# Degree Bounds for the Division Problem in Polynomial Ideals

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## 1. Introduction

Let $f_1, \dots, f_m$ be polynomials in $n$ variables of degree at most $D$ over an arbitrary field, and let $I$ be the ideal they generate. We are interested in the following problem: If $P \in I$, what is the smallest integer $\delta$ such that we can write

$$P = f_1 g_1 + \cdots + f_m g_m \tag{1.1}$$

with $\deg g_j \leq \delta$? In general $\delta$ can be quite large; for example, [16] constructs examples for which $\delta$ is larger than $D^{a^n}$, where $a$ is a positive constant (see also [2]). The doubly exponential estimate $\delta \leq (mD)^{2^n} + \deg P$ was given in 1926 by Hermann [11] (see the Appendix in [16]). For the case of the complex field, Berenstein and Yger [4] gave a better estimate for $\delta$ when the zero locus of $I$ is zero-dimensional; another estimate for $\delta$ when $\{f_1, \dots, f_m\}$ is a regular sequence is given in [3, Thm. 4.1]. In this paper we give sharp bounds for $\delta$ under the assumption that the zero locus of $I$ is zero-dimensional at all finite and infinite points (Theorem 2). It remains an open problem whether this bound remains valid under the weaker hypothesis of [4]. Our method involves replacing the $f_j$ by their homogenizations in $n+1$ variables and localizing in projective $n$-space. We first show (Theorem 1) that if $I$ is homogeneous and of height $n-1$ then $I$ is $(nD-n+1)$-regular; in particular, $I$ is $(nD-n+1)$-saturated, which allows us to localize. (A result similar to Theorem 1 is given by Briançon [5, Prop. 4].)

In the special case $P = 1$, in which (1.1) is called the *Bezout equation,* much better estimates can be found. Estimates for the Bezout equation have been given by Brownawell [6; 7], Masser and Wüstholz [15], Thompson [18], and recently by Caniglia, Galligo, and Heintz [8]. After the first version of this paper was completed, Kollár [12] obtained a sharp degree bound for the Nullstellensatz that yields the sharp result $\delta \leq D^{\min(m,n)} - D$ for the Bezout equation when $D \geq 3$.

Throughout this paper, we let $K$ denote an algebraically closed field. We let $\mathbf{A}^n = \mathbf{A}^n_K$ and $\mathbf{P}^n = \mathbf{P}^n_K$ denote affine $n$-space and projective $n$-space (respectively) over $K$. By a *point* in $\mathbf{A}^n$ or $\mathbf{P}^n$ we mean a closed point. If $V$ is an

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Received February 4, 1988. Revisions received July 19, 1988 and August 9, 1988.
Research partially supported by the National Science Foundation.
Michigan Math. J. 36 (1989).

algebraic set in $\mathbf{P}^n$ we let id $V$ denote the ideal of polynomials in $K[z_0, \dots, z_n]$ that vanish on $V$. For a polynomial $P \in K[z_1, \dots, z_n]$, we let ${}^hP \in K[z_0, \dots, z_n]$ denote the homogeneous polynomial given by

$$(1.2) \qquad {}^hP = z_0^{\deg P} P(z_1/z_0, \dots, z_n/z_0).$$

Suppose $I$ is a homogeneous ideal in $K[z_0, \dots, z_n]$ (when $z_0, \dots, z_n$ are indeterminates). We let $I_\delta$ denote the homogeneous elements of $I$ of degree $\delta$, and we let $Z(I)$ denote the zero locus of $I$ in $\mathbf{P}^n$. The ideal $I$ defines a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbf{P}^n}$, where $\mathcal{I}_x$ is generated by

$$\bigcup \{f\mathcal{O}_{x,\mathbf{P}^n}(-\deg f) : f \in I, \ f \text{ homogeneous}\}$$

for $x \in \mathbf{P}^n$. If $I = (P_1, \dots, P_m)$, where $P_j$ is homogeneous of degree $d_j$, then

$$(1.3) \qquad \mathcal{I} = \sum_{j=1}^{m} P_j \mathcal{O}_{\mathbf{P}^n}(-d_j).$$

(Note that supp $\mathcal{O}_{\mathbf{P}^n}/\mathcal{I} = Z(I)$.) By (1.3) the map

$$\tau : \bigoplus_{j=1}^{m} \mathcal{O}(-d_j) \to \mathcal{I}$$

given by

$$(1.4) \qquad \tau(\varphi_1, \dots, \varphi_m) = \sum_{j=1}^{m} P_j \otimes \varphi_j$$

is surjective. By an elementary cohomology argument we see that, for $\delta$ sufficiently large, the map

$$\tau(\delta)_* : \Gamma\left(\mathbf{P}^n, \bigoplus_{j=1}^{m} \mathcal{O}(\delta - d_j)\right) \to \Gamma(\mathbf{P}^n, \mathcal{I}(\delta))$$

is surjective, and thus

$$(1.5) \qquad I_\delta = \Gamma(\mathbf{P}^n, \mathcal{I}(\delta)).$$

However, if $Z(I) = \phi$ then $\mathcal{I} = \mathcal{O}_{\mathbf{P}^n}$, and Macaulay's theorem ([14], see also [13, p. 169] or [17, p. 85]) states that (1.5) is valid for

$$(1.6) \qquad \delta \geq d_1 + \cdots + d_n + d_m - n,$$

where we assume that $d_1 \geq \cdots \geq d_m$. (To obtain inequality (1.6) from the usual formulation, one chooses homogeneous polynomials $R_0, \dots, R_n$ in $I$ with no common zeros in $\mathbf{P}^n$ such that $R_0 = P_m$ and $\deg R_j = d_j$ for $1 \leq j \leq n$.)

## 2. The Division Problem

We first note the following well-known consequence of Macaulay's theorem.

PROPOSITION 1. *Let $f_1, \dots, f_m \in K[z_1, \dots, z_n]$ such that $Z({}^hf_1, \dots, {}^hf_m) = \phi$. Let $d_j = \deg f_j$ and assume that $d_1 \geq \cdots \geq d_m$. If $P \in K[z_1, \dots, z_n]$ then there exist $g_1, \dots, g_m$ satisfying* (1.1) *such that*

$$\deg f_j g_j \leq \max(\deg P, d_1 + \cdots + d_n + d_m - n).$$

*Proof.* Let $I=({}^hf_1,\dots,{}^hf_m)$ and let

(2.1) $$\delta=\max(\deg P,\, d_1+\cdots+d_n+d_m-n).$$

By Macaulay's theorem,

(2.2) $${}^hPz_0^{\delta-\deg P}\in I_\delta.$$

Therefore there exist homogeneous polynomials $Q_1,\dots,Q_m$ such that

$$\deg f_jQ_j=\delta \quad \text{for } 1\le j\le m$$

and

(2.3) $${}^hPz_0^{\delta-\deg P}={}^hf_1Q_1+\cdots+{}^hf_mQ_m.$$

We then let $g_j=Q_j(1,z_1,\dots,z_n)$. □

We shall extend Macaulay's theorem to ideals with zero-dimensional zero loci. Recall that a homogeneous ideal $I$ in $K[z_0,\dots,z_n]$ is $k$-regular if the $j$th syzygy module of $I$ is generated by elements of degree at most $k+j$, for $j\ge 0$ ([1], [9]).

THEOREM 1. *Let $I$ be an ideal in $K[z_0,\dots,z_n]$ generated by homogeneous polynomials $P_1,\dots,P_m$ (where $m>n$), and suppose that* $\dim Z(I)=0$. *Let $d_j=\deg P_j$ for $1\le j\le m$, and assume that $d_1\ge\cdots\ge d_m\ge 1$. Then $I$ is $k$-regular, where*

$$k=d_1+\cdots+d_{n+1}-n.$$

COROLLARY 1. *Let $I$ and $d_j$ be as in Theorem 1, and let $\mathcal{I}$ be the ideal sheaf in $\mathcal{O}_{\mathbf{P}^n}$ given by $I$. Then*

$$I_\delta=\Gamma(\mathbf{P}^n,\mathcal{I}(\delta)) \quad \textit{for } \delta\ge d_1+\cdots+d_{n+1}-n.$$

*Proof.* By Theorem 1, $I$ is $\delta$-saturated (see [1, p. 3]). Thus

$$\Gamma(\mathbf{P}^n,\mathcal{I}(\delta))=I_\delta^{\text{sat}}=I_\delta.$$ □

(In fact, Corollary 1 is one of the ingredients of Theorem 1, and is proved directly as part of the proof of Theorem 1.)

REMARK. If $m=n$ then $P_1,\dots,P_n$ is a regular sequence and therefore $I$ is saturated, which means that the conclusion is valid for all $\delta\ge 0$.

A consequence of Corollary 1 is the following sharp degree estimate for the Nullstellensatz for the case where $Z(I)$ is zero-dimensional.

COROLLARY 2. *Let $I$ and $d_j$ be as in Theorem 1, and suppose that $n\ge 2$. Then*

$$(\sqrt{I}\,)^{d_1\cdots d_{n-1}d_m}\subset I$$

*unless $n=2$ and $d_m=1$, in which case $(\sqrt{I}\,)^{d_1+d_{m-1}-1}\subset I$.*

*Proof.* If $n=2$ and $d_m=1$, we can assume $P_m=z_2$ and reduce to the case $n=1$, which is elementary. Thus we assume $n\ge 3$ or $d_m\ge 2$. By a standard argument, we can choose homogeneous polynomials $R_0, \dots, R_n$ in $I$ such that $R_0 = P_m$, $\deg R_j = d_j$ for $1 \le j \le n$, $\{R_0, \dots, R_m\}$ is a regular sequence, and $Z(R_0, \dots, R_n) = Z(I)$. Let $J = (R_0, \dots, R_n)$ and let $\mathcal{J}$ be the ideal sheaf in $\mathcal{O}_{\mathbf{P}^n}$ given by $J$. Let $\delta = d_1 \cdots d_{n-1} d_m$. It follows from the Bezout theorem (see [10, p. 145]) that $(\sqrt{\mathcal{J}_x})^\delta \in \mathcal{J}_x$ for all $x \in \mathbf{P}^n$. Let $g_1, \dots, g_\delta \in \sqrt{I} = \sqrt{J}$ and let $f = g_1 \cdots g_\delta$. We must show that $f \in I$. We can assume that the $g_j$ are homogeneous. Then $f \in \Gamma(\mathbf{P}^n, \mathcal{J}(N))$, where $N = \deg f \ge \delta$. One can easily check that $\delta \ge d_1 + \cdots + d_n + d_m - n$. It follows from Corollary 1 applied to $J$ that $f \in J \subset I$. □

We use Corollary 1 to obtain the following degree estimate for the division problem.

THEOREM 2. *Let* $f_1, \dots, f_m \in K[z_1, \dots, z_n]$ *such that* $\dim Z({}^hf_1, \dots, {}^hf_m) = 0$. *Let* $d_j = \deg f_j$ *and assume that* $d_1 \ge \cdots \ge d_m \ge 1$. *If* $P \in (f_1, \dots, f_m)$, *then there exist* $g_1, \dots, g_m \in K[z_1, \dots, z_n]$ *such that*

$$P = f_1 g_1 + \cdots + f_m g_m$$

*and* $\deg f_j g_j \le \delta$ *for* $1 \le j \le m$, *where*

(i) $\delta = d_1 \cdots d_{n-1} d_m + \deg P$ *if* $n \ge 3$,
(ii) $\delta = \max(d_1 d_m + \deg P, d_1 + d_2 + d_3 - 2)$ *if* $n = 2$,
(iii) $\delta = \max(\deg P, d_1 + d_m - 1)$ *if* $n = 1$,

*and furthermore*

(iv) $\delta = \max(\deg P, d_1 + \cdots + d_{n+1} - n)$ *if* $Z(z_0, {}^hf_1, \dots, {}^hf_m) = \phi$, $m > n$,
(v) $\delta = \deg P$ *if* $Z(z_0, {}^hf_1, \dots, {}^hf_m) = \phi$, $m = n$.

REMARKS. If the coefficients of $P, f_1, \dots, f_m$ lie in a subfield of $K$, then the $g_j$ can be chosen with coefficients in this subfield. For the case $K = \mathbf{C}$, Berenstein and Yger [4] gave the estimate

$$\delta \le (n+1)^3 \max(\deg P, d_1)^{3n+1}$$

under the weaker hypothesis that the set of common zeroes (in affine space) of the $f_j$ is zero-dimensional.

*Proof* (assuming Theorem 1). Case (iv) follows from Corollary 1 exactly as in the proof of Proposition 1. Case (v) is an immediate consequence of the remark following Corollary 1. Case (iii) is elementary, so we now consider cases (i) and (ii). Let $I = ({}^hf_1, \dots, {}^hf_m)$ and let $\mathcal{J} \subset \mathcal{O}_{\mathbf{P}^n}$ be given by $I$. By the Bezout theorem (as in the proof of Corollary 2),

$$ {}^hP z_0^{d_1 \cdots d_{n-1} d_m} \in \Gamma(\mathbf{P}^n, \mathcal{J}(d_1 \cdots d_{n-1} d_m + \deg P)) \tag{2.4}$$

and hence

$$ {}^hP z_0^{\delta - \deg P} \in \Gamma(\mathbf{P}^n, \mathcal{J}(\delta)). \tag{2.5}$$

One easily checks that, except for the case

$$n = 3, \quad d_1 = d_2 = d_3 = d_4 = 2, \quad d_m = 1, \quad \deg P = 0,$$

we have

$$\delta \geq d_1 + \cdots + d_{n+1} - n,$$

and the conclusion follows as in the proof of Proposition 1. In the exceptional case, we can assume that $f_m = z_3$ and eliminate $z_3$. Since

$$d_1 d_2 \geq \max(d_1 d_{m-1}, d_1 + d_2 + d_3 - 2),$$

the conclusion follows from case (ii). □

COROLLARY 3. *Let $f_j, d_j$ be as in Theorem 2. If $f_1, \dots, f_m$ have no common zeroes, then there exist $g_1, \dots, g_m \in K[z_1, \dots, z_n]$ such that*

$$f_1 g_1 + \cdots + f_m g_m = 1$$

*and* $\deg f_j g_j \leq \delta$, *where*

$$\delta = \begin{cases} d_1 \cdots d_{n-1} d_m & \text{for } n \geq 3, \\ \max(d_1 d_m, d_1 + d_2 + d_3 - 2) & \text{for } n = 2, \\ d_1 + d_m - 1 & \text{for } n = 1. \end{cases}$$

Note that in the conclusion of Corollary 3, $\delta \leq d_1^n$ for $n \geq 2$. This estimate is slightly sharper than that given in [5, p. 556].

*Proof of Theorem 1.* We use a modification of the spectral sequence argument in [17, pp. 85–87]. Let $I$, $P_j$, $d_j$ and $k$ be given as in the theorem, and let $\mathcal{I}$ be the ideal sheaf in $\mathcal{O}_{\mathbf{P}^n}$ given by $I$. We must show that

(2.6) $$I_\delta = \Gamma(\mathbf{P}^n, \mathcal{I}(\delta)) \quad \text{for } \delta \geq k$$

and

(2.7) $$H^q(\mathbf{P}^n, \mathcal{I}(\delta)) = 0 \quad \text{for } \delta \geq k - q,\ q \geq 1$$

(see [1] or [9]). Consider the locally free sheaf on $\mathbf{P}^n$

(2.8) $$\mathcal{E} = \bigoplus_{j=1}^{m} \mathcal{O}(-d_j),$$

and let $\tau: \mathcal{E} \to \mathcal{I}$ be given by

(2.9) $$\tau(\varphi_1, \dots, \varphi_m) = \sum_{j=1}^{m} P_j \otimes \varphi_j.$$

Note that $\tau$ is surjective by definition. For an arbitrary integer $\delta$, consider the modified Koszul complex

(2.10) $$0 \longrightarrow \mathcal{S}^0 \xrightarrow{\sigma^0} \mathcal{S}^1 \longrightarrow \cdots \longrightarrow \mathcal{S}^{m-1} \xrightarrow{\sigma^{m-1}} \mathcal{S}^m \longrightarrow 0,$$

where

(2.11) $$\mathcal{S}^p = (\Lambda^{m-p}\mathcal{E})(\delta) \quad \text{for } 0 \leq p \leq m-1,$$

(2.12) $$\mathcal{S}^m = \mathcal{S}(\delta),$$

and $\sigma^p$ is interior multiplication by the section

$$s = P_1 \oplus \cdots \oplus P_m \in \Gamma(\mathbf{P}^n, \mathcal{E}^\vee). \tag{2.13}$$

In particular, $\mathcal{S}^{m-1} = \mathcal{E}(\delta)$ and

$$\sigma^{m-1} = \tau(\delta) \colon \mathcal{E}(\delta) \to \mathcal{S}(\delta). \tag{2.14}$$

To verify (2.6) and (2.7), we consider the two hypercohomology spectral sequences

$$_I E_1^{p,q} = H^q(\mathbf{P}^n, \mathcal{S}^p) \Rightarrow \underline{H}^{p+q}(\mathbf{P}^n, \mathcal{S}); \tag{2.15}$$

$$_{II} E_2^{p,q} = H^p(\mathbf{P}^n, H^q\mathcal{S}^\cdot) \Rightarrow \underline{H}^{p+q}(\mathbf{P}^n, \mathcal{S}). \tag{2.16}$$

Note that $_I E_2^{p,q} = H^p(H^q(\mathbf{P}^n, \mathcal{S}^\cdot))$.

The identity (2.6) is equivalent to the surjectivity of the map

$$\sigma_*^{m-1} \colon \Gamma(\mathbf{P}^n, \mathcal{E}(\delta)) \to \Gamma(\mathbf{P}^n, \mathcal{I}(\delta)), \tag{2.17}$$

which in turn is equivalent to the vanishing of $_I E_2^{m,0}$ for $\delta \geq k$. We note that $H^m\mathcal{S}^\cdot = 0$ by definition, and thus $_{II} E_2^{0,m} = 0$. We note also that

$$\operatorname{Supp} H^q\mathcal{S}^\cdot \subset Z(I), \tag{2.18}$$

since $Z(I)$ is the zero set of the section $s$ given by (2.13). Since $\dim Z(I) = 0$, it follows from (2.18) that

$$_{II} E_2^{p,q} = 0 \quad \text{for } p \neq 0. \tag{2.19}$$

Therefore, by (2.16),

$$\underline{H}^{m+q}(\mathbf{P}^n, \mathcal{S}) = {}_{II} E_2^{0,m+q} = 0 \quad \text{for } q \geq 0. \tag{2.20}$$

Therefore, by (2.15),

$$_I E_\infty^{m,q} = 0. \tag{2.21}$$

To evaluate $_I E_1^{p,q}$, we note that

$$\mathcal{S}^p = \bigoplus \mathcal{O}\left(\delta - \sum_{\mu=1}^{m-p} d_{j_\mu}\right) \quad (1 \leq j_1 < \cdots < j_{m-p} \leq m) \tag{2.22}$$

for $0 \leq p \leq m-1$. Thus

$$_I E_1^{p,q} = 0 \quad \text{for } p \neq m,\ q \neq 0, n. \tag{2.23}$$

We consider the differentials

$$d_r \colon {}_I E_r^{m-r,q+r-1} \to {}_I E_r^{m,q}, \quad r \geq 1. \tag{2.24}$$

By (2.23), $_I E_r^{m-r,q+r-1} = 0$ for $r \neq n-q+1$ if $q > 0$ and for $r \notin \{1, n+1\}$ if $q = 0$. We now suppose $\delta \geq k - q$, $q \geq 0$. By (2.22), $\mathcal{S}^{m-n+q-1}$ is the direct sum of terms of the form $\mathcal{O}(t)$, where

$$t = \delta - \sum_{\mu=1}^{n-q+1} d_{j_\mu} \geq \delta - \sum_{i=1}^{n-q+1} d_i \geq -n. \tag{2.25}$$

Therefore,

$$_I E_1^{m-n+q-1,n} = H^n(\mathbf{P}^n, \mathcal{S}^{m-n+q-1}) = 0. \tag{2.26}$$

We first consider the case $q=0$. By (2.26), ${}_IE_2^{m-n-1,n}=0$ and hence, by (2.21), ${}_IE_2^{m,0}={}_IE_\infty^{m,0}=0$, verifying (2.6). Next consider $q>0$. By (2.12), (2.21), and (2.26),

$$H^q(\mathbf{P}^n, \mathcal{G}(\delta))={}_IE_1^{m,q}={}_IE_\infty^{m,q}=0,$$

which verifies (2.7), completing the proof of Theorem 1. □

## 3. Nullstellensatz Estimates

We shall give a short proof of the Nullstellensatz degree estimate for an ideal given by a regular sequence (Lemma 2), which we apply to the division problem (Proposition 2). The results in this section are generally well known.

Let $f_1, \dots, f_m \in K[z_1, \dots, z_n]$ and suppose we are given a point $x \in \mathbf{A}^n$ such that $f_1, \dots, f_m$ defines a regular sequence in the local ring $\mathcal{O}_x = \mathcal{O}_{x,\mathbf{A}^n}$. Let $V$ denote the zero locus of $(f_1, \dots, f_m)$, and let $V_1, \dots, V_r$ denote the (reduced) irreducible components of $V$ that contain $x$. Recall that the regular sequence assumption means that $\dim V_\alpha = n-m$, for $1 \le \alpha \le r$. Let $D_j$ denote the divisor given by $f_j$ and, for $1 \le \alpha \le r$, let $p_\alpha$ denote the intersection multiplicity $i(V_\alpha, D_1 \bullet \cdots \bullet D_m)$. Recall that the $p_\alpha$ are given by the identity

$$D_1 \bullet \cdots \bullet D_m = \sum_{\alpha=1}^{r} p_\alpha V_\alpha + W \tag{3.1}$$

in the Chow ring $A_{n-m}(V)$, where $W$ is a cycle not containing $x$ (see [10, 2.4.2, 7.1.10]). We consider the ideal

$$I_x = (f_1, \dots, f_m)\mathcal{O}_x \tag{3.2}$$

in the local ring $\mathcal{O}_x$. This ideal has a decomposition of the form

$$I_x = Q_1 \cap \cdots \cap Q_r,$$

where $Q_\alpha$ is primary and

$$\sqrt{Q_\alpha} = (\mathrm{id}\, V_\alpha)\mathcal{O}_x \tag{3.3}$$

for $1 \le \alpha \le r$.

LEMMA 1. *Let $Q_\alpha, p_\alpha$ be given as above. Then*

$$(\sqrt{Q_\alpha})^{p_\alpha} \subset Q_\alpha$$

*for* $1 \le \alpha \le r$.

*Proof.* Fix $\alpha$ and let $R = \mathcal{O}_{V_\alpha, \mathbf{A}^n}/(f_1, \dots, f_m)$. By [10, 7.1.10], $p_\alpha$ equals the length of $R$. Let $M$ denote the maximal ideal in $R$; then $M^{p_\alpha}=0$. Let $J = (\sqrt{Q_\alpha})^{p_\alpha}$ and let $\tau: \mathcal{O}_x \to R$ denote the natural homomorphism. Since

$$\tau(\sqrt{Q_\alpha}) \subset M,$$

it follows that $\tau(J)=0$. Let $I'$ and $J'$ denote the ideals in $\mathcal{O}_{V_\alpha, \mathbf{A}^n}$ generated by $I_x$ and $J$, respectively. Since $\tau(J)=0$, it follows that $J' \subset I'$. Hence there exists a polynomial $h \in K[z_1, \dots, z_n] - \mathrm{id}\, V_\alpha$ such that $hJ \subset I_x \subset Q_\alpha$. Since $Q_\alpha$ is primary and $h \notin \sqrt{Q_\alpha}$, it follows that $J \subset Q_\alpha$. □

LEMMA 2. *Suppose that* $f_1, \ldots, f_m \in K[z_1, \ldots, z_n]$ *and* $x$ *is a point in* $\mathbf{A}^n$ *such that* $f_1, \ldots, f_m$ *defines a regular sequence in* $\mathcal{O}_x$. *Let* $I_x = (f_1, \ldots, f_m)\mathcal{O}_x$. *Then*

$$(\sqrt{I_x})^N \subset I_x,$$

*where*

$$N = \prod_{j=1}^{m} \deg f_j.$$

*Proof.* By the Bezout theorem (see [10]), $N \geq p_\alpha$, where $p_\alpha$ is given by (3.1), and the conclusion follows from Lemma 1. □

The following degree bound for the Nullstellensatz for complete intersection ideals is an immediate consequence of Lemma 2.

COROLLARY 4. *Let* $f_1, \ldots, f_m$ *be a regular sequence in* $K[z_1, \ldots, z_n]$ *and let* $I = (f_1, \ldots, f_m)$. *Then*

$$(\sqrt{I})^N \subset I,$$

*where*

$$N = \prod_{j=1}^{m} \deg f_j.$$

We conclude by stating the sharp degree bound for the division problem for regular sequences that are also "regular at infinity".

PROPOSITION 2. *Let* $f_1, \ldots, f_m \in K[z_1, \ldots, z_n]$ *such that* $\{{}^h f_1, \ldots, {}^h f_m\}$ *is a regular sequence. If* $P \in (f_1, \ldots, f_m)$ *then there exist* $g_1, \ldots, g_m \in K[z_1, \ldots, z_n]$ *satisfying* (1.1) *such that*

$$\deg f_j g_j \leq \prod_{i=1}^{m} \deg f_i + \deg P$$

*for* $1 \leq j \leq m$.

*Proof.* If $m = n+1$ then Proposition 1 is sharper, so we may assume $m \leq n$. Let $I = ({}^h f_1, \ldots, {}^h f_m)$ and let $\mathcal{I} \subset \mathcal{O}_{\mathbf{P}^n}$ be given by $I$. By Lemma 2,

$$ {}^h P z_0^N \in \Gamma(\mathbf{P}^n, \mathcal{I}(N + \deg P)), \tag{3.4}$$

where $N = \prod \deg f_j$. Since $I$ is saturated, ${}^h P z_0^N \in I$. (See the remark following Corollary 1.) The conclusion follows immediately as in the proof of Proposition 1. □

A special case of Proposition 2 is the following degree estimate for certain Bezout equations.

COROLLARY 5. *Let* $f_1, \ldots, f_m$ *be as in Proposition 2. If* $f_1, \ldots, f_m$ *also have no common zeroes then there exist* $g_1, \ldots, g_m \in K[z_1, \ldots, z_n]$ *such that* $f_1 g_1 + \cdots + f_m g_m = 1$ *and*

$$\deg f_j g_j \leq \prod_{i=1}^{m} \deg f_i$$

*for* $1 \leq j \leq m$.

*Added in proof:* F. Amoroso [*Tests d'appartenance d'après un Théorème de Kollár,* manuscript] recently used the methods of Kollár [12] to obtain conclusions (i)–(iii) of Theorem 2 under the weaker hypothesis that dim $Z(f_1, \ldots, f_m) = 0$ (but with the added assumption of Kollár that $d_j \neq 2$) as well as the estimate of Proposition 2 for regular sequences.

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