COMPLEMENTATION OF PRINCIPAL IDEALS IN WEIGHTED (FN)-ALGEBRAS OF ENTIRE FUNCTIONS

Jürgen Wolter

Introduction. In this paper we study (FN)-algebras

$$A_p^0(\mathbf{C}^N) := A_p^0 := \{f \in H(\mathbf{C}^N) \mid \text{for all } D>0\colon \sup_{z\in\mathbf{C}^N} |f(z)|e^{-Dp(z)} < \infty\}$$

for nonnegative plurisubharmonic functions p such that $\log(1+|z|^2) = o(p(z))$. For the sake of simplicity we will assume in the introduction that $p(z) = |z|^\alpha$, $\alpha > 0$. Then each principal ideal $I(G)$ generated by $G \in A_p^0$ is closed in A_p^0. Our main result is the following.

THEOREM. *For $G \in A_p^0(\mathbf{C}^2)$ and $G(z,w) := f(z) - \sum_{i+j\le m} a_{i,j} z^i w^j$ ($m \in \mathbf{N}$, $a_{i,j} \in \mathbf{C}$, $a_{0,m} \neq 0$), the following statements are equivalent:*

(1) *f is a polynomial*;
(2) *$I(G)$ is complemented in $A_p^0(\mathbf{C}^2)$ (i.e., there exists a continuous linear projection on $A_p^0(\mathbf{C}^2)$ with range $I(G)$)*;
(3) *for some $H \in A_p^0(\mathbf{C}^2)\setminus\{0\}$ the ideal $I(GH)$ is complemented in $A_p^0(\mathbf{C}^2)$.*

For $\alpha > 1$, the A_p^0 are isomorphic to the strong duals of weighted (DFN)-spaces of entire functions by Fourier–Borel transformation, the pointwise multiplication carried over to form the convolution product. Therefore, the results of this paper imply that certain convolution operators have no continuous linear right inverses.

The main point of our theorem is (2) ⇒ (1). To prove it, we assume that f is not a polynomial. Note then that $A_p^0(\mathbf{C}^2)$ is a power series space of infinite type. Hence by a theorem of Zahariuta [27], $A_p^0(\mathbf{C}^2)$ cannot contain a subspace isomorphic to a power series space of finite type. We shall find such a subspace E in $A_p^0(\mathbf{C}^2)/I(G)$. However, (2) would imply that $A_p^0(\mathbf{C}^2)/I(G)$, and hence E, are subspaces of $A_p^0(\mathbf{C}^2)$.

We must find E only in the case $G(z,w) = f(z) - w$, as the others follow by a substitution argument. For such G we have a canonical isomorphism of locally convex algebras $A_p^0(\mathbf{C}^2)/I(G) \to A_q^0(\mathbf{C})$, with $q(z) := p(z, f(z))$ by a variant of an interpolation theorem of Berenstein and Taylor [3].

An extension of results of Meise and Taylor [14] shows that, for certain closed ideals J in $A_q^0(\mathbf{C})$, the quotient $A_q^0(\mathbf{C})/J$ is a power series space of finite type. The essential step in the proof is to find such an ideal J that is complemented in $A_q^0(\mathbf{C})$. To do this, we construct first a sequence of subharmonic functions and use them, together with Hörmander's L^2-theory of the $\bar\partial$-operator, to find for a certain ideal J a Schauder basis of a complement $E \cong A_q^0(\mathbf{C})/J$.

Received December 4, 1987. Revision received August 1, 1988.
Michigan Math. J. 35 (1988).

(1) $\Rightarrow$ (2) was proved by Djakov and Mityagin [4], and (2) $\Leftrightarrow$ (3) is an easy calculation involving the transposes of the operators $f \mapsto Gf$ and $f \mapsto Hf$ on $A_p^0(\mathbf{C}^2)$.

In Section 1 we gather basic definitions and the needed results on interpolation and division with entire functions of restricted growth. Section 2 serves to construct the J in a given $A_q^0(\mathbf{C})$, while the rest of the theorem's proof is carried out in Section 3. In Section 4 we outline its consequences for convolution operators on (DFN)-algebras of entire functions.

The author thanks the Studienstiftung des deutschen Volkes for financial support and Prof. R. Meise for encouragement and guidance during the work on his thesis [26], of which this article presents a part.

1. Preliminaries. In this section we introduce weight functions, weighted (FN)-algebras of entire functions, and some results about interpolation and division that will be used in the following sections.

DEFINITION 1.1. We call $p: \mathbf{C}^N \to [0, \infty)$ $(N \in \mathbf{N})$ a *projective* [resp. *inductive*] *weight function* if it has the following properties:

(1) p is plurisubharmonic;

(2) there exist $r: \mathbf{C}^N \to [0, \infty)$, $r = o(p)$ [resp. $r = O(p)$], and $C < \infty$ such that for all $\xi, z \in \mathbf{C}^N$ with $\log|z-\xi| \leq -r(z)$ we have $p(\xi) \leq Cp(z) + C$;

(3) $\log(1+|z|^2) = o(p(z))$ [resp. $\log(1+|z|^2) = O(p(z))$].

We call p *componentwise radial* if $p(z_1, \ldots, z_N) = p(|z_1|, \ldots, |z_N|)$ for all $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$. We call it *radial* if p depends only on the Euclidean norm of z.

DEFINITION 1.2. For an (inductive or projective) weight function

$$p: \mathbf{C}^N \to [0, \infty)$$

and for $f \in H(\mathbf{C}^N)$ we define:

$$\|f\|_{D,p} := \sup_{z \in \mathbf{C}^N} |f(z)| e^{-Dp(z)} \quad \text{and}$$

$$|f|_{D,p} := \left(\int_{\mathbf{C}^N} |f|^2 e^{-2Dp} \, d\lambda \right)^{1/2}$$

(where λ denotes the Lebesgue measure on $\mathbf{C}^N$);

$$B_{D,p} := \{ f \in H(\mathbf{C}^N) \mid \|f\|_{D,p} < \infty \} \quad \text{and}$$

$$W_{D,p} := \{ f \in H(\mathbf{C}^N) \mid |f|_{D,p} < \infty \}.$$

If p is projective, we set

$$A_p^0(\mathbf{C}^N) := A_p^0 := \{ f \in H(\mathbf{C}^N) \mid \text{for all } D > 0,\ |f|_{D,p} < \infty \};$$

if p is inductive,

$$A_p(\mathbf{C}^N) := A_p := \{ f \in H(\mathbf{C}^N) \mid \text{there is } D < \infty \text{ with } |f|_{D,p} < \infty \}.$$

Compare 1.1 to [6, Thm. 1]. The next proposition follows by standard arguments, which we omit.

PROPOSITION 1.3. *The following statements hold:*

(1) *with the topology induced by* $|\cdot|_{D,p}$, $W_{D,p}$ *is a Hilbert space*;

(2) *there exists* $K(p) < \infty$ *and, for all* D, $C(D,p), \tilde{C}(D,p) < \infty$ *such that*

$$\|\cdot\|_{DK(p),p} \le C(D,p)|\cdot|_{D,p} \quad and \quad |\cdot|_{D,p} \le \tilde{C}(D,p)\|\cdot\|_{D/2,p}$$

on A_p^0 *for all* D;

(3) *as a projective limit of the* $W_{D,p}$ $(D>0)$ (*or, equivalently, of the* $B_{D,p}$ $(D>0)$), A_p^0 *is a nuclear Fréchet space*;

(4) A_p^0 *with this topology is a locally convex algebra under pointwise multiplication*;

(5) $f \in A_p^0$ *implies* $\partial_i f := df/dz_i \in A_p^0$ $(i := 1, \dots, N)$.

Similar statements hold for A_p, which becomes a (DFN)-space.

DEFINITION 1.4. For $F := (F_1, \dots, F_R) \in H(\mathbf{C}^N)^R$ and $r: \mathbf{C}^N \to \mathbf{R}$ we define

$$S_r(F) := \left\{ z \in \mathbf{C}^N \,\middle|\, \left(\sum_{i=1}^R |F_i(z)|^2 \right)^{1/2} < e^{-r(z)} \right\}.$$

PROPOSITION 1.5 ([3], [10]). *Let* $p: \mathbf{C}^N \to [0, \infty)$ *be a projective weight function, take* r *as in* 1.1(2) *for* p, *and let* $s: \mathbf{C}^N \to [0, \infty)$ *be an inductive weight function with* $r = O(s)$ *and* $s = o(p)$. *Take* $F \in W_{D,s}$ *for some* D *and assume*

$$-\log\left(\sum_{i=1}^N |\partial_i F|^2 \right) = O(s) \quad on \;\; V(F) := \{z \,|\, F(z) = 0\}.$$

Then, for each $\epsilon > 0$ *and* $\tilde{f} \in H(V(F))$ *with*

$$\sup_{z \in V(F)} |\tilde{f}(z)| e^{-\epsilon p(z)} < \infty,$$

there exists $f \in W_{2C\epsilon,p}$ (C *from* 1.1(2) *for* p) *such that* $f|_{V(F)} = \tilde{f}$.

Proof. Assume $r \le \gamma_0 q + \beta$. From [10, Lemmas 6.2, 6.3] we have $\Theta, \gamma \in \mathbf{R}_+$ such that for $t := \gamma s + \Theta$ there exists a holomorphic retraction $\pi: S_t(F) \to V(F)$ with the property:

$$\sup_{z \in S_t(F)} |z - \pi(z)| e^{\gamma_0 s(z)} < e^{-\beta}.$$

Hence, for $z \in S_t(F)$ and $f^*(z) := \tilde{f}(\pi(z))$ we have:

$$\sup_{z \in S_t(F)} |f^*(z)| e^{-C\epsilon p(z)} \le \sup_{z \in S_t(F)} |\tilde{f}(\pi(z))| e^{-\epsilon p(\pi(z)) + \epsilon C} < \infty.$$

An elementary (though tedious) argument yields an upper semicontinuous ρ: $\mathbf{C}^N \to [0, \infty)$ with $\rho > t$ and $\rho = o(p)$, as well as a $\gamma \in C^\infty(\mathbf{C}^N)$ with the properties $0 \le \gamma \le 1$, $\gamma|_{S_\rho(F)} \equiv 1$, $\gamma|_{S_t(F)^c} \equiv 0$, and $\log|\bar{\partial}\gamma| = o(p)$. As $\bar{\partial}(f^*\gamma) = f^*\bar{\partial}\gamma$, by analysis of the proof of [6, Thm. 7] we obtain a $(0,1)$-form h on $\mathbf{C}^N$ such that $\bar{\partial} h = 0$, $Fh = \bar{\partial}(f^*\gamma)$, and

$$\int_{\mathbf{C}^N} |h|^2 e^{-3\epsilon C p} \, d\lambda < \infty.$$

Hence, [7, Thm. 4.4.2] implies the existence of a function u on $\mathbf{C}^N$ with $\bar{\partial}u = h$ and

$$\int_{\mathbf{C}^N} |u|^2 e^{-4\epsilon Cp}\, d\lambda < \infty.$$

Finally, we have $\bar{\partial}(f^*\gamma - Fu) = \bar{\partial}(f^*\gamma) - Fh = 0$, and $f := f^*\gamma - Fu$ has the properties: $|f|_{2C\epsilon, p} < \infty$, $f|_{V(F)} = f^*\gamma|_{V(F)} = \tilde{f}$. □

More general results on interpolation of holomorphic functions on varieties with or without multiplicity can be found in [1], [2], [3], and [10].

For later application of this proposition we need the following.

LEMMA 1.6 (Mittag-Leffler lemma [12, Lemma 1.3]). *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_j & \xrightarrow{\varphi_j} & Y_j & \xrightarrow{\psi_j} & Z_j & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & X_{j+1} & \xrightarrow[\varphi_{j+1}]{} & Y_{j+1} & \xrightarrow[\psi_{j+1}]{} & Z_{j+1} & \longrightarrow & 0 \end{array}$$

be exact sequences of $\mathbf{C}$*-vector spaces for* $j \in \mathbf{N}$ *such that* $\varphi_j(x) = \varphi_{j+1}(x)$ *and* $\psi_j(y) = \psi_{j+1}(y)$ *for all* $j \in \mathbf{N}$, $x \in X_{j+1}$, *and* $y \in Y_{j+1}$. *Let* X_j *be Fréchet spaces and let the inclusions* $X_{j+1} \to X_j$ *be continuous. Then the projective limit of these sequences,*

$$0 \longrightarrow \bigcap_j X_j \longrightarrow \bigcap_j Y_j \longrightarrow \bigcap_j Z_j \longrightarrow 0,$$

is exact provided that, for all $j \in \mathbf{N}$, X_{j+2} *is dense in* X_{j+1} *with respect to the topology of* X_j.

This lemma's last hypothesis leads to the following.

DEFINITION 1.7. A projective weight function p has property (ML) (i.e., Mittag-Leffler) if there exists a decreasing sequence $(D(n))_{n \in \mathbf{N}}$ of strictly positive numbers with limit 0 such that, for all $n \in \mathbf{N}$, $W_{D(n+2),p}$ is dense in $W_{D(n+1),p}$ with respect to the topology of $W_{D(n),p}$.

REMARK 1.8. If p is componentwise radial then the monomials are an orthogonal Schauder basis in each $W_{D,p}$, as is shown in [14, 1.16]. [22] proves that, for p convex and for many other cases, the polynomials are dense in each $W_{D,p}$. Hence all these p have property (ML).

A consequence of Nevanlinna theory is the following.

PROPOSITION 1.9. *Let p be a componentwise radial projective weight function on* $\mathbf{C}^N$ *with* $p(2z) = O(p(z))$ *and* $F, G \in A_p^0$, F/G *entire. Then* $F/G \in A_p^0$.

Proof. For g meromorphic on $\mathbf{C}^N$ and for $z \in \mathbf{C}^N$, define $T(g,z)$ as the Nevanlinna characteristic function of the restriction of g to the subspace generated by z, evaluated in z, and (in case f is entire)

$$M(f,z) := \sup\{|f(e^{i\varphi}z)| \mid 0 \le \varphi \le 2\pi\}.$$

It is well known that there exist $C_1, C_2 \in \mathbf{R}_+$ depending only on f such that $T(f,z) - C_1 \leq \log M(f,z) \leq 3T(f,2z) + C_2$. We also know that $T(1/g,z) = T(g,z)$ in case $g \neq 0$, and that $T(fg,z) \leq T(f,z) + T(g,z)$ (see [23, pp. 196–205]). Therefore we have $\log M(F/G,z) = o(p(2z)) = o(p(z))$. □

DEFINITION 1.10. (1) For $F_1, \ldots, F_R \in A_p^0$ we define $I(F_1, \ldots, F_R)$ as the ideal algebraically generated by $F_1, \ldots, F_R$ in A_p^0.

(2) We call an ideal I in A_p^0 *localized* if the following holds: $F \in A_p^0$ is in I, provided that its germs $F|_a$ for all $a \in \mathbf{C}^N$ are in the ideal generated by the elements of I in O_a, the ring of germs of analytic functions in a.

(3) We call a closed ideal I in A_p^0 *complemented* if there exists a closed subspace G in A_p^0 such that $A_p^0 = I \oplus G$ as a topological vector space.

REMARK 1.11. Cartan's theorem B together with 1.9 implies that, for p componentwise radial and $p(2z) = O(p(z))$, any principal ideal I in A_p^0 is localized. It then follows from the closure-of-modulus theorem that I is closed, too (see [7] for these theorems).

At last we have a look at the linear topological structure of the A_p^0.

DEFINITION 1.12. For $\rho \in (-\infty, \infty]$ and for $\alpha := (\alpha_j)_{j \in \mathbf{N}}$ an unbounded increasing sequence of positive real numbers, we define

$$\Lambda_\rho(\alpha) := \left\{ (x_j)_j \in \mathbf{C}^{\mathbf{N}} \,\middle|\, \text{for all } r < \rho \colon \|(x_j)_j\|_r := \sum_{j=1}^{\infty} |x_j| e^{r\alpha_j} < \infty \right\}.$$

With the topology induced by the $\|\cdot\|_r$ $(r < \rho)$, this becomes an (FN)-space. The $\Lambda_\infty(\alpha)$ are called power series spaces of infinite type, the $\Lambda_\rho(\alpha)$ $(\rho < \infty)$ power series spaces of finite type.

REMARK 1.13. Assume that the projective weight function p on $\mathbf{C}^N$ has the property:

(1) there exist $A, B < \infty$ such that $p(2z) \leq Ap(z) + B$ and $2p(z) \leq p(Az) + B$ for all $z \in \mathbf{C}^N$.

Then the proof of [14, 1.16] can easily be applied to our situation and yields:

$$A_p^0(\mathbf{C}^N) \cong H(\mathbf{C}^N) \cong \Lambda_\infty((j^{1/N})_j)$$

as locally convex spaces (not algebras).

PROPOSITION 1.14 ([27]; for an easier proof due to Vogt see [9, 21.7.6]). *A power series space of finite type can never be isomorphic to a closed subspace of a power series space of infinite type.*

2. Algebras restricted to graphs. In this section we investigate weighted (FN)-algebras of entire functions in two variables, restricted to the graphs of entire functions in one variable. We will show that for nice weight functions the graph is that of a polynomial if and only if the restricted algebra does not contain a closed, complemented ideal of infinite codimension. This result forms the main argument in Section 3.

DEFINITION 2.1. For this section and the next, we fix a componentwise radial, continuous projective weight function $p: \mathbf{C}^2 \to [0, \infty)$ with 1.13(1) and:

(1) an r for 1.1(2) can be chosen as an inductive weight function of the form $r(z) = F\log(1+|z|^2)+F \quad (F>0)$.

Furthermore, we define $p|: z \mapsto p(z, 0)$ for $z \in \mathbf{C}$. For $f \in A^0_{p|}$ not a polynomial, we set $q: z \mapsto p(z, f(z))$.

REMARK 2.2. If one puts $2^\alpha := A$ then 2.1(1) implies that there exist $\alpha, B, C < \infty$ such that, for all $z \in \mathbf{C}^2$, $(1/C)|z|^{1/\alpha} - B \le p(z) \le C|z|^\alpha + B$.

Note that for all $z, w \in \mathbf{C}$ we have $p(z, 0) \le p(z, w)$, as p is plurisubharmonic and componentwise radial. Hence we get $p| \le q$.

Examples for weight functions with the properties of 2.1 are

$$p: (z, w) \mapsto \varphi_1(|z|) + \varphi_2(|w|) + \sum_{k=1}^{L} \varphi_{2k+1}(|z|)\varphi_{2k+2}(|w|),$$

where $\varphi_j: [0, \infty) \to [0, \infty)$ $(j = 1, \dots, 2L+2)$ are strictly increasing functions such that $s \mapsto \varphi_j(e^s)$ is convex and there exists $A < \infty$ with

$$\varphi_j(2t) \le A\varphi_j(t) + A, \qquad 2\varphi_j(t) \le \varphi_j(At) + A$$

for all $t \in [0, \infty)$, $j = 1, \dots, 2L+2$. Such φ_j could be, for example, functions like

$$t \mapsto t^{\rho(1)} \log^{\rho(2)}(1+t^2) \log^{\rho(3)} \log(3+t^2)$$

for $\rho(1) > 0$ and $\rho(2), \rho(3) \ge 0$.

LEMMA 2.3. *q is a continuous projective weight function with property* (ML). *An r for* 1.1(2) *can be chosen as an inductive weight function with*

$$(\log^2(1+|z|^2)+1)^2 \le r(z) \quad (z \in \mathbf{C}).$$

Proof. 1.1(1) and 1.1(3) are obviously true for q. Using the arguments of [14, Lemma 2.3], we obtain radial inductive weight functions $\varphi_1, \varphi_2: \mathbf{C} \to [0, \infty)$ and $D < \infty$ such that, for all $z \in \mathbf{C}$ and $i = 1, 2$,

$$\varphi_i(2z) \le D\varphi_i(z) + D, \qquad \varphi_1 = o(p|),$$

$$\log|f|, \log|\partial f|, F\sqrt{2}(\log(1+|\cdot|^2)+1)^2 \le \varphi_1,$$

$$\varphi_2 = o(p(0, \cdot)), \qquad F\sqrt{2}(\log(1+|\cdot|^2)+1)^2 \le \varphi_2.$$

Define $s(z) := D + \sqrt{2} + D\varphi_1(z) + \varphi_2(f(z))$. By the mean value theorem, it follows for $\log|z-\xi| \le -s(z)$, $|z| \ge 1$, that

$$|(z, f(z)) - (\xi, f(\xi))| \le e^{-r(z, f(z))}.$$

With C as in 1.1(2) for p we get $q(\xi) \le Cq(z) + C$.

As $s = o(q)$, 1.1(2) holds for q. It is easy to check (with the mean value theorem) that s is an inductive weight function.

If we have proved the following claim, we can combine it with 1.3(2) to obtain (ML) for q:

(*) For each $\epsilon > 0$ the polynomials in z and $f(z)$ are dense in $B_{\epsilon,q}$ with respect to the norm $\|\cdot\|_{2CK(p)\epsilon,q}$.

By 1.5 each $f \in B_{\epsilon,q}$, seen as a function on the graph $\Gamma := \{(z,w) \mid f(z)=w\}$, can be extended to an $f \in W_{2C\epsilon,p}$. By [13, Prop. 2.8], f can be approximated with polynomials in z, w with respect to $|\cdot|_{2C\epsilon,p}$, and hence *a fortiori* with respect to $\|\cdot\|_{2CK(p)\epsilon,p}$. The restriction of this approximation to Γ gives (*). □

The main result of this section is the following.

THEOREM 2.4. *There exists $g \in A^0_{p|}$ with infinitely many zeros, each of order* 1, *such that the ideal $I(g)$ in A^0_q is closed, localized, and complemented in A^0_q. Furthermore, $A^0_q/I(g)$ is isomorphic to a power series space of finite type.*

It is easily checked that, for each polynomial $f \in H(\mathbf{C})$, there is a radial weight function $\tilde{p}: \mathbf{C} \to [0,\infty)$ as in 1.13(1) such that $A^0_{\tilde{p}}$ and A^0_q are isomorphic as locally convex algebras. Hence by 1.13 $A^0_q \cong H(\mathbf{C})$, and because of 1.14 there is no way to find a g as in 2.4 for this q. Indeed, [14, 3.4] shows that there are no closed complemented infinite-codimensional ideals in A^0_q at all.

Lemmas 2.5 to 2.7 make up the greater part of the theorem's proof.

LEMMA 2.5. *Take $t>0$, $K := \{z \in \mathbf{C} \mid \frac{1}{2}t < |z| < 2t\}$, $C>0$, $D \geq 0$, and h a real-valued function on δK, integrable with respect to the arc-length. Define $\Gamma := \{\frac{1}{2}te^{i\varphi} \mid \alpha - \pi/C \leq \varphi \leq \alpha + \pi/C\}$* [or $\Gamma := \{2te^{i\varphi} \mid \alpha - \pi/C \leq \varphi \leq \alpha + \pi/C\}$].

If $h|_\Gamma \leq D$ and $h|_{\delta K \setminus \Gamma} \leq C$, then for all $z \in K$ with $\arg z = \alpha$ *and $\frac{1}{2}t \leq |z| \leq \frac{1}{2}t + \frac{1}{4}C^{-2}t$* [*or* $2t - \frac{1}{4}C^{-2}t \leq |z| \leq 2t$] *the harmonic extension $\hat{h}$ of h to K fulfills $\hat{h}(z) \leq 2 + D$.*

To prove the lemma, one transforms the problem into an equivalent one on the unit disk and verifies the estimate by straightforward calculations involving the Poisson kernel or the related Möbius transformations.

The main trick in proving 2.4 is contained in the following.

LEMMA 2.6. *There exists a sequence $(a_j)_{j\in\mathbf{N}}$ in $\mathbf{C}$ with $|a_j|$ unbounded and increasing, a sequence $(s_j)_{j\in\mathbf{N}}$ in $\mathbf{R}_+$ with*

(1) $-\log(s_j) = o(q(a_j))$,

and a sequence $(\varphi_j)_{j\in\mathbf{N}}$ of continuous, plurisubharmonic functions on $\mathbf{C}$ bounded from below with the property

(2) $\forall\, k \in \mathbf{N}$, $\exists\, C_k < \infty$ *and $m(k) \in \mathbf{N}$ $\forall\, j \in \mathbf{N}$ and $z, y \in \mathbf{C}$ with $|y-z| < s_j$:*

$$\varphi_j(z) + \frac{1}{m(k)} q(a_j) \leq C_k + \frac{1}{k} q(z) + \varphi_j(y).$$

Proof. As f is not a polynomial and $r(z) = F\log(1+|z|^2)$, we can find a sequence $(c_j)_{j\in\mathbf{N}}$ in $\mathbf{C}$ with $r_j := |c_j|$ unbounded and increasing and $\log|f(c_j)| < -r(2r_j)$. In the sequel we will drop a finite number of c_j in a finite number of steps without changing the notation.

For $M(t) := \sup_{|z|=t} |f(t)|$ we have $\log_+ r_j = o(\log_+ M(r_j))$.

We define $A_j := \{z \in \mathbf{C} \mid \frac{1}{2}r_j < |z| < 2r_j\}$. Using Harnack's inequality we find an $R < \infty$ such that for all $j \in \mathbf{N}$, h harmonic and positive on A_j, and $|z_1| = |z_2| = r_j$, we have the estimate

$$R^{-1}h(z_1) \le h(z_2) \le Rh(z_1).$$

Furthermore, we define

$$\psi_j(z) := \sup\{g(z) \mid g \text{ subh. on } \mathbf{C}, \forall z \notin A_j : g(z) \le \max(\log|f(z)|, -r(2r_j))\}.$$

ψ_j is harmonic on A_j. Without loss of generality, $\psi_j > -r(2r_j)$ on A_j for all $j \in \mathbf{N}$. On $\mathbf{C}$, ψ_j is continuous and subharmonic, and $\psi_j \ge -r(2r_j)$. Therefore we have $\psi_j(c_j) + r(2r_j) \ge R^{-1}(\log M(r_j) + r(2r_j))$. Hence, again without loss of generality,

(1) $\psi_j(c_j) \ge \frac{1}{2}R^{-1}\log M(r_j)$ for all $j \in \mathbf{N}$.

Let a_j be a point in $\mathbf{C}$ where

$$z \mapsto \psi_j(z) - \max(\log|f(z)|, -r(2r_j))$$

takes its maximum. The following claims hold:

(2) $a_j \in A_j$, with obvious argument.

From the maximum principle we get

(3) $|f(a_j)| = e^{-r(2r_j)}$, as $\max(\log|f(z)|, -r(2r_j))$ is harmonic in all points except those where $|f(z)| = e^{-r(2r_j)}$, while ψ_j is harmonic on A_j.

(4) $\psi_j(a_j) \ge \frac{1}{2}R^{-1}\log M(r_j)$ because of (1).

(5) $\psi_j(a_j) = o(p|(a_j))$ follows from $\psi_j(a_j) \le \sup_{z \in \delta A_j} \log_+|f(z)|$, $\log_+|f(z)| = o(p|)$, and $\sup_{z \in \delta A_j} p|(z) \le A^2 p|(a_j) + AB + B$.

(6) $-\log d(a_j, \delta A_j) = o(p|(a_j))$. To prove this, we define $(D_j)_j$ so that

$$\sup_{z \in A_j}|f(z)|, \sup_{z \in A_j}|\partial f(z)|, e \le D_j \quad (j \in \mathbf{N}), \quad D_j = o(p|(a_j)).$$

This is possible because of 2.1(1). Set $C_j := 4\pi r_j D_j$, and suppose there exists a subsequence of $(a_j)_{j \in \mathbf{N}}$ (which we call $(a_j)_j$ as well) such that $d(a_j, \delta A_j) \le \frac{1}{4}r_j C_j^{-2}$.

For $|y - a_j| \le 4r_j \pi C_j^{-1}$ we would get $||f(y)| - |f(a_j)|| \le |y - a_j| D_j \le 1$. Because $|f(a_j)| \le 1$, for $\alpha_j :=$ arc a_j we have on $\{\frac{1}{2}r_j e^{i\varphi} \mid \alpha_j - \pi/C_j \le \varphi \le \alpha_j + \pi/C_j\}$ [or $\{2r_j e^{i\varphi} \mid \alpha_j - \pi/C_j \le \varphi \le \alpha_j + \pi/C_j\}$] the estimate $\log_+|f| \le \log 2$. Combined with 2.5, this yields $\psi_j(a_j) \le 2 + \log 2$, contradicting (4). Thus, for all j large enough,

$$-\log d(a_j, \delta A_j) \le \log(4C_j^2 r_j^{-1}) = o(p|(a_j)).$$

For $\alpha \ge 1$ from 2.2, we pick s with $0 < s < 1/\alpha$ and define

$$b_j := p|(a_j)(\psi_j(a_j))^{-1}.$$

From (5) we get $b_j \to \infty$; hence, without loss of generality, $b_j \ge 1$ for all $j \in \mathbf{N}$.

Finally, we may put:

$$s_j := |a_j|^{-\alpha} d(a_j, \delta A_j),$$

$$\varphi_j(z) := b_j \psi_j(z) - b_j s^{-1}(\log b_j - \log s),$$

$$\rho_j(z) := b_j \max(\log|f(z)|, -r(2r_j)) - b_j s^{-1}(\log b_j - \log s).$$

2.6(1) is an immediate consequence of (6). To prove 2.6(2), we also need the estimates:

(7) $|f(z)|^s \geq \rho_j(z)$ for all $j \in \mathbf{N}$ and $z \in \mathbf{C}$. This follows from the elementary inequality $a^s \geq b \log_+ a - bs^{-1}(\log b - \log s)$, which holds for $a \geq 0$, $b \geq 1$, $1 > s > 0$.

$$(8) \qquad -\rho_j(a_j) = \left(\frac{r(2r_j)}{\psi_j(a_j)} + \frac{(\log b_j - \log s)}{s\psi_j(a_j)}\right) p|(a_j) = o(p|(a_j)),$$

which follows immediately from $\log p|(a_j) = O(\log|a_j|)$ and (4).

Hence, by omission of finitely many j,

(9) $p|(a_j) \geq \varphi_j(a_j) \geq \frac{1}{2} p|(a_j)$ $(j \in \mathbf{N})$.

$\psi_j + r(2r_j)$ is positive and harmonic on $\{y \,||y - a_j| < d(a_j, \delta A_j)\}$; hence Harnack's inequality (on disks) yields, for $|y - a_j| < s_j$,

$$\begin{aligned} \varphi_j(a_j) - \varphi_j(y) &= b_j(\psi_j(a_j) + r(2r_j)) - b_j(\varphi_j(y) + r(2r_j)) \\ &\leq p|(a_j)\left(1 + \frac{r(2r_j)}{\psi_j(a_j)}\right)\left(1 - \left(\frac{1 - |a_j|^{-\alpha}}{1 + |a_j|^{-\alpha}}\right)^2\right) \\ &= p|(a_j)\left(1 + \frac{r(2r_j)}{\psi_j(a_j)}\right)\left(\frac{4|a_j|^\alpha}{(|a_j|^\alpha + 1)^2}\right) =: E_j. \end{aligned}$$

$(E_j)_{j \in \mathbf{N}}$ is bounded by $E < \infty$ because of 2.2 and (4).

Now we define, for $k \in \mathbf{N}$:

$$C_{1,k} := \max_{z \in \mathbf{C}} \left(|f(z)|^s - \frac{1}{k} q(z)\right) < \infty \quad (\text{as } s < 1/\alpha),$$

$$C_{2,k} := \max_{j \in \mathbf{N}} \left(-\rho(a_j) - \frac{1}{k} p\left|\left(\frac{1}{4} a_j\right)\right.\right) < \infty \quad (\text{because of (8)}),$$

$$m(k) := 3kA^2C \quad (C \text{ from 1.1(2)}),$$

$$C_k := C_{1,3k} + C_{2,3k} + E + (A+1)B + C \quad (A, B \text{ from 2.1(1)}).$$

We now verify 2.6(2).

First case. If $z \in A_j$ then the following inequalities hold:

$$\text{(A)} \qquad \varphi_j(z) - \rho_j(z) \leq \varphi_j(a_j) - \rho_j(a_j) \leq E + \varphi_j(y) - \rho_j(a_j),$$

$$\text{(B)} \qquad 0 \leq C_{2,3k} + \frac{1}{3k} p\left|\left(\frac{1}{4} a_j\right) + \rho_j(a_j) \leq C_{2,3k} + \frac{1}{3k} q(z) + \rho_j(a_j),\right.$$

$$\text{(C)} \qquad \rho_j(z) \leq |f(z)|^s \leq \frac{1}{3k} q(z) + C_{1,3k},$$

$$\text{(D)} \qquad \begin{aligned} \frac{1}{m(k)} q(a_j) &= \frac{1}{3kCA^2} p(|a_j|, e^{-r(2r_j)}) \leq \frac{1}{3kA^2} p|(a_j) + C \\ &\leq \frac{1}{3k} q(z) + (A+1)B + C. \end{aligned}$$

Addition of the leftmost and rightmost terms of these inequalities yields 2.6(2).

Second case. If $z \notin A_j$ then we have:

(A) $$\varphi_j(z) = \rho_j(z) \leq |f(z)|^s \leq \frac{1}{3k} q(z) + C_{1,3k},$$

(B) $$\frac{1}{m(k)} q(a_j) \leq \frac{1}{2C} q(a_j) = \frac{1}{2C} p(|a_j|, -r(2r_j))$$
$$\leq \frac{1}{2} p|(a_j) + C \leq \varphi_j(a_j) + C \leq \varphi_j(y) + E + C,$$

(C) $$0 \leq \frac{2}{3k} q(z) + C_{2,3k} + (A+1)B.$$

Addition of the leftmost and rightmost terms of these again yields 2.6(2). □

Now we construct entire functions g_j whose modulus has about the same growth estimates from above as e^{φ_j}.

LEMMA 2.7. *There exist a sequence $(a_j)_{j \in \mathbf{N}}$ in $\mathbf{C}$ with $|a_j|$ unbounded and increasing, and a sequence $(g_j)_{j \in \mathbf{N}}$ in A_q^0 with*
(1) $g_j(a_j) = 1$ $(j \in \mathbf{N})$;
(2) $\forall\, k \in \mathbf{N}$, $\exists\, D_k < \infty$ *and* $m(K) \in \mathbf{N}$ $\forall\, j \in \mathbf{N}$ *and* $z \in \mathbf{C}$:
$$|g_j(z)| \leq D_k \exp\left[\frac{1}{k} q(z) - \frac{1}{m(k)} q(a_j)\right].$$

Proof. For $\mathbf{D} := \{z \mid |z| \leq 1\}$ we find $\gamma \in C^\infty(\mathbf{C})$ with $0 \leq \gamma \leq 1$, $\gamma|_{(1/2)\mathbf{D}} \equiv 1$, $\gamma|_{\mathbf{D}^c} \equiv 0$ and put $D := \sup_{z \in \mathbf{D}} |\bar\partial \gamma(z)|$. Take $(a_j)_j$, $(s_j)_j$, $(\varphi_j)_j$ from 2.6 and assume without loss of generality that $s_j < 1$ $(j \in \mathbf{N})$.

We define $\gamma_j(z) := \gamma(s_j^{-1}(z - a_j))$. Using [7, Thm. 4.4.2] we obtain functions $(v_j)_j$ on $\mathbf{C}$ with $\bar\partial v_j(z) = (z - a_j)^{-1} \bar\partial \gamma_j(z)$ and the estimate

(x) $$\int_{\mathbf{C}} |v_j(z)|^2 e^{-2\varphi_j(z)} (1 + |z|^2)^{-2}\, d\lambda(z) \leq \int_{\mathbf{C}} |z - a_j|^{-2} |\bar\partial \gamma_j(z)|^2 e^{-2\varphi_j(z)}\, d\lambda(z).$$

For $g_j(z) := \gamma_j(z) - (z - a_j) v_j(z)$ we have, by the Sobolev lemma, $g_j \in H(\mathbf{C})$ and $g_j(a_j) = \gamma_j(a_j) = 1$. It follows from (x) and the elementary estimate
$$d_j := 1 + (1 + |a_j|)^2 \geq \sup_{z \in \mathbf{C}} \frac{|z - a_j|^2}{1 + |z|^2}$$
that

(xx) $$\left(\int_{\mathbf{C}} |g_j(z)|^2 e^{-2\varphi_j(z)} (1 + |z|^2)^{-3}\, d\lambda(z)\right)^{1/2}$$
$$\leq \left(\int e^{-2\varphi_j(z)}_{\{z \mid |z - a_j| \leq s_j\}}\, d\lambda(z)\right)^{1/2}$$
$$+ \left(\int_{\mathbf{C}} |v_j(z)|^2 e^{-2\varphi_j(z)} (1 + |z|^2)^{-3} |z - a_j|^2\, d\lambda(z)\right)^{1/2}$$
$$\leq (1 + 2Ds_j^{-2} d_j) \left(\int e^{-2\varphi_j(z)}_{\{z \mid |z - a_j| \leq s_j\}}\, d\lambda(z)\right)^{1/2} \leq$$

$$\leq (s_j + 2Ds_j^{-1}d_j)\sqrt{\pi}e^{-t_j},$$

where $t_j := \inf\{\varphi_j(y) \,|\, |y-a_j| < s_j\}$.

2.6(2) states that:

$$-\frac{1}{l}q(z) \leq -\varphi_j(z) - \frac{1}{m(l)}q(a_j) + C_l + t_j \quad (l, j \in \mathbf{N});$$

hence we can apply 1.3(2) and (xx) to get:

$$\begin{aligned}
\|g_j\|_{1/k,q} &\leq C(k)\left(\int_{\mathbf{C}} |g_j|^2 \exp\left[-\frac{2}{K(q)k}q\right] d\lambda\right)^{1/2} \\
&\leq L_k\left(\int_{\mathbf{C}} |g_j(z)|^2 \exp\left[-\frac{1}{K(q)k}q(z)\right](1+|z|^2)^{-3}\, d\lambda(z)\right)^{1/2} \\
&\leq L_k(s_j + 2Dd_js_j^{-1})\sqrt{\pi}\exp\left[C_{2K(q)k} - \frac{1}{m(2K(q)k)}q(a_j)\right],
\end{aligned}$$

where $C(k)$ and L_k are independent of j. From 1.1(3), 2.6(1), and $s_j < 1$ we finally obtain $(D_k)_{k\in\mathbf{N}}$ independent of j such that:

$$\|g_j\|_{1/k,q} \leq D_k \exp\left[-\frac{1}{2m(2K(q)k)}q(a_j)\right]. \qquad \square$$

Proof of 2.4. (A similar construction with a radial weight function and much simpler g_j can be found in [16].)

I. Construction of g. We choose $K \geq 1$ and a subsequence of the $(a_j)_{j\in\mathbf{N}}$ from 2.7 (which we again write as $(a_j)_{j\in\mathbf{N}}$) such that, for

$$g(z) := K\prod_{j=1}^{\infty}\left(1 - \frac{z}{a_j}\right),$$

the following hold:

(1) $|a_1| \geq 1$;
(2) $|a_{j+1}| \geq 10|a_j|$ for all $j \in \mathbf{N}$;
(3) $\max_{j\in\mathbf{N}}(\log|g(z)(1-z/a_j)^{-1}|) = O(r(z)) = o(p|(z))$ (where r is from 2.3 for q);
(4) $|\partial g(a_j)| = |K/a_j|\prod_{j\neq k}|1-a_j/a_k| \geq 1$ for all $j \in \mathbf{N}$.

II. Choose r according to 2.3 and put

$$\rho(z) := r(z) + \log(1+|z|^2) + \log 5.$$

It is elementary to check that

$$S_\rho(g) \subset \{z \,|\, |z| \leq |a_1|\} \cup \bigcup_{k=2}^{\infty} \{z \,|\, |z-a_k| < e^{-r(z)}\}.$$

III. Define $\pi : A_q^0 \to \mathbf{C}^{\mathbf{N}}$ by $f \mapsto (f(a_j))_j$. Then

(1) $\ker \pi = I(g)$ and
(2) $\operatorname{Im}\pi = \Lambda_0 := \{(z_j)_j \,|\, \text{for all } k \in \mathbf{N}\text{: } \|(z_j)_j\|_k := \sup_j |z_j|e^{-(1/k)q(a_j)} < \infty\}$.

To prove (1) we must show that, for all $f \in A_q^0$ with $f|_{V(g)} \equiv 0$, we have $f/g \in A_q^0$. This is an easy consequence of the maximum-modulus principle, applied to f/g on $S_\rho(g)$ (see [1, proof of Prop. 3] for the idea). Hence $I(g)$ is localized and closed.

It is obvious that $\operatorname{Im}\pi \subset \Lambda_0$. On the other hand, take $(z_j)_j \in \Lambda_0$ and define a holomorphic function on $V(g)$ by $f(a_j) := z_j$. From 1.5 we have that

$$0 \longrightarrow gW_{2\epsilon,q} \longrightarrow \{f \in W_{\epsilon,q} + gW_{2\epsilon,q} \mid \pi f \in \Lambda_0\} \xrightarrow{\pi} \Lambda_0 \longrightarrow 0$$

is an exact sequence for each $\epsilon > 0$. (Note that, by a similar argument as for III(1), $f \in W_{\epsilon,q}$ and $\pi f \equiv 0$ together imply $f/g \in W_{2\epsilon,q}$.) Endowing $gW_{2\epsilon,q}$ with the Hilbert space topology of $W_{2\epsilon,q}$, we then can apply 1.6 because q has (ML) by 2.3. Hence $\pi : A_q^0 = \bigcap_{\epsilon>0} W_{\epsilon,q} \to \Lambda_0$ is surjective.

IV. With the topology induced by the $\|\cdot\|_k$, Λ_0 becomes a Fréchet space. As $\pi : A_q^0 \to \Lambda_0$ is obviously continuous, it follows from the open mapping theorem that $A_q^0/I(g) \cong \Lambda_0$. Hence Λ_0 is nuclear. As a consequence of [17, 10.1.4 and 10.2.1] Λ_0 is equal to $\Lambda_0((q(a_j))_j)$.

We now define $\xi : \Lambda_0 \to A_q^0$ by

$$\xi((z_j)_j)(z) := \sum_{j=1}^{\infty} z_j \frac{g(z)}{(z-a_j)\partial g(a_j)} g_j(z),$$

where the g_j are taken from 2.7. From 2.7(2) and I(4) we deduce that, for all $l \in \mathbf{N}$,

$$\begin{aligned}
&\sup_{z\in\mathbf{C}} |\xi((z_j)_j)(z)| \exp\left[-\frac{1}{l}q(z)\right] \\
&\quad \le \sup_{z\in\mathbf{C}} \sum_{j=1}^{\infty} \left|\frac{g(z)}{z-a_j}\right| \exp\left[-\frac{1}{2l}q(z)\right] |z_j|\,|g_j(z)| \exp\left[-\frac{1}{2l}q(z)\right] \\
&\quad \le B_l D_{2l} \sum_{j=1}^{\infty} |z_j| \exp\left[-\frac{1}{m(2l)}q(a_j)\right] = B_l D_{2l} \|(z_j)_j\|_{m(2l)},
\end{aligned}$$

where

$$B_l := \sup_{j,z} K|a_j|^{-1} \prod_{k\ne j} \left|1-\frac{z}{a_k}\right| \exp\left[-\frac{1}{2l}q(z)\right] < \infty$$

because of I(3).

We just proved that ξ is well defined and continuous, and it is easy to check that $\pi \circ \xi = \mathrm{id}$ on Λ_0. □

In [14] and [26] there are more general criteria for an ideal I to be localized or closed in A_p^0, as well as more general results concerning the structure of A_p^0/I.

3. Noncomplemented principal ideals. In this section we prove the theorem stated in the introduction. In fact, we give a proof for more general weight functions.

We recall that $p, q, r, f, p|, \ldots$ are defined as in 2.1, unless stated otherwise.

The proof of [4, Prop. 18] shows that the following holds.

PROPOSITION 3.1 ([4]). *Let $p : \mathbf{C}^N \to [0,\infty)$ be a componentwise radial projective weight function with property* 1.13(1). *Any ideal I in A_p^0 generated by finitely many polynomials is closed, localized, and complemented in A_p^0.*

The next proposition gives the relation of complementation to products of principal ideals.

PROPOSITION 3.2. *Let $\tilde{p}$ be any projective weight function, take $F, G \in A^0_{\tilde{p}} \setminus \{0\}$, and let one of the ideals $I(F), I(G), I(FG)$ be closed.*
(1) *Then all three are closed if and only if two of them are closed.*
(2) *This being assumed, $I(FG)$ is complemented if and only if both $I(F)$ and $I(G)$ are complemented.*

Proof. For $H \in A^0_{\tilde{p}}$, define $M_H: A^0_{\tilde{p}} \to A^0_{\tilde{p}}$ by $f \mapsto Hf$ and define T_H as its transposed map. Because of the open mapping theorem, $I(H)$ is closed if and only if M_H is a topological isomorphism onto $I(H)$; (1) is easily deduced from this. To prove (2), one must check that $T_G \circ T_F = T_{GF} = T_{FG} = T_F \circ T_G$, and that a closed $I(H)$ is complemented in $A^0_{\tilde{p}}$ if and only if $T_H: (A^0_{\tilde{p}})'_b \to (A^0_{\tilde{p}})'_b$ has a continuous linear right inverse R_H. This done, if there exists an R_{FG} then $T_G \circ R_{FG}$ and $T_F \circ R_{FG}$ are continuous linear right inverses for T_F and T_G, respectively. If R_F and R_G exist, then $R_F \circ R_G$ is a continuous linear right inverse for T_{FG}. □

We are now going to prove several theorems showing that certain ideals are not complemented in A^0_p. To prepare them, we prove the following.

PROPOSITION 3.3. *For any $f \in A^0_{p|}$, $f \neq 0$, $V(f)$ infinite, and $F(z,w) := f(z)$, $I(F)$ is not complemented in A^0_p.*

Proof. Define $\Phi: A^0_{p|} \to A^0_p$ by $\Phi(g)(z,w) := g(z)$ and define $\phi: A^0_p \to A^0_p$ by $\phi(G)(z,w) := G(z,0)$. Both Φ and ϕ are continuous and linear. It is easily checked that $\operatorname{Im}\Phi + I(F) = (\mathrm{Id} - \phi)^{-1} I(F)$, hence $\operatorname{Im}\Phi + I(F)$ is closed because of 1.11.

For $\Theta: A^0_p \to A^0_p/I(F)$ the canonical quotient map, we have $\ker(\Theta \circ \Phi) = I(f)$ and $A^0_{p|}/I(f) \cong (\operatorname{Im}\Phi + I(F))/I(F)$, the latter being closed in $A^0_p/I(F)$.

By [14, 2.8] $A^0_{p|}/I(f)$ is a power series space of finite type, and hence $A^0_p/I(F)$ cannot be isomorphic to a closed subspace of A^0_p because of 1.13 and 1.14. □

LEMMA 3.4. *Let $F \in A^0_p$ be of the form*

$$F(z,w) = f(z) - v(z,w) = f(z) + wp_1(z) + \cdots + w^{m-1}p_{m-1}(z) + w^m, \quad m \geq 1,$$

with v, p_j polynomials ($j = 1, \ldots, m-1$) but f not a polynomial. Then the following hold:
(1) *any polynomial dividing F is constant;*
(2) *$V(\partial_2 F, F)$ is discrete;*
(3) *$H \in A^0_p$ is in $I(F)$ if and only if $H|_{V(F)} \equiv 0$.*

Proof. (1) Assume $QH = F$, where Q is a polynomial. Then

$$Q(z,w) = q_0(z) + wq_1(z) + \cdots + w^l q_l(z), \quad q_l \neq 0,$$

$$H(z,w) = h_0(z) + wh_1(z) + \cdots + w^n h_n(z), \quad n + l = m.$$

If $l = 0$, we have $1 = q_0 h_m$, hence q_0 is constant.

For $l>0$, we could conclude by induction that

$$\begin{aligned} q_l h_n = 1 &\Rightarrow h_n \text{ is a polynomial,} \\ q_{l-1}h_n + q_l h_{n-1} = p_{m-1} &\Rightarrow h_{n-1} \text{ is a polynomial,} \\ &\vdots \\ q_{l-n}h_n + \cdots + q_l h_0 = p_l &\Rightarrow h_0 \text{ is a polynomial.} \end{aligned}$$

Hence $f = q_0 h_0$ had to be a polynomial, contrary to our hypothesis.

(2) Define $V := \{a \in V(\partial_2 F, F) \mid \dim V(\partial_2 F, F)|_a = 1\}$. Then V is a 1-dimensional subvariety of $V(\partial_2 F)$, which again is algebraic by virtue of the theorems of Chow or Stoll. If $V \neq \emptyset$, by the Hilbert Nullstellensatz we can find a nonconstant polynomial Q dividing $\partial_2 F$ such that $Q\mathbf{C}[z,w] = \{H \in \mathbf{C}[z,w] \mid H|_V \equiv 0\}$. By [19, Prop. 4] this means that Q divides F, which contradicts (1).

(3) "$\Rightarrow$" is trivial. "$\Leftarrow$": If $F|_a$ is reducible, then $a \in V(\partial_2 F, F)$; this set is discrete by (2). H/F is well defined and holomorphic outside of $V(\partial_2 F, F)$ for each $H \in A_p^0$ with $H|_{V(F)} \equiv 0$. Hence it can be extended to an entire function (see e.g. [5, Chap. I, §C.6]), which because of 1.9 is in A_p^0, too. □

For the theorems of Stoll and Chow see [20]; for the Hilbert Nullstellensatz see [8, §1.8, Thm. 16].

Now we state our main results.

THEOREM 3.5. *Take p, $p|$ as in* 2.1. *For $f \in A_{p|}^0$ and $F(z,w) := f(z) - w^m$ ($m \in \mathbf{N}$), $I(F)$ is complemented in A_p^0 if and only if f is a polynomial.*

THEOREM 3.6. *Assume that p, in addition to* 2.1, *has the property:*

(1) *there exists $C < \infty$ such that, for all $z, \xi \in \mathbf{C}^2$ with $|z-\xi| \le 1$,*

$$p(\xi) \le Cp(z) + C.$$

Then for $f \in A_{p|}^0$, v a polynomial in w, and $F(z,w) := f(z) - v(w)$, the ideal $I(F)$ is complemented in A_p^0 if and only if one of the following statements is true:

(a) *f is a polynomial*; *or*

(b) *v is constant and $z \mapsto f(z) - v$ has at most finitely many zeros.*

THEOREM 3.7. *Assume that p, in addition to* 2.1, *is radial. Take $f \in A_{p|}^0$ and set $v(z,w) := \sum_{i+j\le m} a_{i,j} z^i w^j$ ($a_{0,m} \neq 0$). Then, for $F(z,w) := f(z) - v(z,w)$, the ideal $I(F)$ is complemented in A_p^0 if and only if one of the following statements holds:*

(a) *f is a polynomial*; *or*

(b) *$m = 0$ and $z \mapsto f(z) - a_{0,0}$ has at most finitely many zeros.*

REMARK 3.8. Each of the Theorems 3.5, 3.6, and 3.7 states that the $I(F)$ considered are complemented if and only if they are generated by a polynomial.

We now prove Theorem 3.7. The proofs of Theorems 3.5 and 3.6 are analogous, but as v is simpler we have less restrictions on p and the required estimates are easier to obtain.

Proof of Theorem 3.7. "$\Leftarrow$" follows immediately from 3.1.

"$\Rightarrow$": In the case that v is constant, we have from 3.3 that $V(f-v)$ in $\mathbf{C}$ is finite. So the only case left to consider is: f is not a polynomial, $m := \deg v > 0$. For this case we shall show that $A_p^0/I(F)$ contains a closed subspace isomorphic to a power series space of finite type. Because of 1.13 and 1.14 this proves our theorem.

We define

$$T := \{H \in A_p^0 \mid \text{there is a } G \in H(\mathbf{C}^2) \text{ with } H(z,w) = G(z, v(z,w)) \ (z, w \in C)\}.$$

and $p'(z,w) := p(z, |w|^{1/m})$. It is easily checked that p' fulfills 2.1. $\Phi: A_{p'}^0 \to A_p^0$, defined by $\Phi(G)(z,w) := G(z, v(z, w))$, is continuous, and $\operatorname{Im}\Phi \subset T$ because $|z|^2 + |v(z,w)|^{2/m}$ has the same growth as $C_1|z|^2 + C_2|w|^2$ for certain $C_1, C_2 \in \mathbf{R}_+$. If $\Theta: A_p^0 \to A_p^0/I(F)$ is the quotient map and $F'(z,w) := f(z) - w$, then $\ker(\Theta \circ \Phi) = I(F')$ in $A_{p'}^0$.

By [14, 2.3] we find a radial inductive weight function $\tilde{s}$ such that $f \in A_{\tilde{s}}$, $\tilde{s} = o(p|)$. With $s(z,w) := \tilde{s}(z) + \log(1+|w|^2)$ we have

$$-\log\left(\sum_{i=1}^{2} |\partial_i F'(z,w)|\right) \le 0 \le O(s(z,w)).$$

By 1.5 we have that, for $q'(z) := p'(z, f(z))$ and $\pi(G)(z) := G(z, f(z))$,

$$0 \to F'W_{2\epsilon, p'} \to \{G \in W_{\epsilon, p'} + F'W_{2\epsilon, p'} \mid \pi(G) \in A_{q'}^0\} \to A_{q'}^0 \to 0$$

is an exact sequence for each $\epsilon > 0$. Because of 1.6, [14, 1.16] (see 1.8) and the open mapping theorem, π induces an isomorphism $\pi: A_{p'}^0/I(F') \to A_{q'}^0$ of locally convex algebras.

We write $[L]$ for the equivalence class in $A_p^0/I(F)$ of an $L \in A_p^0$. For a sequence $([H_j])_j$ in $(T + I(F))/I(F)$ which converges in $A_p^0/I(F)$, we can assume that $(H_j)_j$ converges to an $H \in A_p^0$ (see [11, v. 2, §33.4(2)]). For all (z,w), $(z,w') \in V(F)$ it is obvious that $H(z,w) = H(z,w')$.

We can define

$$g(z) := H(z,w) \quad \text{if} \quad (z,w) \in V(F)$$

because, for each $z \in \mathbf{C}$, $f(z) - v(z, \cdot)$ has a zero w. For $\eta: (z,w) \mapsto z$ it is easy to see that $(V(F), \eta, \mathbf{C})$ is an analytic cover in the sense of [5, Chap. III, §B.3] and hence $g \in H(\mathbf{C})$. Indeed $g \in A_{q'}^0$, as for all $\epsilon > 0$

$$\begin{aligned}
&\sup_{z \in \mathbf{C}} |g(z)| \exp[-\epsilon p(|z|, |f(z)|^{1/m})] \\
&\quad = \sup_{(z,w) \in V(F)} |H(z,w)| \exp[-\epsilon p(\sqrt{|z|^2 + |v(z,w)|^{2/m}})] \\
&\quad \le \sup_{(z,w) \in V(F)} |H(z,w)| \exp[-\epsilon K p(|z|, |w|) + \epsilon K] < \infty
\end{aligned}$$

with suitable $K < \infty$. As $\pi: A_{p'}^0/I(F') \to A_{q'}^0$ is an isomorphism, there exists $G \in A_{p'}^0$ such that $g(z) = G(z, f(z))$ $(z \in \mathbf{C})$. Hence $\Phi(G) \in [H]$ because of 3.4(3) and $[H] \in (T + I(F))/I(F)$.

We have shown that $(T + I(F))/I(F)$ is closed in $A_p^0/I(F)$ and that

$$(T + I(F))/I(F) \cong A_{p'}^0/I(F') \cong A_{q'}^0.$$

By 2.4, $A^0_{q'}$ has a closed subspace isomorphic to a power series space of finite type, so the same holds for $A^0_p/I(F)$ and our theorem is proved. □

The theorem in the introduction obviously is a combination of 3.1, 3.2, and 3.7 for the special case $p(z):=|z|^\alpha$.

One might conjecture from these theorems that, for $p(z)=|z|^\alpha$, a principal ideal in A^0_p is complemented if and only if it is generated by a polynomial. This is true for $A^0_p(\mathbf{C})$, as is shown in [14]. However, in $A^0_p(\mathbf{C}^2)$ we can give neither a proof nor a counterexample.

Before we consider applications, we should mention another formulation of our main results.

REMARK 3.9. Because of 1.13, all our A^0_p have the linear topological invariants (DN) of Vogt [24] and (Ω) of Vogt and Wagner [25]. Any principal ideal I in A^0_p is isomorphic to A^0_p because of 1.11. Hence the splitting theorem [25, 1.4] states that I is complemented if and only if A^0_p/I has (DN).

The A^0_q from Section 2 thus provide a collection of examples of weighted (FN)-algebras not having (DN), added to those with radial weight functions from [16].

4. Right inverses of convolution operators. In this section we outline the consequences of our results for convolution operators on certain A_p, obtained by Fourier–Borel transformation.

DEFINITION 4.1. For a convex, continuous, componentwise radial inductive weight function $q: \mathbf{C}^N \to [0, \infty)$ with $|z|=o(q(z))$, we define the Young conjugate q^* by

$$q^*(z):=\sup\{\langle t,s\rangle - q(s) \mid s\in[0,\infty)^N\},$$

where $t:=(|z_1|,\ldots,|z_N|)\in\mathbf{R}^N$.

It is easy to check that q^* is well-defined and real-valued.

REMARK 4.2. Take q as in 4.1 with the property

(1) there exist $1<D<A_1<A_2<\infty$ and $B_1, B_2<\infty$ with

$$A_1q(z)-B_1\le q(Dz)\le A_2q(z)+B_2 \quad \text{for all } z\in\mathbf{C}^N.$$

Then q^* is a convex, componentwise radial projective weight function having property (1) (with different constants in general). Furthermore, we have $(q^*)^*=q$, and q and q^* have the property 3.6(1).

Proof. For $s\in[0,\infty)^N$ we have

$$(+)\qquad q^*\left(\frac{A_1}{D}s\right)=\sup_r\{\langle A_1s,r\rangle - q(Dr)\}\le\sup_r\{A_1\langle s,r\rangle - A_1q(r)\}+B_1$$
$$=A_1q^*(s)+B_1,$$

and, from an analogous estimate,

$$q^*\left(\frac{A_2}{D}s\right)\ge A_2q^*(s)-B_2.$$

Iterating (+) we find $\tilde{A}_2, \tilde{B}_2 < \infty$ such that

$$q^*\left(\frac{A_2}{D}s\right) \leq \tilde{A}_2 q^*(s) + \tilde{B}_2,$$

thus proving 4.2(1) for q^*. q^* obviously fulfills 1.1(1) and 1.1(3). For $r, s \in [0, \infty)^N$, $r - s \in [-1, 1]^N$, and $v := (|r_1 - s_1|, \ldots, |r_N - s_N|)$, we obtain

$$\begin{aligned} q^*(r) &= \sup_t \{\langle s + (r-s), t\rangle - q(t)\} \leq \sup_t \{\langle s, t\rangle - \tfrac{1}{2}q(t)\} + \sup_t \{\langle v, t\rangle - \tfrac{1}{2}q(t)\} \\ &= \tfrac{1}{2}q^*(2s) + \tfrac{1}{2}q^*(2v) \leq Cq^*(s) + C \end{aligned}$$

for a suitable C (independent of r, s) because q^* is bounded on $[0, 2]^N$ and (1) holds. Standard theory of convex functions (see [18, Thm. 12.2, p. 104]) gives $(q^*)^* = q$; hence we get 3.6(1) for q if we exchange q and q^* in the previous arguments. □

DEFINITION 4.3. For q as in 4.2 and $\mu \in (A_q)'$, we define the Fourier-Borel transform $\mathfrak{F}(\mu)$ by

$$\mathfrak{F}(\mu)(z) := \langle \mu_\xi, e^{\langle \xi, z\rangle}\rangle \quad (z \in \mathbf{C}^N).$$

PROPOSITION 4.4 ([21]). *For q and $p := q^*$ as in* 4.2,

$$\mathfrak{F} : (A_q)'_b \to A_p^0$$

is a linear topological isomorphism.

Proof. This is a special case of [21, Thm. 5.2]. To see this, one need only check that $(kq)^*(z) = kq^*((1/k)z)$ $(z \in \mathbf{C}^N,\ k \in \mathbf{N})$ and then apply 4.2(1) for q^*. The proof of [14, 4.2] is also easily generalized to give us this proposition. □

DEFINITION 4.5. For q as in 4.2 and $\mu, \nu \in (A_q)'$ we set

$$M_\mu(\nu) := \mu * \nu := \mathfrak{F}^{-1}(\mathfrak{F}(\mu)\mathfrak{F}(\nu)).$$

T_μ is defined as the transpose $A_q \to A_q$ of the continuous linear map M_μ.

Standard arguments yield the following.

PROPOSITION 4.6. *For q as in* 4.2 *the following hold.*

(1) *$(A_q)'_b$ endowed with the product $*$ becomes a commutative* (FN)-*algebra with the unit $\delta_0 : f \mapsto f(0)$.*

(2) *For $\mathfrak{F}(\mu)(w) = \sum_i a_i w^i$ $(i \in (\mathbf{N}_0)^N)$ we have*

$$T_\mu(f) = \sum_i a_i \partial_1^{i_1} \cdots \partial_N^{i_N}(f) \quad \textit{for all } f \in A_q,$$

and this series converges in A_q. Therefore T_μ is a linear differential operator with constant coefficients, possibly of infinite order.

(3) *For all $f \in A_q$, $\mu \in (A_q)'$, and $z \in \mathbf{C}^N$,*

$$T_\mu(f)(z) = \langle \mu_\xi, f(\xi + z)\rangle.$$

Hence T_μ is the convolution operator associated with μ.

(4) *T_μ is surjective for all $\mu \in (A_q)' \setminus \{0\}$.*

Note that (4) is just an interpretation of 1.11 by standard duality theory (see [11, v. 2, §33.4(2)]).

The next proposition gives the connection between principal ideals in A_p^0 and convolution operators on A_q.

PROPOSITION 4.7. *For q as in* 4.2, $p:=q^*$, $\mu\in(A_q)'$, *and* $F:=\mathfrak{F}(\mu)$, $\mathfrak{F}$ *transforms the exact sequence*

$$0\to(\ker T_\mu)^\perp\to(A_q)'\to(A_q)'/(\ker T_\mu)^\perp\to 0$$

into

$$0\to I(F)\to A_p^0\to A_p^0/I(F)\to 0.$$

In particular, $\ker T_\mu$ *is complemented in* A_q *if and only if* $I(F)$ *is complemented in* A_p^0.

Proof. It follows from standard duality theory that $(\ker T_\mu)^\perp=\overline{\mathrm{Im}(M_\mu)}$, and by 4.4 that $\mathfrak{F}(\overline{\mathrm{Im}(M_\mu)})=I(F)$, which proves the proposition. □

It is now obvious how to apply 3.6 and 3.7 to obtain 4.8 and 4.9.

COROLLARY 4.8. *Take* $q:\mathbf{C}^2\to[0,\infty)$ *as in* 4.2 *and a nonzero convolution operator* $T: A_q\to A_q$ *of the form*

$$f\mapsto F(\partial_1)f-P(\partial_2)f$$

for a differential polynomial $P(\partial_2)$ *and a linear differential operator* $F(\partial_1)$ *of possibly infinite order. Then T has a continuous linear right inverse if and only if either* $F(\partial_1)$ *is of finite order too, or P is constant such that* $F(z)-P$ *has at most finitely many zeros.*

COROLLARY 4.9. *Assume* $q:\mathbf{C}^2\to[0,\infty)$ *is as in* 4.2 *and, in addition, is radial. Take a nonzero convolution operator* $T: A_q\to A_q$ *of the form*

$$f\mapsto P(\partial_1,\partial_2)f-F(\partial_1)f$$

for a differential polynomial $P(\partial_1,\partial_2)$ *and a linear differential operator* $F(\partial_1)$ *of possibly infinite order. Assume P is of the form*

$$P(\partial_1,\partial_2)=\sum_{i+j\le m} a_{i,j}\partial_1^i\partial_2^j,\quad a_{0,m}\ne 0.$$

Then T has a continuous linear right inverse if and only if either $F(\partial_1)$ *is of finite order, or* $P=a_{0,0}$ *such that* $F(z)-a_{0,0}$ *has at most finitely many zeros.*

REMARK 4.10. Of course, the other results of Section 3 have analogues in this setting also. We will omit them, as the way to deduce them should be clear.

REFERENCES

1. C. A. Berenstein and B. A. Taylor, *A new look on interpolation theory for entire functions of one variable,* Adv. in Math. 33 (1979), 109–143.
2. ———, *Interpolation problems in* $\mathbf{C}^n$ *with applications to harmonic analysis,* J. Analyse Math. 38 (1980), 188–254.

3. ———, *On the geometry of interpolating varieties,* Lecture Notes in Math., 919, Springer, Berlin, 1982.
4. P. B. Djakov and B. S. Mityagin, *The structure of polynomial ideals in the algebra of entire functions,* Studia Math. 63 (1980), 85–104.
5. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables,* Prentice-Hall, Englewood Cliffs, N.J., 1965.
6. L. Hörmander, *Generators for some rings of analytic functions,* Bull. Amer. Math. Soc. 73 (1967), 943–949.
7. ———, *An introduction to complex analysis in several variables,* North-Holland, Amsterdam, 1973.
8. S. Iitaka, *Algebraic geometry,* Springer, New York, 1981.
9. H. Jarchow, *Locally convex spaces,* B. G. Teubner, Stuttgart, 1981.
10. M. Jarnicki, *A method of holomorphic retraction and pseudoinverse matrices in the theory of continuation of δ-tempered functions,* Polish Acad. Sci. Inst. Math. preprint, 1983.
11. G. Köthe, *Topological vector spaces,* vols. 1 and 2, Springer, Berlin, 1969 and 1979.
12. H. Komatsu, *Ultradistributions I, Structure theorems and a characterization,* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20 (1973), 25–105.
13. R. Meise, *Sequence space representations for (DFN)-algebras of entire functions modulo closed ideals,* J. Reine Angew. Math. 363 (1985), 59–95.
14. R. Meise and B. A. Taylor, *Sequence space representations for (FN)-algebras of entire functions modulo closed ideals,* Studia Math. 85 (1987), 203–227.
15. ———, *Each nonzero convolution operator on the entire functions admits a continuous right inverse,* Math. Z. 197 (1988), 139–152.
16. ———, *Complementation of closed ideals in weighted (FN)-algebras of entire functions,* unpublished manuscript.
17. A. Pietsch, *Nuclear locally convex spaces,* Springer, Berlin, 1972.
18. R. T. Rockafellar, *Convex analysis,* Princeton Univ. Press, Princeton, N.J., 1970.
19. J. P. Serre, *Geométrie algèbrique et geométrie analytique,* Ann. Inst. Fourier (Grenoble) 6 (1955/56), 1–42.
20. G. Stolzenberg, *Volumes, limits and extensions of analytic varieties,* Lecture Notes in Math., 19, Springer, Berlin, 1969.
21. B. A. Taylor, *Some locally convex spaces of entire functions,* Proc. Sympos. Pure Math. II (1968), 431–474.
22. ———, *On weighted polynomial approximation of entire functions,* Pacific J. Math. 36 (1971), 523–539.
23. M. Tsuji, *Potential theory in modern function theory,* Maruzen Co., Tokyo, 1959.
24. D. Vogt, *Charakterisierung der Unterräume von s,* Math. Z. 155 (1977), 109–117.
25. D. Vogt and M. J. Wagner, *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau.* Studia Math. 67 (1980), 225–240.
26. J. Wolter, *Nichtkomplementierte Ideale in gewichteten (FN)-Algebren ganzer Funktionen mehrerer Veränderlicher,* Thesis, Düsseldorf, 1987.
27. K. P. Zahariuta, *On the isomorphisms of Cartesian products of locally convex spaces,* Studia Math. 46 (1973), 201–221.

Mathematisches Institut der Universität Düsseldorf
D-4000 Düsseldorf 1
Federal Republic of Germany