ANALYTIC CONTINUATION OF BIHOLOMORPHIC MAPS

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In this note we give another proof of the following result of Baouendi, Jacobowitz, and Treves [1].

THEOREM. *Let $\Omega_1, \Omega_2 \subset \mathbf{C}^n$ be domains with real analytic boundaries, and let $f: \Omega_1 \to \Omega_2$ be a biholomorphism which extends as a diffeomorphism $f: \bar{\Omega}_1 \to \bar{\Omega}_2$. If $p \in \partial\Omega_1$ and if there is no nontrivial complex variety in $\partial\Omega_2$ passing through $f(p)$, then f extends holomorphically to a neighborhood of p.*

More general results have been obtained by several authors; see Baouendi and Rothschild [2] and Diederich and Fornaess [4] and the references there. The proof given here applies in more general situations. It is evident, for instance, that the proof applies most naturally to the condition that $\partial\Omega_2$ have essentially finite type at p.

We let $\Gamma_f \subset \Omega_1 \times \Omega_2$ denote the graph of f. In what follows, we will show that there is a germ V of an n-dimensional variety in $\mathbf{C}^n \times \mathbf{C}^n$ containing $(p, f(p))$ and Γ_f. It will then follow from Lemma 1 of [3] that f extends holomorphically past p.

We may assume that $p = 0$ and that $\partial\Omega_1 = \{\varphi(\zeta, \bar{\zeta}) = 0\}$ near 0, where $\varphi(\zeta, \bar{\eta})$ is analytic in ζ and $\varphi(\zeta, \bar{\eta}) = \overline{\varphi(\eta, \bar{\zeta})}$. We may assume also that $\varphi = \frac{1}{2}(\zeta_n + \bar{\eta}_n) + \cdots$, so that $\{\operatorname{Re}\zeta_n = 0\}$ is the tangent plane to $\partial\Omega_1$ at 0. Thus

$$E = \{\operatorname{Re}\zeta_1 = \cdots = \operatorname{Re}\zeta_{n-1} = 0\} \cap \Omega_1$$

is a totally real n-manifold, and the reflection about E is given by solving the complexification of the real defining equations: $\zeta_j + \bar{\zeta}_j^* = 0$, $1 \le j \le n-1$, and $\varphi(\zeta, \bar{\zeta}^*) = \frac{1}{2}(\zeta_n + \bar{\zeta}_n^*) + \cdots = 0$.

Thus the reflection about E is an antiholomorphic map of the form:

$$(\zeta_1^*, \ldots, \zeta_n^*) = -(\bar{\zeta}_1, \ldots, \bar{\zeta}_n) + \cdots.$$

We let Ω_1^* denote the image of Ω_1 under this reflection so that $E \subset \partial\Omega_1 \cap \partial\Omega_1^*$ and $T_0\,\partial\Omega_1^* = T_0\,\partial\Omega_1$, although the outward normals point in opposite directions at 0.

Let us start with $\tilde{X}_j = \partial_{z_j} - (\varphi_{z_j}/\varphi_{z_n})\partial_{z_n}$, $1 \le j \le n-1$, and $\tilde{X}^\alpha = \tilde{X}_1^{\alpha_1} \cdots \tilde{X}_{n-1}^{\alpha_{n-1}}$. We then define X_j and X^α by setting $X_j = \tilde{X}_j$ and $X^\alpha = \tilde{X}^\alpha$ on E, and extending them from E by making the coefficients holomorphic in a neighborhood of E. Thus X^α is tangential to $\partial\Omega_1$ at points of $\partial\Omega_1 \cap E$. Although $X^\alpha \ne (X_1)^{\alpha_1} \cdots (X_{n-1})^{\alpha_{n-1}}$, the highest-order parts of both operators are equal to ∂_z^α at 0.

Now let $f(0) = 0$, and let $\psi(w, \bar{w})$ be a defining function for Ω_2. It follows that the (antiholomorphic) operators $\bar{X}^\alpha$ annihilate $\psi(f(z), \overline{f(z)})$ along E. By the chain rule, we obtain an expression of the form:

$$\bar{X}^\alpha \psi(f(z), \overline{f(z)}) = \sum \partial_{\bar{w}}^\gamma \psi P_\gamma \overline{(X^{\alpha_i} f_j(z))},$$

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where P_γ represents a polynomial in the terms $\overline{X^{\alpha_i} f_j}$, with holomorphic coefficients. We use the right-hand side of this expression to define

$$\phi^\alpha(w,z) := \sum \partial_{\bar w}^\gamma \psi(w, \overline{f(z)}) P_\gamma(\overline{X^{\alpha_i} f_j(z)})$$

for $|w| < \epsilon$ and $z \in \Omega_1 \cap \{|z| < \delta\}$. We note that ϕ^α is holomorphic in w and antiholomorphic in z, and that $\phi^\alpha(w,z) = 0$ for $z \in E$ and $w = f(z)$.

We are thus led to define the extension of the graph Γ_f over Ω_1^* as

$$V^* = \bigcap_\alpha \{(z^*, w) \in \Omega_{1,\delta}^* \times \{|w| < \epsilon\} : \phi^\alpha(w,z) = 0\},$$

where $\Omega_{1,\delta}^* = \Omega_1^* \cap \{|z| < \delta\}$. Let $\pi_1 : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n$ denote projection onto the first copy of $\mathbf{C}^n$. Since $\phi^\alpha(w, z^*)$ is holomorpic, V^* is an analytic variety. Without loss of generality, we may assume that the strongly pseudoconvex points of $\partial\Omega$ are dense in E. It is well known (see [5; 6]) that f extends past strongly pseudoconvex boundary points. Thus we have $\bar V^* \supset \bar\Gamma_f \cap \pi_1^{-1} E$, and in particular V^* is nonempty.

CLAIM. *The projection $\pi_1 : V^* \to \Omega_{1,\delta}^*$ is proper for $\delta > 0$ sufficiently small.*

To prove the claim, it suffices to show that the fiber over the point 0 (i.e., $\{w \in \mathbf{C}^n : |w| < \epsilon$ and $\phi^\alpha(w,0) = 0$ for all $\alpha\}$) is discrete. The claim is an immediate consequence of the three lemmas below. In them, we will assume that ψ satisfies the condition $\psi(w,0) = w_n$. This may be achieved by setting $\tilde w_j = w_j$, $1 \le j \le n-1$, $\tilde w_n = \psi(w,0)$ so that in these coordinates $\psi(\tilde w, \bar{\tilde w}) = \tilde w_n + \bar{\tilde w}_n + \text{mixed terms}$.

LEMMA 1. *The variety $W = \{w \in \mathbf{C}^n : |w| < \epsilon$ and $\partial_{\bar w}^\alpha \psi(w,0) = 0$ for all $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 0)\}$ is discrete for $\epsilon > 0$ small.*

Proof. We note that since ψ is real analytic, it is given by the power series

$$\psi(w, \bar w) = \sum \frac{\bar w^\alpha}{\alpha!} \partial_{\bar w}^\alpha \psi(w, 0).$$

By the remark above, we see that $W \subset \{\psi(w,0) = 0\} = \{w_n = 0\}$. If the Lemma does not hold, there is a nontrivial germ W' of a variety $0 \in W' \subset W$. For $w_n = 0$, the power series remains valid if we sum only over $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 0)$. Thus it follows that $W' \subset \{\psi(w, \bar w) = 0\}$, which contradicts the hypothesis of the Theorem. □

LEMMA 2. *$X^\alpha f_n(0) = 0$ for all $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 0)$.*

Proof. We note that

$$\phi^\alpha(w,0) = \psi_{\bar w_n}(w,0)(\overline{X^\alpha f_n(0)}) + \sum \partial_{\bar w}^\beta \psi(w,0) P_\beta(\overline{X^{\alpha_i} f_j(0)}),$$

where the P_β denotes a polynomial in terms of the form $X^{\alpha_i} f_j$, and $|\alpha_i| \le |\alpha|$; when $j = n$, $|\alpha_i| < |\alpha|$. With the normalization of ψ, we have $\partial_{\bar w}^\beta \psi(0,0) = 0$ for $\beta \ne (0, \dots, 0, 1)$. Further, $P_\beta(0) = 0$. Since $\phi^\alpha(0,0) = 0$, an induction argument gives us $X^\alpha f_n(0) = 0$. □

LEMMA 3. *Consider only indices α, β with $\alpha_n = \beta_n = 0$. If w_0 satisfies $\phi^\beta(w_0, 0) = 0$ for $|\beta| \le |\alpha|$, then $\partial_{\bar w}^\beta \psi(w_0, 0) = 0$ for $|\beta| \le |\alpha|$.*

Proof. We may make a linear change of coordinates in the range so that $X_i f_j(0) = \delta_{ij}$ for $1 \le i, j \le n-1$. Thus

$$\phi^\alpha(w,0) = \partial_{\bar w}^\alpha \psi(w,0) + \sum_{|\beta|<|\alpha|} c^\beta \partial_{\bar w}^\beta \psi(w,0)$$

for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 0)$. By Lemma 2, we only sum over β of the form $(\beta_1, \dots, \beta_{n-1}, 0)$. The Lemma then follows by induction. □

Now we observe that $\Omega_{1,\delta}^*$ and V^* were determined by the choice of totally real manifold E. For real $a_1, \dots, a_{n-1}$, we may work with

$$E_a = \{x_1 = a_1, \dots, x_{n-1} = a_{n-1}\} \cap \partial\Omega_1.$$

The vector fields X_j, X^α, the domain Ω_1^*, and the functions ϕ^α all depend real analytically on a. Since

$$\{|w| < \epsilon : \phi_a^\alpha(w,0) = 0 \text{ for all } \alpha\}$$

is discrete for $a = 0$, it follows that it is discrete for $|a|$ small. Thus the projection $\pi_1 : V_a^* \to \Omega_{1,\delta}^*(a)$ is proper for $|a|$ small, and so π_1 has a constant number of pre-images (with multiplicity)

$$\pi_1^{-1}(z) = \{W_a^1(z), \dots, W_a^q(z)\}.$$

Now let $P_j(z_1, \dots, z_n, w_j)$ be

$$\begin{aligned} P_j &= \prod_{i=1}^q (w_j - (W_a^i(z))_j) \\ &= w_j^q + A_j^1(z) w_j^{q-1} + \cdots + A_j^q(z), \end{aligned}$$

which vanishes on V_a^*. It follows that $A_j^i(z)$ is holomorphic on $\Omega_{1,\delta}^*(a)$ and smooth on the closure. Thus the functions $W_a^k(z)$ are smooth on an open dense subset of E_a. As was observed above, $w = f(z)$ must belong to $\pi_1^{-1}(z)$. Now we may take an irreducible component of V_a^* if necessary to obtain that: V_a^* is irreducible for an open dense set of $\{|a| < c\}$, V_a^* varies continuously with a, and V_a^* gives an analytic continuation of Γ_f over a dense open subset of E_a.

We conclude from this that V_a^* is independent of a. Thus $\tilde V^* = \bigcup_{|a|<c} V_a^*$ is an analytic subvariety of $\pi_1^{-1} \bigcup_{|a|<c} \Omega_{1,\delta}^*(a)$. Thus if P_j is as above, then it follows that the coefficients $A_j^k(z)$ are holomorphic in $({}^c\Omega_1) \cap \{|z| < \delta\}$. By a result of Trépreau [4], we conclude that $A_j^k(z)$ extends holomorphically to a neighborhood of 0. Thus

$$V = \{(z,w) : P_j(z, w_j) = 0,\ 1 \le j \le n\}$$

is a germ of an n-dimensional variety containing Γ_f.

REFERENCES

1. M. S. Baouendi, H. Jacobowitz, and F. Treves, *On the analyticity of CR mappings,* Ann. of Math. (2) 122 (1985), 365–400.
2. M. S. Baouendi and L. Rothschild, *Germs of CR maps between real analytic hypersurfaces,* preprint.

3. E. Bedford and S. Bell, *Extension of proper holomorphic mappings past the boundary,* Manuscripta Math. 50 (1985), 1–10.
4. K. Diederich and J.-E. Fornaess, *Proper holomorphic mappings between real-analytic pseudoconvex domains in* $\mathbf{C}^n$, preprint.
5. H. Lewy, *On the boundary behavior of holomorphic mappings,* Accad. Naz. Lincei 35 (1977), 1–8.
6. S. Pinchuk, *On the analytic continuation of holomorphic mappings,* Mat. Sb. (N.S.) 98(140) (1975), Math. USSR Sbornik 27 (1975), 375–392.
7. J.-M. Trépreau, *Sur le prolongement holomorphe des fonctions C-R définies sur une hypersurface réelle de classe* C^2 *dans* $\mathbf{C}^n$, Invent. Math. 83 (1986), 583–592.

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