Continuity Properties of Selectors and Michael's Theorem

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0. Introduction

Recall that if A and B are closed, bounded, convex, nonempty subsets of X (i.e., $A, B \in H(X)$), then the Hausdorff distance between A and B is defined by $d_H(A,B) = \sup(\{d(x,A): x \in B\} \cup \{d(x,B): x \in A\})$. Note that X is always isometric to a subset of $H(X)$ if we identify a point x with the singleton set $\{x\}$. Since no ambiguity arises, we will usually write $d(A,B)$ rather than $d_H(A,B)$. We let $K(X)$ denote the subfamily of compact sets in $H(X)$, also equipped with the Hausdorff metric. Of course $K(X) = H(X)$ when X is finite-dimensional.

Michael's selection theorem [37, Thm. 3.2″] tells us that there is a continuous map $f: H(X) \to X$ such that $f(A) \in A$ for all $A \in H(X)$. For a given Banach space X, can we find a Lipschitz map $f: H(X) \to X$ satisfying the same selection identity? We will refer to selection maps from $H(X)$ to X as *selectors*.

Various authors [40; 43; 45] have observed that this is possible if X is finite-dimensional. Indeed, the Steiner point [46] provides a suitable selector when $X = \mathbf{R}^n$. It is noted in [55] that this is not possible if $X = C[0,1]$. The arguments used in these papers are reasonably elementary.

In this paper, we concern ourselves with the existence of uniformly continuous selectors for general Banach spaces. It follows from [32, Cor. 5] that there is no uniformly continuous selector from $H(X)$ to X whenever X is an infinite-dimensional Hilbert space. This depends on various results which may be found in [24; 32; 38]. From Dvoretzky's theorem it can then be deduced that there is no uniformly continuous selector from $H(X)$ (or even from $K(X)$) to X, whenever X is an infinite-dimensional Banach space. This result could have been proved twenty years ago. Although a special case has already been published [16, p. 245], it does not seem to be very well known. We feel that this problem deserves a thorough exposition. Two new proofs will be presented here, in Sections 2 and 3, indicating how this problem interacts with different areas of analysis.

The original proof of [32, Cor. 5] depended upon a result of Lindenstrauss [32] that if a closed subspace M is the range of a uniformly continuous re-

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tract on the Banach space X, then M^0 is complemented in X^*; and upon a result of Isbell [24] that $H(X)$ is the range of a uniformly continuous retract on any metric space which contains it. Lindenstrauss [32] observed that Isbell's argument actually leads to a Lipschitz continuous retract. (In fact, the Lipschitz constant can always be chosen to be less than 10.9. However, this property of $H(X)$ is not needed by us until the end of §5.)

Our technique is based on the existence of invariant means on abelian semigroups. This technique has been used by Pełczyński [38, pp. 161–2] to give a simpler proof of the just-mentioned result of Lindenstrauss. Apart from this idea, our proofs are independent of the work of Isbell, Lindenstrauss, and Pełczyński. This should not disguise the fact that all known proofs of this result depend on Dvoretzky's theorem, and thus lie fairly deep.

Section 1 contains the necessary background material, as well as the concept of sub-Lipschitz constants for uniformly continuous mappings. A key technical result exhibits the connection between sub-Lipschitz constants of different types of selectors. Sections 2 and 3 give different proofs of our first theorem: There is no uniform selector $K(X)\to X$ unless X is finite-dimensional. Both proofs depend on invariant means and Dvoretzky's theorem. Also in Section 3, we discuss briefly some well-known (but not Lipschitz) finite-dimensional selectors.

This result encourages us to confine our attention to finite dimensions, and to determine the best possible Lipschitz constant, for given Minkowski spaces. In Section 4, the connection between selectors and appropriate linear projections is investigated. Section 5 presents some extension theorems for Lipschitz mappings into finite-dimensional spaces. In Section 6 we examine some other Lipschitz continuous finite-dimensional selectors that are different from the Steiner selector.

Section 7 presents a positive infinite-dimensional result, essentially due to Skaletskii: If X satisfies a certain geometric condition, and if Ω is a bounded subset of $H(X)$, then there is a uniformly continuous selector $\Omega\to X$.

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1. Notation and Preliminaries

Let X and Y be normed spaces, and A a convex subset of X. For any mapping $T: A\to Y$, we can define its modulus of continuity $\omega_T: \mathbf{R}^+\to\mathbf{R}^+\cup\{\infty\}$ as follows: $\omega_T(\delta)=\sup\{\|Tx-Ty\|: x,y\in A,\ \|x-y\|\le\delta\}$. From convexity of A, it follows that ω_T is subadditive. Obviously ω_T is monotonic. If there is a positive δ for which $\omega_T(\delta)$ is finite, we will call T a *uniform* mapping. This is equivalent to requiring $\omega_T(\delta)<\infty$ for all δ. When this is the case, the number

$$C_T=\inf_{\delta>0}\frac{\omega_T(\delta)}{\delta}=\lim_{\delta\to\infty}\frac{\omega_T(\delta)}{\delta}$$

will be called the sub-Lipschitz constant of the mapping T. Clearly T is Lipschitz continuous if and only if $L_T = \sup_{\delta>0}(\omega_T(\delta)/\delta)$ is finite.

A mapping $T: A \to Y$ is uniformly continuous if and only if ω_T is continuous at 0, in which case ω_T is continuous everywhere. Clearly, every uniformly continuous mapping is uniform. However, a uniform mapping need not even be continuous, as shown by a step function from $\mathbf{R}$ to $\mathbf{R}$. Also, the example $f(x) = \sin(x^2)$ shows that a continuous uniform mapping need not be uniformly continuous.

Now suppose that the set A is a wedge in X; that is, suppose that A is closed under vector addition and multiplication by positive scalars. (We do not assume that $-A \cap A = \{0\}$.) The mapping $T: A \to Y$ is said to be *additive* if $T(a+b) = T(a) + T(b)$ for all $a, b \in A$. Standard arguments show that if T is uniform and additive then $T(\lambda a) = \lambda T(a)$ for all $a \in A$, $\lambda \geq 0$. Such maps will be called *linear*. The following simple lemma shows that this is only a slight abuse of terminology.

LEMMA 1.1. *For every uniform, additive mapping T from a wedge A into a Banach space Y, there is a unique bounded linear operator $\tilde{T}: \overline{\operatorname{linsp} A} \to Y$ which extends T. Furthermore, $\tilde{T}$ is just as continuous as T, in the sense that $\omega_T(\delta) = \|\tilde{T}\|\delta$ for all $\delta > 0$, and thus $L_T = C_T = \|\tilde{T}\|$.*

Now fix a Banach space X and consider the metric space $K(X)$. The usual vector operations are defined on $K(X)$ by $A + B = \{a + b : a \in A,\ b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$, for any $A, B \in K(X)$ and $\lambda \geq 0$. (We can also define multiplication by negative scalars, but note that then we have $A + (-A) = \{0\}$ only when A is a singleton.)

It is possible to regard $K(X)$ as a wedge in the Banach space $C(U)$, where U is the unit ball of X^* equipped with the weak* topology. Given $A \in K(X)$, we define the support function $h_A \in C(U)$ by $h_A(f) = \sup f(A)$ for $f \in U$. Weak* continuity of h_A follows easily from the norm compactness of A. It follows from the Hahn-Banach theorem that

$$d(A, B) = \sup_{f \in U} |h_A(f) - h_B(f)|,$$

and so the map $A \mapsto h_A$ is a linear isometric embedding. Moreover, for each $x \in X$, $h_{\{x\}} = \hat{x}$, the evaluation function. Thus this embedding sends X to a linear subspace of $C(U)$. In the sequel, we will regard X as being a subspace of $C(U)$.

Of course $C(U)$ is also a Banach lattice. For all $A, B \in K(X)$ it is easy to verify that $h_A \vee h_B = h_{\operatorname{co}(A \cup B)}$. From the identity

$$(a - b) \vee (c - d) = (b + c) \vee (a + d) - (b + d),$$

which holds for all $a, b, c, d \in \mathbf{R}$, it follows easily that $\operatorname{linsp} K(X)$ is a sublattice of $C(K)$.

Let G be an abelian semigroup, and X a normed space. Denote by $l_\infty(G, X)$ the space of all bounded mappings from G into X, equipped with the usual supremum norm. For any $f \in l_\infty(G,X)$ and $g \in G$, we define the translated function f_g by $f_g(h) = f(g+h)$.

PROPOSITION 1.2. *For every dual Banach space $X = Y^*$ and for every abelian semigroup G, there is a linear operator $M: l_\infty(G,X) \to X$ such that:*

(i) *$M(f_g) = M(f)$ for all $f \in l_\infty(G,X)$, $g \in G$;*
(ii) *$M(f)$ lies in the weak* closed convex hull of $f(G)$, for all $f \in l_\infty(G,X)$; and*
(iii) $\|M\| = 1$.

Proof. First consider the case $X = \mathbf{R}$. It is well known ([11] or [18]) that there is an invariant mean N on $l_\infty(G,\mathbf{R})$, having the specified properties.

For any $y \in Y$, let $\hat{y} \in Y^{**}$ denote the corresponding evaluation functional. Define a map $M: l_\infty(G, Y^*) \to Y^*$ by $M(f)(y) = N(\hat{y} \circ f)$, for $f \in l_\infty(G, Y^*)$ and $y \in Y$.

It is easy to check that $M(f)$ is a bounded linear functional on Y, so M is well defined. Because N is translation invariant and linear, so also is M. Finally, for each $y \in Y$,

$$(Mf)(y) = N(\hat{y} \circ f) \in \overline{\mathrm{co}(\hat{y}(f(G))} = \overline{\hat{y}(\mathrm{co}\, f(G))} = \hat{y}(\overline{\mathrm{co}\, f(G)}),$$

as $\hat{y}$ is weak* continuous. From the separation theorem we obtain $Mf \in \overline{\mathrm{co}\, f(G)}$. □

PROPOSITION 1.3. *For every Banach space X and for every abelian semigroup G, there is a linear operator $M: l_\infty(G,X) \to X^{**}$ such that:*

(i) *$M(f_g) = M(f)$ for all $f \in l_\infty(G,X)$, $g \in G$; and*
(ii) *$M(f)$ lies in the weak* closed convex hull of $f(G)$, for all $f \in l_\infty(G,X)$.*

Proof. Proposition 1.2 gives us an invariant mean $l_\infty(G, X^{**}) \to X^{**}$. The restriction of this operator to $l_\infty(G,X)$ is the mapping we require. □

The operators given by Propositions 1.2 and 1.3 will be called invariant means. This refers to the translation invariance condition $M(f_g) = M(f)$, and is consistent with the usual terminology when $X = \mathbf{R}$. It will sometimes be helpful and suggestive to use the notation $M(f) = \int_G f(g)\, dg$, even though there is no measure defined on G.

The following technical result underpins most of our work in this paper.

PROPOSITION 1.4. *Suppose, for a given Banach space X, that there is a uniform retract $R: K(X) \to X$. Then there is a linear map $T: K(X) \to X^{**}$ with $T(\{x\}) = x$ for all $x \in X$, and $L_T \le C_R$.*

Proof. Under addition, both X and $K(X)$ are abelian semigroups. We define T by

$$T(B) = \int_{K(X)} \int_X R(A+B+x) - R(A+x)\, dx\, dA,$$

where the integrals denote invariant means on $l_\infty(X, X)$ and $l_\infty(K(X), X^{**})$. Since

$$\|R(A+B+x)-R(A+x)\| \leq \omega_R(d(A+B+x, A+x)) = \omega_R(d(B, \{0\})),$$

we see that integrand is bounded, and T is well defined.

For any $B, C \in K(X)$, we have

$$\begin{aligned} T(B+C) &= \int_{K(X)} \int_X R(A+B+C+x)-R(A+x)\,dx\,dA \\ &= \int_{K(X)} \int_X R(A+B+C+x)-R(A+B+x)\,dx\,dA \\ &\quad + \int_{K(X)} \int_X R(A+B+x)-R(A+x)\,dx\,dA \\ &= T(C)+T(B), \end{aligned}$$

using translation invariance. Thus T is additive.

Furthermore, given any $y \in X$ and $A \in K(X)$, we have

$$\begin{aligned} &\int_X R(A+y+x)-R(A+x)\,dx \\ &\quad = y+\int_X R(A+y+x)-x-y+x-R(A+x)\,dx \\ &\quad = y+\int_X R(A+y+x)-R(y+x)\,dx-\int_X R(A+x)-R(x)\,dx \\ &\quad = y, \end{aligned}$$

again using translation invariance. Thus $T(\{y\}) = \int_{K(X)} y\,dA = y$, as required.

Easy calculation shows that $\|T(B)-T(C)\| \leq \omega_R(d(B, C))$ for each $B, C \in K(X)$, and so $\omega_T \leq \omega_R$. Lemma 1.1 completes the proof. □

2. There Is No Uniform Version of Michael's Selection Theorem

Michael's selection theorem [37, Thm. 3.2″] tells us that if S is a metric space and $\Psi: S \to H(X)$ a continuous map, then there is a continuous map $f: S \to X$ such that $f(x) \in \Psi(x)$ for each $x \in S$. In other words, Ψ admits a continuous selection.

Michael's theorem has applications to diverse areas of mathematics, such as differential inclusions [2], control theory [7], mathematical economics [22], operator theory [39; 41], approximation theory [28; 54], and topology [5; 37]. Naturally, the following problem arises ([23, p. 651], [28, p. 349], and [54, p. 265]): If Ψ is Lipschitz continuous, is it possible to choose f to be Lipschitz also? This would be nice if it were true. For instance, Rademacher's theorem [13, 3.1.6] ensures that Lipschitz maps between finite-dimensional Banach spaces are differentiable almost everywhere—something not holding for arbitrary continuous maps. Note that this is equivalent to the problem:

For a given Banach space X, can we find a Lipschitz map $f: H(X) \to X$ satisfying $f(A) \in A$ for all $A \in H(X)$? In this section, we show that this question has a negative answer, for every infinite-dimensional Banach space. In fact, not even a uniform selector can be found.

Recall that a subspace is *complemented* if it is the range of a continuous linear projection. A Banach space is said to be *injective* if it is complemented in any Banach space which contains it. More precisely, it is said to be λ-injective if it is the range of a projection, with norm at most λ, from any superspace. It is easy to show that a complemented subspace of an injective space is again injective.

We will need the well-known result that if X is a closed sublattice of $C(U)$, for some compact Hausdorff space U, then X^{**} is 1-injective. Perhaps the easiest way to see this is as follows. Recall that X is said to have the κ-intersection property if, whenever B_i $(i \in I)$ is a collection of closed balls in X, with card $I \leq \kappa$ and $B_i \cap B_j \neq \emptyset$ for all $i, j \in I$, then $\bigcap_{i \in I} B_i \neq \emptyset$. It is easy to see that X has the κ-intersection property, for every finite κ, if X is a sublattice of $C(U)$. A clever duality argument [30, Thm. 2.16] then shows that X^{**} has the κ-intersection property for all finite κ. Hence, using weak* compactness, X^{**} has the binary intersection property—that is, the κ-intersection property for all cardinals κ. The usual proof of the Hahn–Banach extension theorem then shows that X^{**} is 1-injective.

Our next result is a slight improvement of [32, Cor. 5].

PROPOSITION 2.1. *If there is a uniform retract from $K(X)$ onto X, then X^{**} is injective.*

Proof. We regard $K(X)$ as a wedge in $C(U)$, where U is the unit ball of X^*. Then $Y = \overline{\text{linsp}\, K(X)}$ is a closed sublattice of $C(U)$. Proposition 1.4 and Lemma 1.1 then give us a linear map $T: Y \to X^{**}$ which fixes X. Passing to the second adjoint, we find that there is a projection from Y^{**} onto X^{**}. Since Y^{**} is injective, so is X^{**}. □

Proposition 2.1 shows that if X is, for example, one of the Banach spaces given by [26, Example 1], [31], or [33], then there is no uniform retract from $H(X)$ to X. These examples are not covered by [32, Cor. 5]. It also shows that there is no uniform retract $H(l_p) \to l_p$ for $1 \leq p < \infty$. Nonetheless, it was shown in [32, Thm. 8] that, for fixed $r > 0$, there is a uniformly continuous retract from $\Omega = \{A \in H(X): \text{diam}\, A \leq r\}$ onto X, provided X is uniformly convex.

PROPOSITION 2.2. *If X is an infinite-dimensional Hilbert space, then there is no uniform retract from $H(X)$ onto X.*

Proof. Recall that a Banach space Y has the Dunford–Pettis property if, whenever $f_n \to 0$ weak* in Y^* and $x_n \to 0$ weakly in Y, then $f_n(x_n) \to 0$. It is well known that any space $C(U)$ has the Dunford–Pettis property [10, p. 113], but that no infinite-dimensional Hilbert space has (consider an orthonormal sequence). Since the Dunford–Pettis property is clearly inherited by complemented subspaces, it follows that an infinite-dimensional Hilbert space cannot be injective. □

Let us say that (ϵ, δ) is a common modulus of continuity for a collection of functions $\mathfrak{F}$ if $\omega_f(\delta) < \epsilon$ for every $f \in \mathfrak{F}$.

THEOREM 2.3. *Let $f_n: H(\mathbf{R}^n) \to \mathbf{R}^n$ be any sequence of retracts. Then there is no modulus of uniformity common to every f_n.*

Proof. Suppose that (ϵ, δ) is a common modulus of uniformity. That is, for all n and all $A, B \in H(\mathbf{R}^n)$, we have $\|f_n(A) - f_n(B)\| < \epsilon$ whenever $d(A, B) < \delta$. We identify $\mathbf{R}^n$ with the subspace $\{(\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots): \lambda_i \in \mathbf{R}\}$ in l_2, and let $P_n: l_2 \to \mathbf{R}^n$ be the natural projection. Given an invariant mean on $l_\infty(\mathbf{N}, l_2)$, we define $f: H(l_2) \to l_2$ by $f(A) = \int_{\mathbf{N}} f_n(P_n(A))\, dn$. Routine calculations show that f is uniform, with modulus of uniformity (ϵ, δ), and that $f(\{x\}) = x$ for every $x \in l_2$. (Note that $\bigcap_{n=1}^{\infty} \overline{\text{co}\{P_k x: k > n\}}$ is just $\{x\}$.) But this contradicts Proposition 2.2. □

THEOREM 2.4. *Let X be any infinite-dimensional Banach space. Then there is no uniform selector $H(X) \to X$.*

Proof. Suppose that $f: H(X) \to X$ is a uniform selector, with modulus of uniformity (ϵ, δ). If Y is any closed subspace of X, then $f \mid H(Y)$ is a selector for Y, also with modulus of uniformity (ϵ, δ). But recall Dvortezky's theorem ([14] or [17]): Every infinite-dimensional Banach space contains almost isometric copies of $\mathbf{R}^n$ for every n. This implies that there is a common modulus of uniformity for selectors $H(\mathbf{R}^n) \to \mathbf{R}^n$, contrary to Theorem 2.3. □

It is natural to ask if Theorem 2.4 holds for retracts as well as selectors. In fact, it does not. For if X is injective then there is a Lipschitz continuous retract $H(X) \to X$. It is easy to see where the above proof breaks down for retracts. We suspect that the converse of Proposition 2.1 is true. Lindenstrauss [32] showed (amongst other things) that if $X = C(K)$, K any compact metric space, then there is a Lipschitz retract from $H(X)$ onto X.

3. A Finite-Dimensional Proof

We establish fairly sharp estimates for the Lipschitz constants of selectors $H(\mathbf{R}^n) \to \mathbf{R}^n$, thereby answering a question raised by Saint-Pierre [43, §7]. This gives us a direct proof of Theorem 2.3, independent of the functional analytic results in the previous section. In particular, we do not need to know about injectivity, the Dunford–Pettis property, or intersecting balls. However, we do need the Stone–Weierstrass theorem.

So let X be a Minkowski space—that is, a finite-dimensional Banach space. Let ∂U be the boundary of the unit ball of X^*. Clearly ∂U is a compact metric space. For $A \in K(X) = H(X)$, the restriction of h_A to ∂U will also be denoted by h_A. This should not lead to any ambiguity. As before, the mapping $K(X) \to C(\partial U)$, $A \mapsto h_A$, is a linear isometric embedding. It is obvious that $K(X)$ (in fact, the subset $\{h_{\{x\}} : x \in X\}$) separates points of ∂U. Furthermore, the constant function equal to 1 is the support function of the unit ball of X. Thus the sublattice linsp $K(X)$ is dense in $C(\partial U)$. The following result then follows immediately from Proposition 1.4 and Lemma 1.1.

PROPOSITION 3.1. *If X is finite-dimensional and $R: K(X) \to X$ is a uniform retract, then there is a linear projection $P: C(\partial U) \to X$ with $\|P\| \leq C_R$.*

Now let S^{n-1} denote, as usual, the unit sphere in the Euclidean space $\mathbf{R}^n$. We recall a result proved independently by Rutovitz [42] and Daugavet [9].

PROPOSITION 3.2. *The minimum norm, ranging over all projections from $C(S^{n-1})$ onto $\mathbf{R}^n$, is $2\Gamma(n/2+1)/\sqrt{\pi}\Gamma((n+1)/2)$.*

This immediately gives us a lower bound for the sub-Lipschitz constants of selectors from $H(\mathbf{R}^n) \to \mathbf{R}^n$. Since this lower bound is asymptotic to $\sqrt{2n/\pi}$, we have a somewhat simpler proof of Theorem 2.3. Theorem 2.4 follows as before, using Dvoretzky's theorem. It would be nice to have a proof that did not depend on Dvoretzky's theorem, but we have been unable to find one.

The projection of minimal norm from $C(S^{n-1})$ onto $\mathbf{R}^n$ is not unique, as we shall see in Section 5. One such projection can be written as $P(f) = n\int f(x)x\,d\sigma(x)$, where σ is the normalized Lebesgue measure on S^{n-1}. Let $O(n)$ denote the group of orthogonal transformations of $\mathbf{R}^n$. The measure σ is invariant under the action of $O(n)$. This fact implies that P is orthogonally invariant in the following sense: if we define f_u by $f_u(x) = f(u^*x)$ for $f \in C(S^{n-1})$ and $u \in O(n)$, then $P(f_u) = uP(f)$. Averaging over $O(n)$, we can see that, for any projection from $C(S^{n-1})$ onto $\mathbf{R}^n$, there is an orthogonally invariant projection whose norm is no greater. The crucial point in Daugavet's proof of the minimality of $\|P\|$ is that P is the unique orthogonally invariant projection from $C(S^{n-1})$ onto $\mathbf{R}^n$.

Naturally connected with P is the mapping $s: H(\mathbf{R}^n) \to \mathbf{R}^n$, defined by

$$s(A) = n\int h_A(x)x\,d\sigma(x).$$

In the notation of Lemma 1.1, we have $\tilde{s} = P$. The point $s(A)$ is known as the Steiner point of the convex body A. The Steiner point has been widely studied [20; 21; 35; 43; 44; 46]. As noticed by Shephard [45], s is a selector. We will refer to s as the Steiner selector in the sequel. Daugavet [9] gave a simple calculation to evaluate L_s, which was later rediscovered by Vitale [50]. Vitale also showed, in a certain sense, that there is no continuous extension of the Steiner point to infinite-dimensional Hilbert space.

We have just seen that there are no Lipschitz selectors $H(X) \to X$, when X is infinite-dimensional. The existence of the Steiner point ensures that there are Lipschitz selectors $H(X) \to X$ when X is finite-dimensional. It is appropriate now to consider some other well-known finite-dimensional selectors, and note that they are not Lipschitz continuous.

For $A \in H(X)$, the Chebyshev radius of A (denoted $\operatorname{rad} A$) is the infimum of those real numbers r for which A is contained in some ball, with centre in A and radius r. Clearly $\operatorname{rad} A \leq \operatorname{diam} A$. If $a \in A \subseteq B(a, \operatorname{rad} A)$, then a is said to be a *Chebyshev centre* of A. If A is weakly compact, the existence of at least one Chebyshev centre is assured; if X is strictly convex, each set can have at most one Chebyshev centre. Thus if X is finite-dimensional and strictly convex then each $A \in H(X)$ has a unique centre, which we will denote by $\zeta(A)$. It is routine to show that the map $A \mapsto \zeta(A)$ is continuous. However, even when X is a two-dimensional Hilbert space, simple examples [54, Lemma 3] show that this map is not uniformly continuous.

Provided that $A \in H(X)$ has nonempty interior, we may define its barycentre by $b(A) = (1/m(A)) \int_A x \, dm(x)$. Here m is the Lebesgue measure, calculated with respect to some basis of X. If we restrict our attention to sets with nonempty interior, this map can be shown to be continuous, but not uniformly continuous [47]. For sets with empty interior, we could define $b(A)$ with respect to Lebesgue measure on the subspace spanned by A. However, b, so extended to all of $H(X)$, is not continuous. (Consider the sequences of triangles in the Euclidean plane with vertices at $(0,0)$, $(1,0)$, and $(0,1/n)$.) Another approach to this problem is taken in [2], where a selector b_1 is defined by $b_1(A) = b(A + B(0,1))$. This selector is well defined and Lipschitz continuous on bounded subsets of $H(X)$, but not Lipschitz continuous on all of $H(X)$.

The minimal selector is the map which, for each $A \in H(X)$, chooses the unique element of A closest to the origin. This is well defined whenever A is weakly compact and X is strictly convex. Even in the finite-dimensional case, the minimal selector is not Lipschitz continuous [2, §1.7].

4. Selection Constants

We have seen, from Proposition 1.4 and its consequences, that there is a close relationship between linear and uniform selectors on $K(X)$, and their Lipschitz constants. This leads us to consider selection constants in a wider context.

We say that the mapping $s: K(X) \to X$ is a valuation if, for every $A, B \in K(X)$ such that $A \cup B \in K(X)$, the identity $s(A \cup B) + s(A \cap B) = s(A) + s(B)$ is satisfied. For a comprehensive survey of the topic of valuations, we refer to [35]. The next theorem is due to Spiegel [49].

THEOREM 4.1. *Let us suppose that $s: K(X) \to X$ is a continuous valuation which satisfies the following two conditions:*

(i) $s(A + x) = s(A) + x$ *for every* $A \in K(X)$, $x \in X$; *and*
(ii) $s(2A) = 2s(A)$ *for every* $A \in K(X)$.

Then s is linear.

In the other direction, it is easy to see that every linear mapping on $K(X)$ is a valuation. It is not hard to show that $A\cup B+A\cap B=A+B$ whenever $A, B, A\cup B\in K(X)$ (consider the corresponding identity for support mappings).

The family of all uniform selectors on $K(X)$ will be denoted by $\mathcal{S}(X)$. The subfamily of $\mathcal{S}(X)$ consisting of uniformly continuous valuations (resp., linear selectors) will be denoted by $\mathcal{S}_V(X)$ (resp., $\mathcal{S}_L(X)$). We obviously have $\mathcal{S}_L(X)\subset\mathcal{S}_V(X)\subset\mathcal{S}(X)$.

By the selection constant $S(X)$ of a Minkowski space X, we mean the infimum of the sub-Lipschitz constants of the elements of $\mathcal{S}(X)$. Similarly, we define the valuation selection constant $S_V(X)$ and the linear selection constant $S_L(X)$.

Recall that the projection constant $\lambda(X)$ of a Banach space X is the infimum of all numbers λ such that, whenever X is a subspace of another Banach space Z, then there is a projection of norm at most λ from Z onto X. By the upper projection constant $\bar{\lambda}(X)$, we mean $\sup\{\lambda(Y): Y \text{ is a subspace of } X\}$. By virtue of Dvoretzky's theorem and Proposition 3.2, $\bar{\lambda}(X)$ is finite only when X is finite-dimensional.

The principal purpose of this section is to show that, for any Minkowski space X, $\bar{\lambda}(X)\leq S(X)\leq S_V(X)=S_L(X)$. These relations are also true, but of little consequence, for infinite-dimensional Banach spaces.

LEMMA 4.2. *If the selector* $s: K(X)\to X$ *is a uniformly continuous valuation, then there is a linear selector* $\bar{s}: K(X)\to X$ *with* $L_{\bar{s}}\leq C_s$.

Proof. For fixed $x\in X$ and $n\in\mathbf{N}$, let us define a selector $f(n,x): K(X)\to X$ by $f(n,x)(A)=n^{-1}(s(nA+x)-x)$. From the identities

$$y+(B\cup C)=(y+B)\cup(y+C) \quad\text{and}\quad y+(B\cap C)=(y+B)\cap(y+C),$$

it follows that $f(n,x)$ is a valuation.

Let us regard $\mathbf{N}$ as a semigroup under multiplication. Given invariant means on $l_\infty(X,X)$ and $l_\infty(\mathbf{N},X)$, we define $\bar{s}$ by $\bar{s}(A)=\int_{\mathbf{N}}\int_X f(n,x)(A)\,dx\,dn$. One can easily check that $\bar{s}$ is both a selector and a valuation, and that $\bar{s}(A+x)=\bar{s}(A)+x$. From the $\mathbf{N}$-translation invariance, we see that $\bar{s}(kA)=k\bar{s}(A)$ for all $k\in\mathbf{N}$. Straightforward calculations show that $d(\bar{s}A,\bar{s}B)\leq d(sA,sB)$ for all $A,B\in K(X)$. It follows from Spiegel's theorem that $\bar{s}$ is linear and that $L_{\bar{s}}\leq C_s$. □

LEMMA 4.3. *If* Y *is a subspace of* X, *then* $\lambda(Y)\leq S(X)$.

Proof. Let $s\in\mathcal{S}(X)$. We show that $\lambda(Y)\leq C_s$.

Clearly the restriction of s to $K(Y)$ is a selector $K(Y)\to Y$. Proposition 3.1 then gives us a linear projection $P: C(\partial U)\to Y$ (where U denotes the unit ball of Y^*), with $\|P\|\leq C_S$. It is known [8, Thm. 9] that any such projection satisfies $\|P\|\geq\lambda(Y)$. □

Combining Lemmas 4.2 and 4.3 yields the result we claimed earlier.

THEOREM 4.4. *For any Minkowski space X,*

$$\bar{\lambda}(X) \leq S(X) \leq S_V(X) = S_L(X).$$

From Section 3, we have

$$s_L(\mathbf{R}^n) = \bar{\lambda}(\mathbf{R}^n) = \lambda(\mathbf{R}^n) = \frac{2\Gamma(n/2+1)}{\sqrt{\pi}\Gamma((n+1)/2)} < \sqrt{(2/\pi)(n+1)}.$$

This gives us sharp lower bounds for the sub-Lipschitz constants of selectors on $\mathbf{R}^n$. Taking Proposition 3.1 into account, we obtain even more.

PROPOSITION 4.5. *For any retract $R: K(\mathbf{R}^n) \to \mathbf{R}^n$, we have $C_R \geq \lambda(\mathbf{R}^n)$, and this estimate is sharp.*

We do not know, amongst all Lipschitz continuous selectors $K(\mathbf{R}^n) \to \mathbf{R}^n$, whether only the Steiner selector has its Lipschitz constant equal to $\lambda(\mathbf{R}^n)$. The answer to this question is negative for retracts, as we shall see in Section 5.

For a general Minkowski space X, such simple estimates are not available. Let us define $k(X)$ to be the maximum of those $m \in \mathbf{N}$ for which there is an m-dimensional subspace Y in X with $d(Y, \mathbf{R}^m) \leq 2$. Here $d(Y, Z)$ denotes, as usual, $\inf\|T\| \cdot \|T^{-1}\|$, where T ranges over the isomorphisms between Y and Z.

PROPOSITION 4.6. *There is an absolute constant K such that, for every n-dimensional normed space X,*

$$K\sqrt{\log n} \leq S(X) \leq S_L(X) = S_V(X) < \min\{n, \sqrt{(2/\pi)}(n+1)\}.$$

Proof. From Dvoretzky's theorem [14, Thm. 4.4], there is an absolute constant c such that $k(X) \geq c\log(\dim X)$ for all finite-dimensional X. Given X, let Y be a subspace with $d(Y, \mathbf{R}^{k(X)}) \leq 2$. Routine calculations show that, for $K = \sqrt{c/2\pi}$,

$$\begin{aligned} 2K\sqrt{\log n} \leq \sqrt{2k(X)/\pi} < \lambda(\mathbf{R}^{k(X)}) = S(\mathbf{R}^{k(X)}) &\leq S(Y)d(Y, \mathbf{R}^{k(X)}) \\ &\leq 2S(Y) \leq 2S(X). \end{aligned}$$

This gives us the left inequality.

For the right inequality, recall John's theorem [25]: Given any n-dimensional X, we have $d(X, \mathbf{R}^n) \leq \sqrt{n}$. Thus

$$S_L(X) \leq S_L(\mathbf{R}^n)d(X, \mathbf{R}^n) \leq \lambda(\mathbf{R}^n)\sqrt{n} < n. \qquad \square$$

We feel that the left inequality is not at all sharp. Probably $S_L(X)$ and $S(X)$ are of the order of $\sqrt{\dim X}$. However, the right inequality seems to be reasonably sharp.

For a concrete example, let us recall the well-known result that the n-dimensional l_p space satisfies $d(l_p^n, l_2^n) = n^{|1/2-1/p|}$. Thus there are constants $a, b > 0$ such that, for all n,

$$an^{\min(1/p, 1/q)} \leq S(l_p^n) \leq S_L(l_p^n) \leq bn^{\max(1/p, 1/q)}.$$

We finish this section with a generalization of our main result.

PROPOSITION 4.7. *For a normed space X, let $\Omega(X)$ denote the family of finite-dimensional polytopes in X (i.e., the convex hulls of all finite subsets of X). Suppose that there is a uniform retract $R: \Omega(X) \to X$ such that $R(A) \in \operatorname{linsp} A$ for every $A \in \Omega(X)$. Then X is finite-dimensional.*

We leave the proof of this result to the reader.

5. Extensions of Lipschitz Maps

There is a natural connection between extension and selection problems. Indeed, it is easy to see that every extension problem can be formulated as a selection problem [37]. We will now use some ideas from previous sections to consider various cases of the general extension problem: Given metric spaces $S \subset T$ and a Banach space X, does every Lipschitz function $f: S \to X$ admit a Lipschitz extension $g: T \to X$?

This is so whenever X has the binary intersection property, and we actually have $L_g = L_f$ [1]. When $T \setminus S$ is a singleton, this is easy to see; the general case then follows from a routine application of Zorn's lemma. The Kirszbraun–Valentine theorem asserts that the same conclusion holds when X and T are both Hilbert spaces [51], but the proof is more difficult.

No general extension theorem is possible, because $\lambda(\mathbf{R}^n) \to \infty$ with n. However, further extension results are available, with an increase of Lipschitz constant, if some "finiteness" assumptions are made.

For example, such results are obtained in [27] under the assumption either that S is finite, or that T is a finite-dimensional normed space. The magnitude of the increase in the Lipschitz constant depends on card S, or $\dim T$, respectively.

Here we observe that similar results hold under the assumption that X is injective. This follows from the known result for the case $X = l_\infty(\Gamma)$. However, we are able to give sharp estimates of the increase in the Lipschitz constant, with a constructive proof, in the special case $X = \mathbf{R}^n$. When $n = 1$ this result is well known [4; 36; 52], and the idea of our proof can be traced back to these early works.

PROPOSITION 5.1. *Let T be a metric space, S a subset of T, and X an injective (in particular, finite-dimensional) Banach space. Then every Lipschitz map $f: S \to X$ admits a Lipschitz extension $g: T \to X$, with $L_g \le \lambda(X) L_f$. Furthermore, this estimate is the best possible.*

Proof. We may embed X isometrically into $l_\infty(\Gamma)$, where Γ is a sufficiently large set. Since $l_\infty(\Gamma)$ has the binary intersection property, f admits an extension $f': T \to l_\infty(\Gamma)$, with the same Lipschitz constant. Now let $P: l_\infty \to X$ be a projection with $\|P\| = \lambda(X)$. We take $g = P \circ f'$ and the proof is complete. □

Proposition 5.1 also holds (suitably reformulated) for uniform mappings. This proof is, of course, nonconstructive. In the special case $X=\mathbf{R}^n$, a constructive proof is available. Given f, S, T as above, we define g by

$$g(t)=\int_{S^{n-1}} \sup_{s\in S}[\langle f(s),x\rangle - L_f d(s,t)]x\, d\sigma(x),$$

where σ is as in Section 3. It is routine to verify that g is well defined, and has the required properties.

Proposition 5.1 enables us to verify the claims made earlier, that Steiner projection (retract) is not the only projection (retract) from $C(S^{n-1})$ (from $K(\mathbf{R}^n)$) onto $\mathbf{R}^n$ with norm (Lipschitz constant) equal to λ_n.

PROPOSITION 5.2. *Let X be a finite-dimensional Banach space. For each $y\in X$ with $\|y\|\le 1$, there is a projection $P: C(\partial U)\to X$ with $\|P\|=\lambda(X)$ and $P(1)=y$.*

Proof. Let $\mathcal{B}(X)$ denote the collection of closed balls in X. Clearly $\mathcal{B}(X)$ is a subsemigroup of $K(X)$. Defining $S: \mathcal{B}(X)\to X$ by $S(B(x,r))=x+ry$, it is easy to check that S is nonexpansive. By Proposition 5.1, we may extend S to all of $K(X)$, so that $L_S\le\lambda_n$. Now define $T: K(X)\to X$ by

$$T(C)=\int_{K(X)}\int_{\mathcal{B}(X)} S(A+B+C)-S(A+B)\, dB\, dA,$$

where, as usual, the integral signs denote suitable invariant means. As in the proof of Proposition 1.4, T is linear, $T(B(x,r))=x+ry$ for all x and r, and $L_T\le\lambda_n$. Lemma 1.1 completes the proof. □

The proof of the next result should be clear by now, using the result of Isbell mentioned in the introduction.

COROLLARY 5.3. *Let $S\subset T$ be any metric spaces, X any Banach spaces. Then every Lipschitz continuous map $f: S\to H(X)$ can be extended to a map $g: T\to H(X)$ with $L_g<11L_f$. In the case when X is one-dimensional, we may obtain $L_g=L_f$.*

For the special case when T is a Hilbert space and $X=\mathbf{R}^n$, a version of Corollary 5.4 was given by Bressan and Cortesi [6]. Their proof is quite different, using the Kirszbraun–Valentine theorem.

6. Some Generalized Selectors

Let us review the basic properties of the Steiner point. For any $A\in K(\mathbf{R}^n)$, its support function $h_A\in C(S^{n-1})$ is easily shown to be Lipschitz continuous and hence, by Rademacher's theorem, differentiable almost everywhere. As in [46], the Gauss–Green formula [13, 4.5.6] gives us

$$s(A) = \frac{1}{V(B_n)} \int_{\partial B_n} h_A(x) n(x)\, d\sigma(x) = \frac{1}{V(B_n)} \int_{B_n} \nabla h_A(x)\, dx,$$

where B_n denotes the unit ball of $\mathbf{R}^n$, $n(x)$ denotes the unit outward normal to B_n at x, and $V(\cdot) = V_n(\cdot)$ denotes the volume function in $\mathbf{R}^n$. The left equality shows that $s(A)$ is a Lipschitz continuous function of A. Elementary convex analysis shows that $\nabla h_A(x) \in A$. Hence the right equality shows that $s(A) \in A$. Thus, the properties of the Steiner point in which we are most interested follow solely from the identity above. A moment's thought shows that this formula can be generalized to obtain other Lipschitz selectors. Such selectors will be the subject of this section.

Let X be any Minkowski space, not necessarily Euclidean. In what follows, B^* is the unit ball of X^*, and the volumes, surface measures and outward normals are calculated with respect to an arbitrary predetermined Euclidean basis for X^*. For any $A \in K(X)$ we have, by the same reasoning as before,

$$(\triangle) \qquad \frac{1}{V(B^*)} \int_{\partial B^*} h_A(x) n(x)\, d\sigma(x) = \frac{1}{V(B^*)} \int_{B^*} \nabla h_A(x)\, dx.$$

Hence $(\triangle)$ defines a Lipschitz continuous selector $K(X) \to X$, which we shall denote by $s_X(A)$. Our first task will be to investigate the Lipschitz constant for this selector. Of course, this will be equal to the norm of the projection $P\colon C(\partial B^*) \to X$ defined by $P(f) = (1/V(B^*)) \int_{\partial B^*} f(x) n(x)\, d\sigma(x)$.

We recall from [53, Ch. 9] that a base norm space is a Banach space whose unit ball is the (closed) convex hull of two opposite faces. An order unit space is a Banach space equipped with a vector ordering, such that the unit ball is an order interval. For finite-dimensional spaces, these concepts are in complete duality: A finite-dimensional space is an order unit space if and only if its dual is a base norm space.

THEOREM 6.1. *For an n-dimensional Minkowski space X, the selector s_X just defined has Lipschitz constant $L_{s_X} \le n$. Furthermore, equality holds if and only if X is a base norm space.*

Proof. Let us denote by p the Euclidean norm on X^*, and by $d(0, \text{face}(x))$ the Euclidean distance from the origin to the largest face of B^* containing x. We note that $\|x\| = h_B(x)$ whenever $x \in X^*$, and so

$$n(x) = (p(\nabla h_B(x)))^{-1} \nabla h_B(x)$$

whenever it exists. Thus

$$\begin{aligned} V(B^*)\|P\| &= \sup_{\|f\| \le 1} \left\| \int_{\partial B^*} f(x) n(x)\, d\sigma(x) \right\| \\ &= \sup_{\|f\| \le 1} \sup_{u \in \operatorname{ext} B^*} \left| \int_{\partial B^*} f(x) u(n(x))\, d\sigma(x) \right| \\ &= \sup_{u \in \operatorname{ext} B^*} \sup_{\|f\| \le 1} \left| \int_{\partial B^*} f(x) u(n(x))\, d\sigma(x) \right| = \end{aligned}$$

$$= \sup_{u \in \text{ext } B^*} \int_{\partial B^*} |u(n(x))| \, d\sigma(x)$$

$$= \sup_{u \in \text{ext } B^*} \int_{\partial B^*} |u(\nabla h_B(x))| \frac{1}{p(\nabla h_B(x))} \, d\sigma(x)$$

$$\leq \int_{\partial B^*} \frac{1}{p(\nabla h_B(x))} \, d\sigma(x).$$

Now if X is a polyhedral space, and x is a relative interior point of some $(n-1)$-dimensional face of ∂B^*, then it is easy to see that $d(0, \text{face}(x)) = 1/p(\nabla h_B(x))$. The set of points in ∂B^* which belong to two or more maximal faces has measure zero, and can be ignored. Thus

$$\begin{aligned}
\int_{\partial B^*} \frac{1}{p(\nabla h_B(x))} \, d\sigma(x) &= \int_{\partial B^*} d(0, \text{face}(x)) \, d\sigma(x) \\
&= \sum_{\text{faces } F} \int_F d(0, \text{face}(x)) \, d\sigma(x) \\
&= \sum_{\text{faces } F} V_{n-1}(F) \, d(0, F) \\
&= \sum_{\text{faces } F} n V_n(\text{co}(F \cup \{0\})) \\
&= n V_n(B^*).
\end{aligned}$$

A standard approximation argument then shows that

$$\int_{\partial B^*} d(0, \text{face}(x)) \, d\sigma(x) = n V_n(B^*),$$

even when X is not polyhedral. It now follows that $V(B^*) \|P\| \leq n V_n(B^*)$, as required.

Equality occurs precisely when there exists a $u \in X^*$ with $|u(\nabla h_B(x))| = 1$ for almost all x. (For, it is enough to observe that the supremum over ext B^* dealt with above is actually a maximum.) This means that, for each face F of B^*, either $u \in F$ or $-u \in F$. In other words, X^* must be an order unit space. □

Using a similar argument, we see that

$$\int_{\partial B^*} |u(n(x))| \, d\sigma(x) = 2p(u) V_{n-1}(Q_u(B^*)),$$

where Q_u denotes the projection onto the subspace orthogonal to u. This leads to the result that

$$\|P\| = \sup_{u \in \text{ext } B^*} \frac{2p(u) V_{n-1}(Q_u(B^*))}{V_n(B^*)},$$

which makes the calculation of $\|P\|$ simpler in some special cases.

It is natural to investigate L_{s_X} for the common finite-dimensional spaces. When $X=l_1^n$ or $X=l_\infty^n$, we have $L_{s_X}=n$, as we do for every other base norm space. For $X=l_2^n$, we obviously have $L_{s_X}=\lambda(\mathbf{R}^n)$. Next we investigate the case of two-dimensional spaces, whose unit balls are regular polygons.

PROPOSITION 6.2. *Suppose that X is two-dimensional, and that its unit ball is a regular m-gon. If m is divisible by 4, then $L_{s_X}=8/(m\sin(2\pi/m))$. If m is not divisible by 4, then $L_{s_X}=4/(m\sin(\pi/m))$.*

Proof. This follows easily from the remarks above. □

Let X be two-dimensional, with a regular hexagon for its unit ball. Then, taking note of [19], we have $\lambda(X)=s_L(X)=\frac{4}{3}$.

This new selector we have defined on finite-dimensional spaces suggests a method for defining a selector on the finite-dimensional sets in $K(X)$, even when X is infinite-dimensional. We could simply define $s(A)$ to be $s_X(A)$, where X is the finite-dimensional subspace spanned by A. (It can be shown that the right side of ($\triangle$), and hence also the left side, is independent of the choice of scalar product on X. Thus the selector s_X depends only on the affine structure of the unit ball of X.) Unfortunately, this is not a consistent definition, as the following example shows.

Let $X=l_1^3$, let A be the convex hull of $(0,0,0)$, $(1,1,0)$ and $(0,0,1)$, and let Y be the linear span of A. Straightforward computations now show that $s_Y(A)=\frac{1}{16}(7,7,5)$, whereas $s_X(A)=\frac{1}{48}(19,19,17)$.

Of course, for Euclidean spaces, the Steiner point is consistent. That is, when $X=\mathbf{R}^n$, $s(A)$ may be calculated with respect to any subspace of X containing A.

We finish this section by introducing another Lipschitz selector in two dimensions. It does not seem to generalize well to higher dimensions. Given $A\in H(\mathbf{R}^2)$, we denote by $s_0(A)$ the centre of the smallest rectangle containing A which has its sides parallel to the coordinate axes. Elementary geometry shows that $s_0(A)\in A$. In fact s_0 satisfies the formula

$$s_0(A)=\tfrac{1}{2}(h_A(e_1)-h_A(-e_1))e_1+\tfrac{1}{2}(h_A(e_2)-h_A(-e_2))e_2,$$

and so s_0 is a Lipschitz continuous selector. For l_∞^2, s_0 is a selector with the least possible Lipschitz constant: $\lambda(l_\infty^2)=s_L(l_\infty^2)=1$. Similarly, one can obtain a selector corresponding to any nondegenerate parallelogram.

PROPOSITION 6.3. *Given linearly independent, norm-one elements $x,y\in\mathbf{R}^2$, let us put $k(x,y)=(1-\langle x,y\rangle^2)^{-1}(x-\langle x,y\rangle y)$. Then the identity*

$$s_{x,y}(A)=\tfrac{1}{2}(h_A(x)-h_A(-x))k(x,y)+\tfrac{1}{2}(h_A(y)-h_A(-y))k(y,x)$$

defines a Lipschitz continuous selector from $H(\mathbf{R}^2)$ to $\mathbf{R}^2$. Geometrically, $s_{x,y}(A)$ is the centre of the smallest parallelogram with sides perpendicular to x and y.

7. A Positive Result

We have seen that the most simple-minded uniform version of Michael's theorem fails to hold. This represents not the end but rather the beginning of this field of research. Another interesting question is: Given a reasonable subset Ω of $H(X)$, is there a uniformly continuous selector $\Omega \to X$?

Frolík et al. [15] gave some results about uniformly continuous selections, within the general context of uniform spaces. When their results are specialized to Banach spaces, they are not very interesting.

As remarked in Section 2, Lindenstrauss [32, Thm. 8] showed that there is a uniformly continuous retract, when X is uniformly convex, and that $\Omega = \{A \in H(X): \operatorname{diam} A \leq r\}$ for some $r > 0$. Skaletskii [48] improved this by establishing the existence of a selector, whilst also weakening the first assumption to "X has uniformly normal structure". In fact, Skaletskii gave a more technical result concerning Fréchet spaces. Rather than deal with that level of generality, we present a variation of Skaletskii's argument for the case of Banach spaces.

Recall from Section 3 the definition of $\operatorname{rad} A$. Then X is said to have uniformly normal structure if $\sup\{\operatorname{rad} A/\operatorname{diam} A: A \in H(X), A \text{ infinite}\}$ is strictly less than 1. It is well known that every uniformly convex space has uniformly normal structure. Following Skaletskii, we will say that a subset Ω of $H(X)$ has uniformly normal structure if $\sup\{\operatorname{rad} A/\operatorname{diam} A: A \in \Omega, A \text{ infinite}\} < 1$.

If Ω does have uniformly normal structure, we can find $\alpha > 0$ such that

$$g_A(t) = A \cap \bigcap_{x \in A} B(x, (1-\alpha t) \operatorname{diam} A)$$

is nonempty for all $A \in \Omega$, $t \leq 1$. Note that, for each A, $g_A: (-\infty, 1] \to H(X)$ is a monotonic function. Before proving anything, we shall need to recall the integration theory for such functions. This was developed first by Isbell [24], who used it to show that $H(X)$ is the range of a uniformly continuous retract on any metric space which contains it.

Note that $H(X)$ is a complete metric space, and is equipped with a partial order—namely, that given by inclusion. The metric, order, and wedge structures are related by an order unit: an element $e \in H(X)$ such that $d(a, b) \leq r$ if and only if $a \leq b + re$ and $b \leq a + re$. (This is consistent with the definition of order units given in §6.) In this case, the order unit is just the unit ball, $e = B(0, 1)$. These properties of $H(X)$ are just what one needs to set up a rudimentary integration theory. Let $f: [0, 1] \to H(X)$ be an increasing function. For any partition $\alpha: 0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$, we may define the upper sum $U_\alpha = \sum_{i=1}^n (t_i - t_{i-1}) f(t_i)$ and also the lower sum $L_\alpha = \sum_{i=1}^n (t_i - t_{i-1}) f(t_{i-1})$, as well as the mesh of α, $|\alpha| = \max_i |t_i - t_{i-1}|$. One may easily check that $L_\alpha \leq U_\alpha \leq L_\alpha + |\alpha|(f(1) - f(0))$, and so $d(U_\alpha, L_\alpha) \leq |\alpha| d(f(1), f(0))$. If β is any partition refining α, then $L_\alpha \leq L_\beta \leq U_\beta \leq L_\alpha$.

It follows that (U_α) and (L_α) are Cauchy nets, with a common limit, which we denote by $\int_0^1 f$ or $\int_0^1 f(t)\,dt$. Standard arguments show that this elementary Riemann integral has all the properties expected of it. One could investigate the integrability of nonmonotonic functions, but this is not necessary for our purposes.

LEMMA 7.1. *In the notation just established,*

$$d\left(\int_0^1 g_A, \int_0^1 g_C\right) \le \left(1+\frac{3}{\alpha}\right) d(A, C)$$

whenever $A, C \in \Omega$ *and* $d(A, C) < (\alpha/3)\min(\operatorname{diam} A, \operatorname{diam} C)$.

Proof. By symmetry, it suffices to show that

$$\int_0^1 g_A \subseteq \int_0^1 g_C + B\left(\delta\left(1+\frac{3}{\alpha}\right)\right),$$

where $\delta = d(A, C)$. We put $\theta = 3\delta/(\alpha \operatorname{diam} C)$, so $0<\theta<1$. Note that, for any subset S of X and any $k>0$,

$$\bigcap_{x\in S} B(x, k) \cap [C + B(0,\delta)] \subseteq \bigcap_{x\in S} [B(x, k+\delta)\cap C] + B(0,\delta).$$

Thus

$$\begin{aligned} g_A(t) &= A \cap \bigcap_{x\in A} B(x, (1-\alpha t)\operatorname{diam} A) \\ &\subseteq [C+B(0,\delta)] \cap \bigcap_{x\in C} B(x, (1-\alpha t)(\operatorname{diam} C+\delta)+\delta) \\ &\subseteq \bigcap_{x\in C} [B(x, (1-\alpha t)(\operatorname{diam} C+\delta)+2\delta)\cap C] + B(0,\delta) \\ &\subseteq \bigcap_{x\in C} [B(x, (1-\alpha(t-\theta))\operatorname{diam} C)\cap C] + B(0,\delta) \\ &= g_C(t-\theta) + B(0,\delta). \end{aligned}$$

For any $a<b<c$, the identity $\int_a^c f = \int_a^b f + \int_b^c f$ leads to the inclusion $\int_a^b f \subseteq \int_a^c f - \int_b^c f$. Then

$$\begin{aligned} \int_0^1 g_A &\subseteq \int_{-\theta}^{1-\theta} g_C + B(0,\delta) \\ &\subseteq \int_0^1 g_C + \int_{-\theta}^0 g_C + \int_{1-\theta}^1 (-g_C) + B(0,\delta) \\ &\subseteq \int_0^1 g_C + \int_{-\theta}^0 C + \int_{1-\theta}^1 (-C) + B(0,\delta) \\ &= \int_0^1 g_C + \theta(C-C) + B(0,\delta) \\ &\subseteq \int_0^1 g_C + B(0, \theta \operatorname{diam} C) + B(0,\delta) = \end{aligned}$$

$$= \int_0^1 g_C + B\left(0, \delta\left(1+\frac{3}{\alpha}\right)\right),$$

as required. □

THEOREM 7.2. *Let Ω be a convex subset of $H(X)$, with uniformly normal structure. Suppose also that $A \in H(X)$, $B \in \Omega$, and $A \subset B \Rightarrow A \in \Omega$. Then there is a selector $F: \Omega \to X$ with the property that, for each $r > 0$, the restriction of F to $\{A \in \Omega: \operatorname{diam} A \le r\}$ is uniformly continuous.*

Proof. We define $f: \Omega \to \Omega$ by $f(A) = \int_0^1 g_A$. Note that $f(A) \subseteq A$. From Lemma 7.1 and the convexity of Ω, it follows that f is Lipschitz continuous. Furthermore,

$$\operatorname{diam} f(A) \le \int_0^1 \operatorname{diam} g_A \le \int_0^1 (1-\alpha t) \operatorname{diam} A \, dt = \left(1 - \frac{\alpha}{2}\right) \operatorname{diam} A.$$

Then $\operatorname{diam} f^n(A) \le (1-\alpha/2)^n \operatorname{diam} A \to 0$, and so $F(A) = \bigcap_{n=1}^{\infty} f^n(A)$ is a singleton set. We may regard F as a selector from Ω to X. Now

$$d(F(A), f^n(A)) \le \operatorname{diam} f^n(A) \le (1-\alpha/2)^n \operatorname{diam} A,$$

and so $f^n(A) \to F(A)$ uniformly on $\{A \in \Omega: \operatorname{diam} A \le r\}$. Since each f^n is Lipschitz continuous, F must be uniformly continuous on $\{A \in \Omega: \operatorname{diam} A \le r\}$. □

We remark that convexity of Ω is not needed in the last theorem, but the proof is more difficult in the general case. As an example, Skaletskii showed that for fixed n, $\Omega_n = \{A \in H(X): \dim A < n\}$ has uniformly normal structure. In fact, it follows easily from Helly's theorem [29] that rad A/diam $A \le 1 - 1/n$ for each $A \in \Omega_n$.

COROLLARY 7.3. *If X is isomorphic to a Banach space with uniformly normal structure, then there is a selector $H(X) \to X$ which is uniformly continuous on each of the sets $\{A \in H(X): \operatorname{diam} A \le r\}$.*

Proof. If X has uniformly normal structure, then this is a special case of Theorem 7.2. But the conclusion here is unaffected if we put an equivalent norm on X. □

Bae [3] and Maluta [34] independently observed that uniform normal structure implies reflexivity. It has been conjectured that uniform normal structure implies superreflexivity. It is worth noting that every superreflexive space has an equivalent uniformly convex norm [12]. Thus far, no example is known of a Banach space which satisfies the hypotheses of Corollary 7.3 but which is not already uniformly convex.

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