SUBNORMAL TUPLES QUASI-SIMILAR TO THE SZEGÖ TUPLE

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In what follows, if $\mathcal{H}$ is a Hilbert space then $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators in $\mathcal{H}$. All the Hilbert spaces occurring below are separable and all the measures are compactly supported positive regular Borel measures on $\mathbf{C}^m$. Recall that if $S_1, \dots, S_m$ are m commuting elements in $\mathcal{B}(\mathcal{H})$, then the operator tuple $S=(S_1, \dots, S_m)$ is called *subnormal* on $\mathcal{H}$ if there exist a Hilbert space $\mathcal{H}' \supset \mathcal{H}$ and m commuting normal elements $N_1, \dots, N_m$ in $\mathcal{B}(\mathcal{H}')$ such that $N_j \mathcal{H} \subset \mathcal{H}$ and $N_j / \mathcal{H} = S_j$ for $1 \leq j \leq m$. If $H^2(\mathbf{B}^{2m})$ denotes the Hardy space of the open unit ball $\mathbf{B}^{2m}$ in $\mathbf{C}^m$ [i.e., $H^2(\mathbf{B}^{2m})$ is the completion of polynomials in $L^2(\sigma)$, σ being the surface area measure on the unit sphere $\mathbf{S}^{2m-1}$], and $M_{z_j}^{(\sigma)}$ denotes multiplication by z_j on $H^2(\mathbf{B}^{2m})$; then the multiplication tuple $M_z^{(\sigma)} = (M_{z_1}^{(\sigma)}, \dots, M_{z_m}^{(\sigma)})$, hereafter referred to as the Szegö tuple, is an example of a subnormal tuple. Moreover, $M_z^{(\sigma)}$ is cyclic. Recall that an operator tuple $S=(S_1, \dots, S_m)$ on $\mathcal{H}$ is called *cyclic* if there exists a vector u in $\mathcal{H}$ (called a *cyclic vector* for S) such that the smallest subspace of $\mathcal{H}$ containing u and invariant under $S_1, \dots, S_m$ is all of $\mathcal{H}$. The constant function 1 of course serves as a cyclic vector for $M_z^{(\sigma)}$. The following proposition is a well-known fact about cyclic subnormal tuples [3].

PROPOSITION 0. *Suppose* $S=(S_1, \dots, S_m)$ *is a subnormal tuple on* $\mathcal{H}$ *with a cyclic vector of norm one. Then there exists a probability measure* μ *with compact support in* $\mathbf{C}^m$ *and a unitary operator* U *from* $\mathcal{H}$ *onto* $H^2(\mu)$ [$H^2(\mu)$ *is the completion of polynomials in* $L^2(\mu)$] *such that* $\mathsf{U}u = 1$ *and* $S_j = \mathsf{U}^* M_{z_j}^{(\mu)} \mathsf{U}$, $1 \leq j \leq m$; *where* $M_{z_j}^{(\mu)}$ *is multiplication by* z_j *on* $H^2(\mu)$.

DEFINITION. Let $S = (S_1, \dots S_m)$ be a subnormal tuple on $\mathcal{H}$, and let $T = (T_1, \dots T_m)$ be a subnormal tuple on $\mathcal{K}$. We say that S is *quasi-similar to* T if there exist bounded linear operators $A: \mathcal{H} \to \mathcal{K}$ and $B: \mathcal{K} \to \mathcal{H}$ such that $\operatorname{Ker} A = \{0\}$, $\operatorname{Ker} B = \{0\}$, $\overline{\operatorname{Ran} A} = \mathcal{K}$, $\overline{\operatorname{Ran} B} = \mathcal{H}$, and $AS = TA$, $SB = BT$; that is, $AS_j = T_j A$ and $SB_j = BT_j$ for $1 \leq j \leq m$.

A function-theoretic characterization of subnormal tuples quasi-similar to the multiplication tuple on the Hardy space of the unit polydisc was obtained in [3]. In this note we observe that a similar characterization holds for subnormal tuples quasi-similar to the Szegö tuple. Our characterization allows us in particular to recapture a result in [2] that the Bergman tuple is not quasi-similar to the Szegö tuple. [If $A^2(\mathbf{B}^{2m})$ denotes the completion of polynomials in $L^2(V)$, V being the volumetric measure on the closed unit ball $\overline{\mathbf{B}^{2m}}$, then the multiplication tuple on $A^2(\mathbf{B}^{2m})$ is called the Bergman tuple.]

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Crucial for our purposes is the following specialized version of an approximation theorem related to the solution of the inner function problem on the unit ball $\mathbf{B}^{2m}$ (Theorem 3.5 in [4]).

THEOREM 0. *Suppose that*

(a) $\chi: \overline{\mathbf{B}^{2m}} \to (0, \infty)$ *is continuous, and*
(b) θ *is a positive measure on* $\mathbf{S}^{2m-1}$.

Then there exists a sequence $\{p_j\}$ *of polynomials such that*

(1) $|p_j| < \chi$ *on* $\overline{\mathbf{B}^{2m}}$,
(2) $\lim_{j\to\infty} p_j(z) = 0$ *uniformly on every compact subset of* $\mathbf{B}^{2m}$, *and*
(3) $\lim_{j\to\infty} |p_j(\xi)| = \chi(\xi)$ *a.e.* $[\theta]$.

PROPOSITION 1. *Let* $S = (S_1, \ldots, S_m)$ *and* $T = (T_1, \ldots, T_m)$ *be subnormal tuples on* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively and let* $A: \mathcal{H} \to \mathcal{K}$ *and* $B: \mathcal{K} \to \mathcal{H}$ *be bounded linear operators such that* $\overline{\operatorname{Ran} A} = \mathcal{K}$ *and* $\overline{\operatorname{Ran} B} = \mathcal{H}$, *with* $AS = TA$ *and* $SB = BT$. *If* T *is cyclic then so is* S, *and* A *and* B *are injective; in particular,* S *is quasi-similar to* T.

Proof. The proof is a straightforward generalization of Lemma 2.4 in [1] and depends crucially on the fact that the commutant $\{T_1, \ldots, T_m\}'$ of the polynomial algebra generated by $T_1, \ldots, T_m$ is equal to $\{M_\varphi : \varphi \in H^2(\mu) \cap L^\infty(\mu)\}$, where M_φ denotes multiplication by φ on $H^2(\mu)$ and where μ is the measure associated with T as in Proposition 0. (The last-mentioned fact is in turn a straightforward generalization to cyclic subnormal tuples of a result of Yoshino [5] for cyclic subnormal operators.) □

Propositions 0 and 1 show that to discuss subnormal tuples S quasi-similar to the Szegö tuple, we need only consider $S = M_z^{(\mu)}$ for some compactly supported measure μ on $\mathbf{C}^m$. We choose to call a function f in $H^2(\mu)$ *cyclic* if f is a cyclic vector for $M_z^{(\mu)}$. *Hereafter,* ν *will always stand for a fixed positive regular Borel measure on* $\mathbf{S}^{2m-1}$.

PROPOSITION 2. *Let* μ *be a compactly supported measure on* $\mathbf{C}^m$. *Suppose there exists an operator* $B: H^2(\nu) \to H^2(\mu)$ *with dense range such that* $M_z^{(\mu)} B = B M_z^{(\nu)}$. *Then* $\mu | \mathbf{S}^{2m-1}$ *is absolutely continuous with respect to* ν *and there is a cyclic function* g *in* $H^2(\mu)$ *such that* $\int |p|^2 |g|^2 \, d\mu \le \int |p|^2 \, d\nu$ *for every* m*-variable polynomial* p.

Proof. We may assume B has norm one. If $g = B1$, then clearly g is cyclic in $H^2(\mu)$. Define, for any Borel set $E \subset \mathbf{S}^{2m-1}$, $\eta(E) = \int_E |g|^2 \, d\mu$. Then

$$\int |p|^2 \, d\eta \le \int |p|^2 |g|^2 \, d\mu = \int |Bp|^2 \, d\mu \le \int |p|^2 \, d\nu.$$

If χ is any positive continuous function on $\mathbf{S}^{2m-1}$ then it can be extended to a positive continuous function on $\overline{\mathbf{B}^{2m}}$ (still denoted χ), and one can choose a sequence $\{p_j\}$ of polynomials corresponding to $\theta = \eta + \nu$ as in Theorem 0. In view of (1) and (3) of Theorem 0, it is clear that $\int \chi \, d\eta \le \int \chi \, d\nu$. This shows that η is absolutely continuous with respect to ν. Thus $\int_E |g|^2\,d\mu = 0$ for every Borel set $E \subset \mathbf{S}^{2m-1}$ such that $\nu(E) = 0$. Since g is cyclic, however, g does not vanish on a set of positive μ-measure and it follows that $\mu(E) = 0$ for $\nu(E) = 0$. □

PROPOSITION 3. *Let μ be a measure supported on $\overline{\mathbf{B}^{2m}}$. Suppose also there exists an operator $A: H^2(\mu) \to H^2(\nu)$ such that A has dense range and $AM_z^{(\mu)} = M_z^{(\nu)}A$. Then ν is absolutely continuous with respect to $\mu \mid \mathbf{S}^{2m-1}$ and there exists a cyclic function f in $H^2(\nu)$ such that $\int |p|^2|f|^2\,d\nu \leq \int |p|^2\,d(\mu \mid \mathbf{S}^{2m-1})$ for every m-variable polynomial p.*

Proof. We may assume A has norm one. If $f = A1$, then clearly f is cyclic in $H^2(\nu)$. Let $\chi = 1$ in Theorem 0 and choose a sequence $\{p_j\}$ of polynomials corresponding to $\theta = \nu + (\mu \mid \mathbf{S}^{2m-1})$. Then, for any m-variable polynomial p,

$$\int |Ap|^2\,d\nu = \lim_{j\to\infty} \int |p_j|^2|Ap|^2\,d\nu = \lim_{j\to\infty} \int |Ap_jp|^2\,d\nu \leq \lim_{j\to\infty} \int |p_jp|^2\,d\mu$$
$$= \lim_{j\to\infty} \int |p_jp|^2\,d(\mu \mid \mathbf{S}^{2m-1}) = \int |p|^2\,d(\mu \mid \mathbf{S}^{2m-1}).$$

Thus $\int |p|^2|f|^2\,d\nu \leq \int |p|^2\,d(\mu \mid \mathbf{S}^{2m-1})$ for any m-variable polynomial p, and appealing to Theorem 0 again we conclude that $\int \chi|f|^2\,d\nu \leq \int \chi\,d(\mu \mid \mathbf{S}^{2m-1})$ for any positive continuous function χ on $\mathbf{S}^{2m-1}$. Since f is cyclic, however, f does not vanish on a set of positive ν-measure and it follows that ν is absolutely continuous with respect to $\mu \mid \mathbf{S}^{2m-1}$. □

THEOREM 1. *Let μ be a positive regular Borel measure with compact support in $\mathbf{C}^m$. Then $M_z^{(\mu)}$ is quasi-similar to $M_z^{(\nu)}$ if and only if*

(a) *there exists a cyclic function f in $H^2(\nu)$ such that*

$$\int |p|^2|f|^2\,d\nu \leq \int |p|^2\,d(\mu \mid \mathbf{S}^{2m-1})$$

for every m-variable polynomial p, and

(b) *there exists a cyclic function g in $H^2(\mu)$ such that*

$$\int |p|^2|g|^2\,d\mu \leq \int |p|^2\,d\nu$$

for every m-variable polynomial p.

Proof. Suppose $M_z^{(\mu)}$ is quasi-similar to $M_z^{(\nu)}$. By Proposition 1 in [3], μ has its support in $\overline{\mathbf{B}^{2m}}$. Condition (a) in Theorem 1 now follows from Proposition 3 and condition (b) from Proposition 2.

Conversely, suppose (a) and (b) are true. We define $A: H^2(\mu) \to H^2(\nu)$ and $B: H^2(\nu) \to H^2(\mu)$ by requiring $Ap = fp$ and $Bp = gp$ for every m-variable polynomial p. [The boundedness of A and B is of course a consequence of conditions in (a) and (b).] Since f and g are cyclic, A and B are seen to have dense range. Now apply Proposition 1. □

REMARK 1. The characterization of subnormal tuples quasi-similar to the Szegö tuple is obtained by choosing ν in Theorem 1 to be the surface area measure σ on $\mathbf{S}^{2m-1}$.

REMARK 2. Since the restriction of the volumetric measure V on $\overline{\mathbf{B}^{2m}}$ to $\mathbf{S}^{2m-1}$ is zero, σ is not absolutely continuous with respect to $V|\mathbf{S}^{2m-1}$ and our observations above show that the Bergman tuple is not quasi-similar to the Szegö tuple.

REMARK 3. Condition (b) in Theorem 1 actually guarantees that μ has its support in $\overline{\mathbf{B}^{2m}}$. [Justification: Let α be any vector in $\mathbf{C}^m$ with the Hermitian norm $\|\alpha\|$ less than or equal to one. If (b) holds and $\cdot$ denotes the Hermitian inner product on $\mathbf{C}^m$, then for any positive integer n we have

$$\int |z\cdot\alpha|^{2n}|g|^2\,d\mu \le \int |z\cdot\alpha|^{2n}\,d\nu \le \int \|z\|^{2n}\|\alpha\|^{2n}\,d\nu \le \nu(\mathbf{S}^{2m-1}).$$

This shows that $|g|^2\,d\mu$ has its support in $\bigcap_{\|\alpha\|\le 1}\{z : |z\cdot\alpha|\le 1\} = \overline{\mathbf{B}^{2m}}$. Because g is cyclic in $H^2(\mu)$, however, μ has its support in $\overline{\mathbf{B}^{2m}}$.]

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