in a space X (which doesn't necessarily cover X) then, for $c \in C$,

$$i(c, C) = \{y \in c \mid y \notin \bar{c}' \text{ for } c' \in C - \{c\}\}.$$

If $m, n \in N$ with $m < n$, let $[m, n] = \{m, m+1, \ldots, n\}$. A function $f: [m', n'] \twoheadrightarrow [m, n]$ is called a (*light*) *pattern* provided $|f(i+1) - f(i)| \leq 1$ (respectively, $|f(i+1) - f(i)| = 1$) for $i = m', \ldots, n'-1$. (The symbol $\twoheadrightarrow$ indicates that f is an onto function.) If $V = \{V_{m'}, V_{m'+1}, \ldots, V_{n'}\}$ and $U = \{U_m, U_{m+1}, \ldots, U_n\}$ are chain covers of the compactum X, and $f: [m', n'] \twoheadrightarrow [m, n]$ is a pattern, we will say that *V follows the pattern f in U* provided $V_i \subseteq U_{f(i)}$ for each $i \in [m', n']$. We call f a *pattern on U*.

The chain $C = C[0, a]$ is *crooked* in the chain $D = D[0, b]$ if C refines D and if—whenever $k, m \in [0, a]$ $(k < m)$, $c(k) \subseteq d(p)$, $c(m) \subseteq d(q)$, and $|p - q| \geq 3$—there are $s, t \in [0, a]$ such that $k < s < t < m$, $c(s)$ is contained in a link adjacent to $d(p)$, and $c(t)$ is contained in a link adjacent to $d(q)$.

We will use the following well-known fact.

FACT. Suppose that $A_1, A_2, \ldots$ is a sequence of open chains in the plane such that

(1) $\lim_i \operatorname{mesh} A_i = 0$,
(2) A_{i+1} closure refines A_i, and
(3) A_{i+1} is crooked in A_i.

Then $\bigcap_i A_i = P$ is a pseudoarc.

We will make the following notational conventions: chains will be denoted with upper-case letters (and possibly additional symbols), and links of chains with the associated lower-case letters, associated symbols, and link numbers. So, for example,

$$C_1 \equiv \{c(1,0), \ldots, c(1,m)\} \equiv C_1[0,m];$$

$$\tilde{D}_2 \equiv \{\tilde{d}(2,k), \ldots, \tilde{d}(2,l)\} \equiv \tilde{D}_2[k,l];$$

$$F \equiv \{f(1), \ldots, f(n)\} \equiv \{f_1, \ldots, f_n\} \equiv F[1,n].$$

If A and B are collections of sets, then $A \cap B = \{a \cap b \mid a \in A, b \in B\}$.

Suppose that d is a metric on the pseudoarc P that is compatible with its topology. For $x \in P$ and $\epsilon > 0$, $D_\epsilon(x) = \{y \in P \mid d(x, y) < \epsilon\}$. Also, if H and K

are closed subsets of the pseudoarc, $d(H,K)=\min\{d(x,y) \mid x \in H, y \in K\}$. If $H=\{x\}$ then we will write $d(x,K)$ for $d(\{x\},K)$.

We need the following background theorem, which is due to Oversteegen and Tymchatyn [5].

THEOREM OT. *Let X be a hereditarily indecomposable compactum and let $U=\{U_1,\ldots,U_n\}$ be an open taut chain cover of X such that there exists a continuum $Z \subseteq X$ with $Z \cap i(U_1,U) \neq \emptyset \neq Z \cap i(U_n,U)$. Let $f\colon [1,m] \to [1,n]$ be a pattern on U. Then there exists an open taut chain cover $V=\{V_1,\ldots,V_m\}$ of X such that V follows the pattern f in U.*

There are several different equivalent definitions of topological entropy. The one we give is due to Bowen and can be found in [6, p. 168]. Suppose (X,d) is a compactum and $f\colon X \to X$ is a continuous map. If $\epsilon>0$ and $n \in \mathbf{N}$ then the set $K \subseteq X$ is *(n,ϵ)-separated* (under f) provided that, for each x,y in K $(x \neq y)$, there is a $k \in [0,n-1]$ such that $d(f^k(x),f^k(y)) \geq \epsilon$. Let

$$S(n,\epsilon,f)=\max\{\text{cardinality}(K) \mid K \text{ is } (n,\epsilon)\text{-separated}\}.$$

(Intuitively, $S(n,\epsilon,f)$ represents the greatest number of orbit segments $\{x,f(x),\ldots,f^{n-1}(x)\}$ of length n that can be distinguished from each other if we can only distinguish between points that are ϵ or more apart.) Let $h(f,\epsilon)=\lim_{n\to\infty}\sup \ln S(n,\epsilon,f)/n$ and let $h(f)=\lim_{\epsilon\to 0} h(f,\epsilon)$. The number $h(f)$ is called the *topological entropy of f*. It is independent of the (compatible) metric involved and it is an invariant of topological conjugacy. (A map $f\colon X\to X$ and a map $g\colon Y\to Y$ are *topologically conjugate* if there exists a homeomorphism $\theta\colon X\to Y$ such that $g\theta=\theta f$. If the maps f and g are topologically conjugate then they share the same dynamics.)

We will not directly use the definition of topological entropy, but rather make use of the following facts (more detail can be found in [6]):

(1) If C is a Cantor set and r is a nonnegative number, then there is $\alpha \in H(C)$ such that $h(\alpha)=r$.

(2) If X is a compactum, $f \in H(X)$, and A is a closed subset of X such that $f(A)=A$, then $h(f|A) \leq h(f)$.

Define $D=(-1,1)\times(-1,1)$.

THEOREM 1. *If C is a Cantor set and h is a homeomorphism from C onto itself, then C can be embedded in a pseudoarc P in such a way that the homeomorphism on the image of C in P induced by h can be extended to a homeomorphism f on P.*

Proof. Let us (without loss of generality) consider C to be a subset of $\mathbf{R}^2$. Our strategy is to build a pseudoarc P in $\mathbf{R}^2$ so that $C \subseteq P$, building at the same time the homeomorphism f on P which will extend h.

There exists a finite collection $U_1=\{u(1,0),\ldots,u(1,a_1)\}$ of open sets in $\mathbf{R}^2$ such that:

(1) $\bigcup_{i=0}^{a_1} u(1,i) \supseteq C$;
(2) mesh $U_1 \leq \frac{1}{2}$; and
(3) $\overline{u(1,i)} \cap \overline{u(1,j)} = \emptyset$ if $i \neq j$.

Let $\hat{u}(1,i) = u(1,i) \cap C$ for $i \in [0, a_1]$. There exist an open taut chain $D = \{d(1,0), \dots, d(1,n_1)\}$ and a finite subsequence $\{0 = \alpha_1(0), \alpha_1(1), \dots, \alpha_1(a_1) = n_1\}$ of $\{0, 1, \dots, n_1\}$ such that:

(4) $\underline{i(d(1, \alpha_1(i)), D_1)} \supseteq \hat{u}(1,i)$ for $i \in [0, a_1]$;
(5) $\overline{d(1,j)} \cap \hat{u}(1,i) = \emptyset$ for $j \neq \alpha_1(i)$;
(6) mesh $D_1 < \frac{1}{2}$ and $D_1^* \cong D$; and
(7) for $i \in [0, a_1)$ and $j \in [0, a_1]$, some link d of $D_1[\alpha_1(i), \alpha_1(i+1)]$ is contained in $u(1,j)$.

Now choose an open taut chain $E_1 = E_1[0, n_1]$ in the plane with the following properties:

(8) $\bar{E}_i^* \subseteq D_1^*$ and $E_1^* \cong D$;
(9) for $i \in [0, a_1)$ and $j \in [0, a_1]$, some link e of $E_1[\alpha_1(i), \alpha_1(i+1)]$ intersects $u(1,j)$;
(10) $\underline{i(e(1, \alpha_1(i)), E_1)} \supseteq h(\hat{u}(1,i))$ for $i \in [0, a_1]$; and
(11) $\overline{e(1,j)} \cap h(\hat{u}(1,i)) = \emptyset$ for $j \neq \alpha_1(i)$.

(Note that no mesh control is possible here.)

There exists a finite collection $U_2 = \{u(2,0), \dots, u(2,a_2)\}$ of open sets in $\mathbf{R}^2$ such that:

(12) $\bigcup_{i=0}^{a_2} u(2,i) \supseteq C$;
(13) mesh $U_2 < \frac{1}{4}$;
(14) for $i \in [0, a_2]$, $\hat{u}(2,i) = u(2,i) \cap C \subseteq \hat{u}(1,j) \cap h(\hat{u}(1,k))$ for some j, k;
(15) $\overline{u(2,i)} \cap \overline{u(2,j)} = \emptyset$ if $i \neq j$; and
(16) if $u \in U_2$ then there is e in E_1 such that $\bar{u} \subseteq i(e, E_1)$.

Choose an open taut chain $E_2 = E_2[0, n_2]$ and a finite subsequence $\{0 = \alpha_2(0), \dots, \alpha_2(a_2) = n_2\}$ of $[0, n_2]$ as follows:

(17) E_2 is crooked in E_1, E_2 closure refines E_1, and $\bar{E}_2^* \subseteq E_1^*$;
(18) E_2 has mesh less than $\frac{1}{4}$ and $E_2^* \cong D$;
(19) for $i \in [0, a_2)$ and $j \in [0, a_2]$, some link e of $E_2[\alpha_2(i), \alpha_2(i+1)]$ is contained in $u(2,j)$, some link e' of $E_2[\alpha_2(i), \alpha_2(i+1)]$ is contained in $i(e(1,0), E_1)$, and some link $\tilde{e}$ of $E_2[\alpha_2(i), \alpha_2(i+1)]$ is contained in $i(e(1,n_1), E_1)$;
(20) $\underline{i(e(2, \alpha_2(i)), E_2)} \supseteq \hat{u}(2,i)$ for $i \in [0, a_2]$; and
(21) $\overline{e(2,j)} \cap \hat{u}(2,i) = \emptyset$ for $j \neq \alpha_2(i)$.

Now E_2 follows some pattern ξ_1 in E_1. Choose the open taut chain D_2 with the following properties:

(22) D_2 follows ξ_1 in D_1, and D_2 closure refines D_1;
(23) $\bar{D}_2^* \subseteq E_2^*$ and $D_2^* \cong D$;

(24) for $i \in [0, a_2)$ and $j \in [0, a_2]$, some link d of $D_2[\alpha_2(i), \alpha_2(i+1)]$ intersects $u(2, j)$, some link d' of $D_2[\alpha_2(i), \alpha_2(i+1)]$ is contained in $i(d(1,0), D_1)$, and some link $\tilde{d}$ of $D_2[\alpha_2(i), \alpha_2(i+1)]$ is contained in $i(d(1, n_1), D_1)$;

(25) $i(d(2, \alpha_2(i)), D_2) \supseteq h^{-1}(\hat{u}(2, i))$ for $i \in [0, a_2]$; and

(26) $\overline{d(2, j)} \cap h^{-1}(\hat{u}(2, i)) = \emptyset$ for $j \neq \alpha_2(i)$.

Continue this process, obtaining sequences of chains $D_1, D_2, \ldots$ and E_1, $E_2, \ldots$ and a sequence $U_1, U_2, \ldots$ of collections of pairwise disjoint open sets with the following properties:

(27) $U_i = \{u(i, j) \mid j \in [0, a_i]\}$ and, for $j \in [0, a_i]$, $\hat{u}(i, j) = u(i, j) \cap C$;

(28) D_{i+1} follows ξ_i in D_i and E_{i+1} follows ξ_i in E_i;

(29) $D_{i+1}(E_{i+1})$ is crooked in and closure refines $D_i(E_i)$;

(30) $D_1^* \supseteq \bar{E}_1^* \supseteq E_1^* \supseteq \bar{E}_2^* \supseteq E_2^* \supseteq \bar{D}_2^* \cdots$;

(31) $\lim_i \operatorname{mesh} E_i = \lim_i \operatorname{mesh} D_i = 0$;

(32) $D_i = D_i[0, n_i]$ and $E_i = E_i[0, n_i]$; and

(33) there is a subsequence $\{0 = \alpha_i(0), \ldots, \alpha_i(a_i) = n_i\}$ of $[0, n_i]$ such that

(a) for $j \in [0, a_i)$ and $k \in [0, a_i]$, some link $d(e)$ of $D_i[\alpha_i(j), \alpha_i(j+1)](E_i[\alpha_i(j), \alpha_i(j+1)])$ intersects $u(i, k)$;

(b) $i(d(i, \alpha_i(j)), D_i) \supseteq \hat{u}(i, j)$ for $j \in [0, a_i]$, i odd;

(c) $i(d(i, \alpha_i(j)), D_i) \supseteq h^{-1}(\hat{u}(i, j))$ for $j \in [0, a_i]$, i even;

(d) $i(e(i, \alpha_i(j)), E_i) \supseteq h(\hat{u}(i, j))$ for $j \in [0, a_i]$, i odd;

(e) $i(e(i, \alpha_i(j)), E_i) \supseteq \hat{u}(i, j)$ for $j \in [0, a_i]$, i even;

(f) $\overline{d(i, j)} \cap \hat{u}(i, k) = \emptyset$ and $\overline{e(i, j)} \cap h(\hat{u}(i, k)) = \emptyset$ for i odd and $j \neq \alpha_i(k)$;

(g) $\overline{d(i, j)} \cap h^{-1}(\hat{u}(i, k)) = \emptyset$ and $\overline{e(i, j)} \cap \hat{u}(i, k) = \emptyset$ for i even and $j \neq \alpha_i(k)$; and

(h) for $j \in [0, a_i)$ and $i > 1$, some link $d'(e')$ of $D_i[\alpha_i(j), \alpha_i(j+1)]$ $(E_i[\alpha_i(j), \alpha_i(j+1)])$ is contained in $i(d(i-1, 0), D_{i-1})$ $(i(e(i-1, 0), E_{i-1}))$, and some link $\tilde{d}(\tilde{e})$ of $D_i[\alpha_i(j), \alpha_i(j+1)]$ $(E_i[\alpha_i(j), \alpha_i(j+1)])$ is contained in $i(d(i-1, n_{i-1}), D_{i-1})$ $(i(e(i-1, n_{i-1}), E_{i-1})$.

Then $\bigcap D_i^* = \bigcap E_i^* = P$ (a pseudoarc), and a homeomorphism f has been induced on P where f is defined as follows: If $x \in P$ then there is an infinite sequence $j(x, 1), j(x, 2), \ldots$ of integers such that, for each i,

(34) $x \in d(i, j(x, i))$ and

(35) $\xi_i(j(x, i+1)) = j(x, i)$.

Let $f(x) = \bigcap_{i=1}^{\infty} e(i, j(x, i))$. (For more details on why f is a homeomorphism, see [3].) Further, if $x \in C$ then $j(x, i)$ is unique and there is $k(x, i) \in [0, a_i]$ such that $j(x, i) = \alpha_i(k(x, i))$ and $h(x) \in e(i, \alpha_i(k(x, i)))$; thus $f(x) = h(x)$. □

REMARK 2. In Theorem 1, C does not separate P; in fact, no composant of P contains more than one point of C.

Proof. Suppose that the composant G contains no more than one point c of C. If x and y are in G and if D is the subcontinuum of G irreducible between x and y, then $D-\{c\}$ is connected. Thus, we will be done if we can prove that no composant of P contains more than one point of C. Suppose that F is a composant of P that contains two points, p and q, of C. There is a proper subcontinuum K of P which contains p and q and is irreducible between them. Also, there is i odd, $i>1$, such that p and q are not contained in the same member of U_i and K does not intersect one of the end links of D_{i-1}. Let us say that $p \in u(i,j)$ and $q \in u(i,k)$. Then

$$p \in i(d(i,\alpha_i(j)),D_i) \quad \text{and} \quad q \in i(d(i,\alpha_i(k)),D_i),$$

and K must intersect every link of some $D_i[\alpha_i(l),\alpha_i(l+1)]$; thus K intersects every link of D_{i-1}. This is a contradiction, and so it must be the case that no composant of P contains more than one point of C. □

COROLLARY 3. *If P is a pseudoarc then there is a homeomorphism f on P such that $h(f)>0$.*

Proof. There is a homeomorphism α on C such that $h(\alpha)=r$. Applying Theorem 1, we can embed C in P in such a way that the homeomorphism induced by α on the embedded copy of C can be extended to a homeomorphism f on P. Let us (without loss of generality) think of C as a subset of P and f as an actual extension of α. Then $h(f)\geq h(\alpha)$ [6, pp. 167, 178]. □

COROLLARY 4. *Suppose that $n_1, n_2, \ldots$ is a sequence of positive integers. Then there is a homeomorphism f on a pseudoarc P such that there is a sequence $p_1, p_2 \ldots$ of points of P with:*

(1) *$|O(p_i)|=n_i$ for each i, $O(p_i)=\{f^n(p_i)\mid n\in \mathbf{Z}\}$;*
(2) *$O(p_i)\cap O(p_j)=\emptyset$ for $i\neq j$; and*
(3) *no two points of $O=\{O(p_i)\mid i\geq 1\}^*$ are on the same composant of P.*

Proof. There is a Cantor set C, a homeomorphism α on C, and a sequence $p_1, p_2, \ldots$ of points of C such that:

(1) $|O(p_i)|=n_i$ for each i, $O(p_i)=\{a^n(p_i)\mid n\in \mathbf{Z}\}$; and
(2) $O(p_i)\cap O(p_j)=\emptyset$ for $i\neq j$.

We can think of C as being a subset of a pseudoarc to which the homeomorphism α on C can be extended to the homeomorphism f. Furthermore, this can be done so that no two points of C are on the same composant of P. □

Corollary 4 gives us another way of seeing that a pseudoarc homeomorphism can have many different periodic points without actually being a periodic homeomorphism. (The author found such homeomorphisms in [2], but these all had the property that each composant of the pseudoarc was mapped to itself.) But we can go still further in this direction and construct a homeomorphism on the pseudoarc with the property that, for each positive integer n, $|\{p\in P\,||O(p)|=n\}|=c$, as Corollary 5 demonstrates.

COROLLARY 5. *Suppose that $n_1, n_2, \ldots$ is a sequence of positive integers. Then there is a homeomorphism f on a pseudoarc P such that there is a sequence $C_1, C_2, \ldots$ of mutually disjoint Cantor sets in P such that:*

(1) *$f(C_i)=C_i$ and $f|C_i$ is periodic of period n_i for each i; and*
(2) *no two points of $\bigcup_{i=1}^{\infty} C_i$ are on the same composant of P.*

Proof. There is a Cantor set C in $\mathbf{R}^2$ such that $C=\{p\}\cup C_1\cup C_2\cup\cdots$, where

(1) $C_i\cap C_j=\emptyset$ unless $i=j$;
(2) for each i, C_i is itself a Cantor set; and
(3) $\lim_{i\to\infty} d(p,C_i)=0$ but $p\notin C_i$ for any i.

For each i, there is a homeomorphism $\alpha_i\in H(C_i)$ such that α_i has period n_i. Define $\alpha\in H(C)$ by

$$\alpha(x)=\begin{cases} x & \text{if } x=p,\\ \alpha_i(x) & \text{if } x\in C_i.\end{cases}$$

Now use Theorem 1 to construct both a pseudoarc P containing C and a homeomorphism f on P extending α. ❑

Theorem 1 and the remark that follows it lead one rather naturally to the following question: If a Cantor set C is embedded in a pseudoarc P such that no composant of P contains more than one point of C, and if $\alpha\in H(C)$, can α be extended to all of P? The answer is yes, as the next lemma, theorem, and corollary show us.

LEMMA 6. *Suppose that C_1 is a Cantor set in the pseudoarc P_1 such that no composant of P_1 contains more than one point of C_1, and such that C_2 is a Cantor set in the pseudoarc P_2 with no composant of P_2 containing more than one point of C_2. Suppose further that $D_1[0,n]$ (resp., $E_1[0,n]$) is an open taut chain cover of P_1 (resp., P_2), and that $\{\alpha(0)=0,\alpha(1),\ldots,\alpha(a)=n\}$ is an increasing subsequence of $[0,n]$ such that*

$$C_1\subseteq\bigcup_{i=1}^{a} i(d(1,\alpha(i)),D_1),\qquad C_2\subseteq\bigcup_{i=0}^{a} i(e(1,\alpha(i)),E_1),$$

and, for $i\in[0,a]$,

$$C_1\cap d(1,\alpha(i))\neq\emptyset,\qquad C_2\cap e(1,\alpha(i))\neq\emptyset.$$

Then, if $E_2[0,m]$ is an open taut chain cover of P_2 such that E_2 closure refines E_1, E_2 follows the pattern ξ in E_1, $\{\beta(0)=0,\beta(1),\ldots,\beta(b)=m\}$ is an increasing subsequence of $[0,m]$ such that $C_2\subseteq\bigcup_{i=1}^{b} i(e(2,\beta(i)),E_2)$, and $E_2[\beta(i),\beta(i+1)]$ is properly covered by E_1 for $i\in[0,b-1]$, then there is an open taut chain cover $D_2[0,m]$ of P_1 such that D_2 closure refines D_1, D_2

follows ξ *in* D_1, $C_1 \subseteq \bigcup_{i=0}^{b} i(d(2,\beta(i)),D_2)$, *and* $D_2[\beta(i),\beta(i+1)]$ *is properly covered by* D_1 *for* $i \in [0,b-1]$.

Proof. For $i \in [0,a]$, let $n_i = |\{\beta(j) \mid j \in [0,b] \text{ and } \xi(\beta(j)) = \alpha(i)\}|$. (Without loss of generality, let us assume that $e(2,\beta(j)) \cap C_2 \neq \emptyset$ for $j \in [0,b]$.) Then divide $d(1,\alpha(i)) \cap C_1$ into exactly n_i nonempty mutually disjoint closed sets, and list them: $d(1,\alpha(i)) \cap C_1 = \bigcup_{j \in B_i} K_{\alpha(i)j}$, where $B_i = \{\beta(k) \mid k \in [0,b]$ and $\xi(\beta(k)) = \alpha(i)\}$. Then $C_1 = \bigcup_{i,j} K_{ij}$, and we can list $\{K_{ij} \mid i \in [0,a], j \in B_i\}$ by the ordering on the j's because each j occurs once and only once. We list them: $\{K_{\xi\beta(0),0}, K_{\xi\beta(1),1}, \ldots, K_{\xi\beta(b),b}\}$.

If v_1 is an open set such that $K_{\xi\beta(1),1} \subseteq v_1 \subseteq \bar{v}_1 \subseteq i(d(1,\xi\beta(1)),D_1)$, $H_{0,1} = \{L \mid L$ is a component of $P_1 - v_1$ and $L \cap K^*_{\xi\beta(0),0} \neq \emptyset\}^*$, and $H_1 = \{L \mid L$ is a component of $P_1 - v_1$ and $L \cap (\bigcup_{j \neq 0,1} K_{\xi\beta(j),j}) \neq \emptyset\}^*$, then $H_{0,1}$ and H_1 are mutually exclusive closed sets, and there are open sets G_0 and G_2 such that:

(1) $H_{0,1} \subseteq G_0$ and $H_1 \subseteq G_2$;
(2) $G_0 \cup v_1 \cup G_2 = P_1$; and
(3) $\bar{G}_0 \cap \bar{G}_2 = \emptyset$, $\bar{G}_0 \cap K_{\xi\beta(1),1} = \emptyset$, and $\bar{G}_2 \cap K_{\xi\beta(1),1} = \emptyset$.

Further, we may choose v_1 so that each component contained in $H_{0,1}$ not only intersects each link in D_1 but also, for each link d in D_1, contains a point in $i(d,D_1)$; likewise we may choose v_1 so that, for each $i \in [2,b]$ and $d \in D_1$, each component contained in H_1 contains a point of $i(d,D_1)$. Let $v_1 = G_1$. Then $G = \{G_0, G_1, G_2\}$ is an open taut chain cover of P_1, with $K_{\xi\beta(0),0} \subseteq i(G_0,G)$, $K_{\xi\beta(1),1} \subseteq i(G_1,G)$, and $\bigcup_{i>1} K_{\xi\beta(i),i} \subseteq i(G_2,G)$.

Pick v_2 an open set so that

$$K_{\xi\beta(2),2} \subseteq v_2 \subseteq \bar{v}_2 \subseteq i(d(1,\xi\beta(2)),D_1) \cap i(G_2,G).$$

Let $H_{1,1} = \{L \mid L$ is a component of $P_1 - v_2$ and $L \cap (K_{\xi\beta(0),0} \cup K_{\xi\beta(1),1}) \neq \emptyset\}^*$ and $H_2 = \{L \mid L$ is a component of $P_1 - v_2$ and $L \cap (\bigcup_{j \neq 0,1,2} K_{\xi\beta(j),j}) \neq \emptyset\}^*$. Then $H_{0,1} \cup H_{1,1}$ and H_2 are mutually exclusive closed sets, and there are open sets G_0^2 and G_2^2 such that:

(4) $H_{0,1} \cup H_{1,1} \subseteq G_0^2$ and $H_2 \subseteq G_2^2$;
(5) $G_0^2 \cup v_2 \cup G_2^2 = P_1$; and
(6) $\bar{G}_0^2 \cap \bar{G}_2^2 = \emptyset = \overline{G_0^2} \cap K_{\xi\beta(2),2} = \overline{G_2^2} \cap K_{\xi\beta(2),2}$.

Again, we may choose v_2 so that:

(7) $\bar{v}_2 \subseteq i(G_2,G)$;
(8) for each d in D_1, each component in $H_{1,1}$ contains a point in $i(d,D_1)$; and
(9) for each d in D_1, each component in H_2 contains a point in $i(d,D_1)$.

Then $G^2 = \{G_0 \cap G_0^2, G_1 \cap G_0^2, G_2 \cap G_0^2, v_2, G_2^2\}$ is an open taut chain cover of P_1.

Let $G_0 \cap G_0^2 = g^2(0)$, $G_1 \cap G_0^2 = g^2(1)$, $G_2 \cap G_0^2 = g^2(2)$, $v_2 = g^2(3)$, and $G_2^2 = g^2(4)$. Continue this process until finally we obtain an open taut chain cover $G^b = G^b[0, 2b]$ of P_1, and the increasing subsequence $\{\beta'(0) = 0, \beta'(1), \ldots, \beta'(b)\}$, such that:

(10) for $i \in [0, b]$, $K_{\xi\beta(i),i} \subseteq i(g^b(\beta'(i)), G^b)$; and
(11) for $i \in [0, b-1]$ and $z \in K_{\xi\beta(i),i}$, there is a continuum Q_z contained in $G^b[\beta'(i), \beta'(i+1)]^*$ which is essentially covered by D_1 and which contains z.

We are almost done, for now, using Theorem OT (plus a little) we can conclude that there is an open taut chain cover $D_2[0, m]$ of P_1 such that:

(12) for $i \in [0, b]$, $K_{\xi\beta(i),i} \subseteq i(d(2, \beta(i)), D_2)$;
(13) for $i \in [0, b-1]$, $D_2[\beta(i), \beta(i+1)]$ is properly covered by D_1; and
(14) D_2 closure refines D_1 and follows ξ in D_1.

[By "plus a little" we mean the following: We wish to construct $D_2[0, \beta(1)]$ from $G^b[0,1]$ (since $1 = \beta'(1)$) so that $D_2[0, \beta(1)]$ follows the appropriate restriction of ξ in E_1. There are open sets w_1 and w_2 in P_2 such that $C_2 \cap e(2,0) \subseteq w_1 \subseteq \bar{w}_1 \subseteq i(e(2,0), E_2)$ and $C_2 \cap e(2, \beta(1)) \subseteq w_2 \subseteq \bar{w}_2 \subseteq i(e(2, \beta(1)), E_2)$. Let w_3 denote an open set in P_2 such that

$$\overline{e(2, \beta(1)+1)} \cap e(2, \beta(1)) \subseteq w_3 \subseteq \bar{w}_3 \subseteq \overline{e(2, \beta(1))} - \overline{e(2, \beta(1)-1)}.$$

Let w_4 denote an open set P_2 such that

$$e(2,0) - \bar{w}_1 \subseteq w_4 \quad \text{and} \quad \bar{w}_4 \cap (C_2 \cap e(2,0)) = \emptyset,$$

and let w_5 denote an open set in P_2 such that $e(2, \beta(1)) - (\overline{w_2 \cup w_3}) \subseteq w_5$ and $\bar{w}_5 \cap ((C_2 \cap e(2, \beta(1)) \cup \overline{e(2, \beta(1)+1)}) = \emptyset$. Let

$$E = \{w_1, w_4, e(2,1), \ldots, e(2, \beta(1)-1), w_5, w_2 \cup w_3\}.$$

Then E is an open taut chain cover of $E_2[\beta(0), \beta(1)]^*$ and follows a pattern ξ' in D_1, and $E - \{w_1, w_2 \cup w_3\}$ follows a restriction of ξ' in D_1. In addition, $E^* - \{w_1, w_2, w_3\}^*$ is closed in P_2. There is an open set u_1 in P_1 such that $K_{\xi\beta(0),0} \subseteq u_1 \subseteq \bar{u}_1 \subseteq i(d(1,0), D_1) \cap i(g^b(0), G^b)$ and $\bar{u}_1 \cap (\bigcup_{i>1} K_{\xi\beta(i),i}) = \emptyset$. There is an open set u_2 such that $K_{\xi\beta(1),1} \subseteq u_2 \subseteq \bar{u}_2 \subseteq i(d(1, \xi\beta(1)), D_1) \cap i(g^b(\beta'(1)), G^b)$ and $\bar{u}_2 \cap (\bigcup_{i \neq 1} K_{\xi\beta(i),i}) = \emptyset$. Also, there is an open set u_3 such that $\overline{g^b(2)} \cap g^b(1) \subseteq u_3 \subseteq v_1 \subseteq i(d(1, \xi\beta(1)), D_1)$ and $\bar{u}_3 \cap \overline{g^b(0)} = \emptyset$. Consider $G^b[\beta'(0), \beta'(1)]^* - (u_1 \cup u_2 \cup u_3)$. This is a compactum essentially covered by D_1, and there is an open (in P_1) taut chain cover $D_2'[\beta(0), \beta(1)]$ of this compactum that follows the restriction of ξ' in D_1 such that—if $d(2, \beta(0)) = d'(2, \beta(0)) \cup u_1$, $d(2, i) = d'(2, i)$ for $i \in [1, \beta(1)-1]$, and $d(2, \beta(1)) = d'(2, \beta(1)) \cup u_2 \cup u_3$—then $D_2[\beta(0), \beta(1)]$ follows the required restriction of ξ in D_1. Continue this process.] □

THEOREM 7. *Suppose that C_1 is a Cantor set in the pseudoarc P_1 such that no composant of P_1 contains more than one point of C_1, and suppose that C_2 is a Cantor set in the pseudoarc P_2 such that no composant of P_2 contains more than one point of C_2. Then there is a homeomorphism α: $P_1 \to P_2$ such that $\alpha(C_1) = C_2$.*

Proof. There are open taut chain covers $D_1 = D_1[0, 2] = \{d(1, 0), d(1, 1), d(1, 2)\}$ of P_1 and $E_1 = E_1[0, 2] = \{e(1, 0), e(1, 1), e(1, 2)\}$ of P_2 such that:

(1) $C_1 \cap i(d(1,0), D_1) \neq \emptyset$ and $C_2 \cap i(e(1,0), E_1) \neq \emptyset$;
(2) $C_1 \cap i(d(1,2), D_1) \neq \emptyset$ and $C_2 \cap i(e(1,2), E) \neq \emptyset$; and
(3) $C_1 \cap \overline{d(1,1)} = \emptyset$ and $C_2 \cap \overline{e(1,1)} = \emptyset$.

Let $A_1 = \{0, 2\}$. Choose an open taut chain cover $E_2 = E_2[0, n_2]$ of P_2 and an increasing subsequence $A_2 = \{\alpha_2(0) = 0, \alpha_2(1), \ldots, \alpha_2(a_2) = n_2\}$ of $[0, n_2]$ such that:

(4) mesh $E_2 < \frac{1}{2}$;
(5) $C_2 \cap i(e(2,0), E_2) \cap i(e(1,0), E_1) \neq \emptyset$ and $C_2 \cap i(e(2, n_2), E_2) \cap i(e(1,2), E_1) \neq \emptyset$;
(6) E_2 closure refines E_1 and follows some pattern ξ_1 in E_1;
(7) some link e of $E_2[\alpha_2(i), \alpha_2(i+1)]$ is in $i(e(1,0), E_1)$ and some link e' of $E_2[\alpha_2(i), \alpha_2(i+1)]$ is in $i(e(1,2), E_1)$ for $i \in [0, a_2 - 1]$; and
(8) $C_2 \subseteq \bigcup_{i=0}^{a_2} i(e(2, \alpha_2(i)), E_2)$ and, for $i \in [0, a_2]$, $C_2 \cap i(e(2, \alpha_2(i)), E_2) \neq \emptyset$.

Applying Lemma 6, there is an open taut chain cover D_2 of P_1 such that:

(9) $D_2 = D_2[0, n_2]$ closure refines D_1 and follows ξ_1 in D_1;
(10) $C_1 \subseteq \bigcup_{i=0}^{a_2} i(d(2, \alpha(i)), D_2)$ and, for $i \in [0, a_2]$, $C_1 \cap i(d(2, \alpha(i)), D_2) \neq \emptyset$; and
(11) for $i \in [0, a_2 - 1]$, $D_2[\alpha_2(i), \alpha_2(i+1)]$ is properly covered by D_1.

Now there exist an open taut cover $D_3[0, n_3]$ of P_1 and an increasing subsequence $A_3 = \{\alpha_3(0) = 0, \alpha_3(1), \ldots, \alpha_3(a_3) = n_3\}$ of $[0, n_3]$ such that:

(12) mesh $D_3 < \frac{1}{4}$;
(13) D_3 closure refines D_2 and follows some pattern ξ_2 in D_2;
(14) $C_1 \subseteq \bigcup_{i=1}^{\infty} i(d(3, \alpha_3(i)), D_3)$ and $i(d(3, \alpha_3(i)), D_3) \cap C_1 \neq \emptyset$ for $i \in [0, a_3]$; and
(15) for $i \in [0, a_3 - 1]$, $D_3[\alpha_3(i), \alpha_3(i+1)]$ is properly covered by D_2.

We now choose E_3 using Lemma 6, and continue choosing sequences of chains $D_1[0, n_1], D_2[0, n_2], \ldots$ and $E_1[0, n_1], E_2[0, n_2], \ldots$, as well as subsequences $A_1, A_2, \ldots$, such that:

(16) $\lim_i$ mesh $E_i = \lim_i$ mesh $D_i = 0$;
(17) E_i is an open taut chain cover of P_2 and D_i is an open taut chain cover of P_1;
(18) for $j \in \mathbf{N}$, $C_1 \subseteq \bigcup_{i=0}^{a_j} i(d(j,i), D_j)$ and $C_2 \subseteq \bigcup_{i=0}^{a_j} i(e(j,i), E_j)$; and
(19) for $i \in \mathbf{N}$, D_{i+1} follows ξ_i in D_i and E_{i+1} follows ξ_i in E_i.

In the standard way, then, we have induced a homeomorphism $\alpha: P_1 \to P_2$, where α is defined as follows: For $x \in P_1$ there is an infinite sequence $j(x,1)$, $j(x,2), \ldots$ of integers such that, for each i,

(20) $x \in d(i, j(x,i))$ and
(21) $\xi_i(j(x, i+1)) = j(x,i)$.

Then $\alpha(x) = \bigcap_i e(i, j(x,i))$. Note that $\alpha(C_1) = C_2$. □

COROLLARY 8. *If C is a Cantor set embedded in a pseudoarc P such that no composant of P contains more than one point of C, and if $\alpha \in H(C)$, then there is an extension $f \in H(P)$ of α.*

Proof. Theorem 1 gives us a pseudoarc P' containing C and $f' \in H(P')$ such that $f' | C = \alpha$. By Theorem 7 and its proof, there is a homeomorphism $g: P' \to P$ such that $g(c) = c$ for $c \in C$. Then $f = gf'g^{-1}$ is the desired homeomorphism. □

Mazurkiewicz [4] proved that each nondegenerate indecomposable continuum X contains a Cantor set C such that no composant of X contains two points of C. Cook [1] has shown that no such C can intersect every composant of X, and that $\hat{C} = \{K \mid K \text{ is a composant of } X \text{ and } K \cap C \neq \emptyset\}^*$, the composant saturation of C, is an F_σ-set. Thus, we obtain the following.

COROLLARY 9. *If $\hat{C}$ is the composant saturation of a Cantor set C in a pseudoarc P such that no composant of P contains more than one point of C, then for each $\alpha \in H(C)$ there is an extension $f_\alpha \in H(P)$ of α with $f_\alpha(\hat{C}) = \hat{C}$, and $\hat{C}$ is an F_σ-set, $\hat{C} \neq P$.*

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