AN INTERFERENCE PROBLEM FOR EXPONENTIALS

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1. Introduction. H. L. Montgomery has posed the following problem: For $T>0$ find the function of T,

$$C(T)=\sup_{A,B}\frac{\int_{-T}^{T}|B(t)|^2\,dt}{\int_{-1}^{1}|A(t)|^2\,dt}, \tag{1.1}$$

where

$$A(t)=\sum_{1}^{N}a_k e^{i\lambda_k t},\quad a_k>0;\qquad B(t)=\sum_{1}^{N}b_k e^{i\lambda_k t},\quad |b_k|\le a_k,$$

the λ_k are any N real numbers, and N may be arbitrarily large. This problem will be called Montgomery's second problem, his first problem being the special case $B(t)=A(t)$. In the second problem, $B(t)$ could be, for example, a translate of $A(t)$ that maximizes the ratio. Of course, one could require the λ_k to be positive without affecting the supremum. So by rescaling the functions, estimates of $C(T)$ would apply to power series with positive coefficients. In this connection, Norbert Wiener proved a result quoted by Boas [2] to the effect that if

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,\quad a_n\ge 0,$$

has the unit circle as its circle of convergence with boundary values in L^2 on a (small) arc centered on $z=1$, then the boundary values belong to L^2 on the whole circle. Under the same assumptions, Erdös and Fuchs [4] established the inequality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{i\theta})|^2\,d\theta\le\frac{3}{2\epsilon}\int_{-\epsilon}^{\epsilon}|f(e^{i\theta})|^2\,d\theta,\quad 0<\epsilon\le\pi.$$

Later, Wiener and Wintner [7] proved essentially that if $B(t)=A(t)$ in (1.1) then, as detailed by Bateman and Diamond [1] in their commentary, the ratio is less than $8[T]+1$. (Wiener and Wintner were not concerned with improving the constant.) As remarked by Bateman and Diamond, Wainger [6] showed that the analogous L^p result did not hold for $1\le p<2$ (doing violence to the principle that if positive coefficients don't cause trouble around the origin, then they don't cause trouble anywhere). Wainger remarked, however, that the analogous L^p result does hold for p an even positive integer; that is, the L^2-result applies to $\{f(z)\}^n$. Shapiro [5] showed, in fact, that the even positive integers are the only p for which the L^p result holds. He also showed that the 3 on the right in the Erdös–Fuchs inequality can be replaced by 2. Shapiro obtains the improved constant by replacing $|f(e^{i\theta})|^2$ by a positive positive-definite function $g(\theta)$ of period 2π and poses the problem of finding the best constant (as a function of ϵ) in this modified problem. (I solved this modification of the problem several years ago, unaware

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of Shapiro's paper at that time. The solution is too complicated to state here and will be the subject of a future paper.) The difficulty is that the best constant in the modified problem is only an upper bound for the best constant in the original problem, and they may or may not be equal. We face the same difficulty in a similar modification of the problem here, with the additional complication that the $\{\lambda_k\}$ are not specified.

Observe that if the exponentials $\{e^{i\lambda_k t}\}$ were orthogonal over both intervals $(-T, T)$ and $(-1, 1)$, then the ratio in (1.1) could be made no larger than T, as in the case of a single exponential. So in order for the ratio to be large, the exponentials should interfere (supposedly on both intervals), tending to cancel over $(-1, 1)$. As a measure of the cancellation that can occur with positive coefficients we obtain, from an upper bound for $C(T)$, the inequality

$$\frac{1}{2\tau}\int_{-\tau}^{\tau}\left|\sum_{-\infty}^{\infty} a_k e^{i\lambda_k t}\right|^2 dt \geq \frac{1}{2}\sum_{-\infty}^{\infty} a_k^2, \tag{1.2}$$

provided $a_k > 0$. That is, over any interval centered on the origin, the mean square with positive coefficients cannot be less than half what it would be if the exponentials were orthogonal over that interval. Furthermore, the constant $\frac{1}{2}$ is best possible without further constraints (see Appendix). Taking the λ_k to be integers we obtain the Erdös–Fuchs inequality with Shapiro's improved constant.

Interesting lower bounds for $C(T)$ may be obtained from periodic pulse trains of period slightly larger than 1. Thus if

$$A(t) = \sum_{k=-\infty}^{\infty} p(t - k\tau),$$

where $\tau = 1 + \epsilon$ and $p(t)$ is a "pulse" of norm 1 which vanishes for $|t| > \epsilon$ and has a positive Fourier transform, then

$$\int_{-1}^{1} |A(t)|^2\,dt = 1$$

and

$$\int_{-T_n}^{T_n} |A(t)|^2\,dt = 2n + 1, \quad n = 0, 1, 2, \ldots,$$

where $T_n = n\tau + \epsilon$. Taking $B(t) = A(t)$, we get

$$C(T_n) \geq 2n + 1.$$

Taking $B(t) = A(t + \tau/2)$, we have

$$C(X_n) \geq 2n,$$

where $X_n = (n - \frac{1}{2})\tau + \epsilon$. Letting ϵ tend to 0, we get

$$\begin{aligned} C(n - \tfrac{1}{2}+) &\geq 2n, \\ C(n+) &\geq 2n + 1. \end{aligned} \tag{1.3}$$

These lower bounds are sharp because, as we shall see,

$$C(T) \leq M(T) = \frac{n(n+1)}{2(n-T)}, \quad n-1 \leq 2T \leq n, \tag{1.4}$$

where $n = 1, 2, 3, \ldots$. The function $M(T)$ is a "scalloped-linear" function which is continuous and satisfies

$$M(T) \leq 2T+1, \tag{1.5}$$

with equality holding only for $2T$ an integer. Thus equality holds in (1.3).

Now the question arises as to whether or not the function $C(T)$ is continuous, especially at the points $2T = 1, 2, 3, \ldots$. The function is not continuous at $T=0$, but it is continuous on the left for $T>0$, since for any $\epsilon > 0$ there exists $A_\epsilon(t;T)$ and $B_\epsilon(t;T)$ such that

$$\frac{\int_{-T}^{T} |B_\epsilon(t;T)|^2 \, dt}{\int_{-1}^{1} |A_\epsilon(t;T)|^2 \, dt} > C(T) - \epsilon.$$

Therefore, since the indefinite integral of $|B_\epsilon(t;T)|^2$ is continuous, there exists a corresponding $\delta > 0$ such that

$$\frac{\int_{-T'}^{T'} |B_\epsilon(t;T)|^2 \, dt}{\int_{-1}^{1} |A_\epsilon(t;T)|^2 \, dt} > C(T) - 2\epsilon,$$

where $T' = T - \delta$. Of course δ may be much, much smaller than ϵ. Nevertheless, we have

$$C(T) = C(T-), \quad T > 0.$$

The question now is: do we have $C(T) = C(T+)$? There seems to be no simple argument for establishing continuity, but we are able to establish by examples that

$$C(T) = 2T+1, \quad T = 1, 3, 5, \ldots. \tag{1.6}$$

Thus $C(T)$ is continuous at the odd integers. At this writing it is not known whether or not $C(T)$ is continuous at the half-integers and even integers, but we can show that

$$C(T) = 2T + 1 - O(T^{-1}) \tag{1.7}$$

if $2T$ is a large integer. Thus, if $C(T)$ is discontinuous at such points then the discontinuity is eventually small.

The method for obtaining upper bounds makes it quite clear how the functions $|A(t)|^2$ and $|B(t)|^2$ must behave if $C(T) = 2T+1$, without indicating how such behavior may be obtained.

2. The mass method for upper bounds. The method of obtaining upper bounds for $C(T)$, called the *mass method,* avoids the mean squares and applies to an extension of Montgomery's problem, namely, the determination of

$$C^*(T) = \sup_{A^*, B^*} \frac{\int_{-T}^{T} B^*(t) \, dt}{\int_{-1}^{1} A^*(t) \, dt}, \tag{2.1}$$

where

$$A^*(t)=\sum_{-N}^{N}\alpha_k e^{i\lambda_k t}\geq 0 \quad \text{with } \alpha_k>0$$

and

$$B^*(t)=\sum_{-N}^{N}\beta_k e^{i\lambda_k t}\geq 0 \quad \text{with } |\beta_k|\leq \alpha_k,$$

and N may be arbitrarily large.

Here the nonnegative functions $A^*(t)$ and $B^*(t)$ have replaced $|A(t)|^2$ and $|B(t)|^2$ with the appropriate constraints on the coefficients of the exponentials. Clearly,

$$C(T)\leq C^*(T). \tag{2.2}$$

The difficulty here is in determining whether or not equality holds in (2.2), and involves all possible factorizations of candidates $A^*(t)$ in the form $|A(t)|^2$. There may not be a factorization where all the coefficients in $A(t)$ are positive. Despite this shortcoming, the mass method seems to afford the only viable means of obtaining upper bounds for $C(T)$, as explained below.

The idea here is to pick a function $p(t)$ which, when convolved with two different mass distributions, will give us good inequalities for the integrals appearing in (2.2). The choice is restricted to a suitable class of functions.

Denote by $PD\,1$ the collection of even, continuous functions $\{p(t)\}$ which vanish outside the interval $(-1,1)$ and have nonnegative Fourier transforms, normalized by the condition $p(0)=1$. In the following it will be understood that $p(t)$ belongs to $PD\,1$.

Being the Fourier integral of a nonnegative function, $p(t)$ takes its maximum value at the origin, and only there. Hence,

$$\int_{-1}^{1}A^*(t)\,dt>\int_{-1}^{1}A^*(t)p(t)\,dt=\sum_{-N}^{N}\alpha_k\hat{p}(\lambda_k), \tag{2.3}$$

where $\hat{p}$ is the Fourier transform of p.

Next, we suppose that p is convolved with a suitable measure with support in $[-T,T]$, giving a resultant

$$q(t)=\int_{-T}^{T}p(t-x)\,d\mu(x) \tag{2.4}$$

satisfying

$$q(t)\geq \chi_T(t), \tag{2.5}$$

where $\chi_T(t)$ is the characteristic function of the interval $(-T,T)$. Then

$$\begin{aligned}\int_{-T}^{T}B^*(t)\,dt&\leq\int_{-\infty}^{\infty}q(t)B^*(t)\,dt\\&=\sum_{-N}^{N}\beta_k\hat{q}(\lambda_k)\leq\sum_{-N}^{N}\alpha_k|\hat{q}(\lambda_k)|,\end{aligned} \tag{2.6}$$

where $\hat{q}(\lambda)$, the Fourier transform of $q(t)$, is given by

$$\hat{q}(\lambda)=\hat{p}(\lambda)\int_{-T}^{T} e^{-i\lambda x}\,d\mu(x).$$

Then

$$|\hat{q}(\lambda)|\le\hat{p}(\lambda)\int_{-T}^{T}|d\mu(x)|. \tag{2.7}$$

Thus we have, from (2.3), (2.6), and (2.7),

$$C(T)\le C^*(T)\le M(T;p), \tag{2.8}$$

where

$$M(T;p)=\int_{-T}^{T}|d\mu(x)|.$$

So the game here is to choose $p(t)=p(t;T)$ in $PD\,1$ so that the satisfaction of (2.5) requires a measure $d\mu(x)=d\mu(x;p,q,T)$ of minimal mass.

From the convolution equation we have the relation

$$\int_{-\infty}^{\infty} q(t)\,dt=\left[\int_{-1}^{1} p(t)\,dt\right]\left[\int_{-T}^{T} d\mu(x)\right]. \tag{2.9}$$

This, with the inequality (2.5), gives

$$M(T;p)>\frac{2T}{\int_{-1}^{1} p(t)\,dt}. \tag{2.10}$$

The integral in (2.10) is maximized over $PD\,1$ for $p(t)=\Delta(t)$, the triangular function, given for $|t|<1$ by

$$\Delta(t)=1-|t|. \tag{2.11}$$

This fact suggests that $p(t;T)=\Delta(t)$ should be a good choice for making $M(T;p)$ small, at least for large T. Indeed, since it turns out that $M(T;\Delta)\le 2T+1$, candidates for minimizing $M(T;p)$ must satisfy

$$\int_{-1}^{1} p(t;T)\,dt\ge\frac{2T}{2T+1}, \tag{2.12}$$

which requires, as is easily verified, that $p(t;T)$ tend to $\Delta(t)$ as T tends to infinity.

THE PACKING PROBLEM. Given $p(t)$ and T, what definition of $q(t)$ should one take in (2.4) in order to minimize the mass $M(T;p,q)$? In referring to $M(T;p)$ as the mass "required" for the satisfaction of (2.5), we have tacitly assumed that it is the minimal mass required for that choice of p, and its determination is a problem in linear programming which, in general, does not have a simple answer. In this problem, one seeks the greatest lower bound for $M(T;p,q)$.

LEMMA. *Suppose*

$$q_0(t)=\int_{-T}^{T} p(t-x)\,d\mu_0(x;T,p), \tag{2.13}$$

where

$$q_0(t)\le 1,\quad -T\le t\le T. \tag{2.14}$$

Then, for any solution $d\mu(x)$ *of* (2.4)-(2.5), *we have*

$$\int_{-T}^{T} |d\mu(x)| \geq \int_{-T}^{T} d\mu_0(x; T, p). \tag{2.15}$$

The lemma is a simple consequence of the identity

$$\int_{-T}^{T} g_1(x)\, d\mu_2(x) = \int_{-T}^{T} g_2(x)\, d\mu_1(x), \tag{2.16}$$

connecting solutions of the integral equations

$$g_i(t) = \int_{-T}^{T} p(t-x)\, d\mu_i(x) \quad (i = 1, 2)$$

satisfying

$$\int_{-T}^{T} |d\mu_i(x)| < \infty,$$

thereby ensuring the continuity of $g_i(t)$. □

So the game here is to *maximize* the integral on the right in (2.15), that is, to pack as much mass as possible (supposing $d\mu_0 \geq 0$) into the interval $[-T, T]$, subject to the constraint (2.14). This objective will be attained when a pair $\{q, q_0\}$ is found giving equality in (2.15), which requires first that $d\mu(x) \geq 0$, and then that equality hold in (2.14) whenever t is in the support of $d\mu(t)$, and finally that equality should hold in (2.5) whenever t is in the support of $d\mu_0(t; T, p)$. This could happen for $q = q_0$.

THEOREM 1. *If the equation* (2.13) *has a solution* $d\mu_0(x; p, T) \geq 0$ *for the special case where equality holds in* (2.14) *throughout the interval* $[-T, T]$, *and also gives* $q_0(t) \geq 0$ *outside the interval, then that solution is a minimal-mass solution of* (2.4)-(2.5).

The theorem follows immediately from the lemma and the considerations for equality to hold in (2.15). The result is obviously of limited utility, but it does apply to the case $p(t) = \Delta(t)$, for L. A. Shepp has shown (private communication) that the hypotheses will be satisfied for any even $p(t)$ that is decreasing and convex on $(0, \infty)$. This fact dismisses from consideration any $p(t)$ in *PD*1 that is decreasing and strictly convex on $(0, 1)$ in favor of $\Delta(t)$, since the latter dominates the former and therefore would give a larger $q(t)$ for the same positive mass distribution.

Notice, when Theorem 1 is applicable, that the conclusion does not rule out the possibility of other essentially different minimal-mass solutions. Indeed, we will see instances of this possibility in the next section. Minimal-mass solutions of (2.4) for which equality holds in (2.5) for the smallest possible set are, of course, very desirable for delineating conditions for equality in (2.6).

3. The mass method with $p(t; T) = \Delta(t)$. In case $p(t; T)$ is the triangular function defined in (2.11), the integral equation

$$q_0(t; T) = \int_{-T}^{T} \Delta(t-x)\, d\mu_0(x; \Delta, T), \tag{3.1}$$

where

$$q_0(t;T)=1, \quad -T\le t\le T,$$

has a solution which is obvious in case $2T$ is an integer, and otherwise may be obtained by a simple trick. The trick involves the piecewise-linear function,

$$f_n(t)=\sum_{k=1}^{n} f_n(k)\Delta(t-k), \tag{3.2}$$

where

$$f_n(k)=k, \quad k=1,2,\dots,n.$$

We have

$$f_n(t)=t, \quad 0\le t\le n, \tag{3.3}$$

and so (trick)

$$\frac{1}{\tau}[f_n(t)+f_n(\tau-t)]=\frac{1}{\tau}(t+\tau-t)=1 \tag{3.4}$$

if $0\le t\le n$ and $\tau-n\le t\le\tau$. Thus for $n-1<2T\le n$ and $\tau=-2T+2n$, we have (cf. Figure 1)

$$q_0(t;T)=\frac{1}{2(n-T)}[f_n(t-T+n)+f_n(-t-T+n)]=1, \quad -T\le t\le T. \tag{3.5}$$

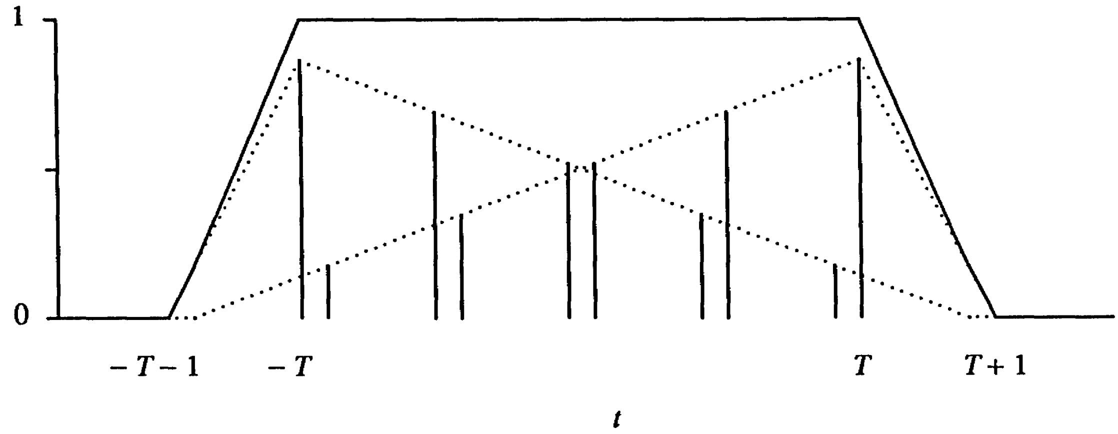

Figure 1 Composition of $q_0(t;T)$; $(T=2.1)$

The solution of (3.1) is then evident from (3.5) and (3.2), and according to Theorem 1 it is a minimal-mass solution giving

$$M(T;\Delta)\equiv M(T)=\frac{2\sum_1^n k}{2(n-T)}=\frac{n(n+1)}{2(n-T)}, \quad n-1<2T\le n, \tag{3.6}$$

where $n=1,2,3,\dots$. The function $M(T)$ is strictly convex for $n-1<2T<n$ and $M(T)=2T+1$ for $2T=n$. Therefore

$$M(T)\le 2T+1, \tag{3.7}$$

with equality only for $2T=0,1,2,\dots$.

OTHER MINIMAL-MASS SOLUTIONS. We are primarily interested in these for $2T$ an integer, but let us note that

$$M(T;\Delta)=\frac{1}{1-T},\quad 0<2T\leq 1,$$

and since

$$\frac{\Delta(t)}{1-T}\geq 1,\quad -T\leq t\leq T<1,$$

another minimal-mass solution for $0<2T<1$ is obtained with a single point mass at the origin.

In case $2T=n$ an integer, the solution of (3.1) consists of $n+1$ unit masses equispaced in the closed interval $[-T,T]$. Other minimal-mass solutions may be obtained by adding to $q_0(t;T)$ appropriate multiples of translates of the function

$$s(t)=2\Delta(t)-\Delta(t-\tfrac{1}{2})-\Delta(t+\tfrac{1}{2}), \tag{3.8}$$

so that the total mass is unaltered and the negative masses are absorbed by the positive masses of the q_0-solution. The function $s(t)$ is piecewise linear between the integers and half-integers, taking the sequence of values $\{0,-\frac{1}{2},0,1,0,-\frac{1}{2},0\}$ for $t=\{-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2}\}$, vanishing for $|t|\geq\frac{3}{2}$. Now consider the piecewise-linear function

$$\sigma_n(t)=\sum_{k=0}^{n}a_k s(t-k)\quad (n=0,1,\ldots), \tag{3.9}$$

which vanishes at the odd half-integers. If the a_k are such that

$$\sigma_n(k)>0,\quad k=0,1,\ldots,n,$$

then

$$\sigma_n(t)\geq 0,\quad -\tfrac{1}{2}\leq t\leq n+\tfrac{1}{2},$$

with equality only at the odd half-integers. Now

$$\sigma_n(k)=a_k-\tfrac{1}{2}(a_{k-1}+a_{k+1}). \tag{3.10}$$

Hence the quadratic in k,

$$a_k=(k+1)(n+1-k),\quad k=0,1,\ldots,n, \tag{3.11}$$

will give

$$\sigma_n(k)=1,\quad k=0,1,\ldots,n. \tag{3.12}$$

Thus, for sufficiently small $\epsilon_n>0$, the function

$$q_s(t;T)=q_0(t;T)+\epsilon_n\sigma_n(t+n/2) \tag{3.13}$$

will correspond to a minimal-mass solution of (2.4)–(2.5) for $p(t)=\Delta(t)$ and $2T=n+1$. We will take ϵ_n as large as possible, the constraint being that the mass m_k at the point $-T+k$ be nonnegative; that is,

$$m_k=1-\epsilon_n(a_k+a_{k-1})\geq 0,\quad k=0,1,\ldots,n+1. \tag{3.14}$$

We see from (3.11) that (a_k+a_{k-1}) is maximum for $k=(n+1)/2$ in case n is odd, and for $k=n/2$ and $(n/2)+1$ in case n is even. So, making $m_k=0$ for the maximizing k, we get

$$\epsilon_{2n+1}=\frac{1}{2(n+1)(n+2)}\quad (T=n+1),$$
$$\epsilon_{2n}=\frac{1}{2n^2+4n+1}\quad (T=n+\tfrac{1}{2}). \tag{3.15}$$

For future reference, the mass m_0 at each end-point of the interval $[-T,T]$ is

$$m_0=1-\frac{1}{n+2}\quad (T=n+1),$$
$$m_0=1-\frac{2n+1}{2n^2+4n+1}\quad (T=n+\tfrac{1}{2}), \tag{3.16}$$

giving

$$q_s(T+\tfrac{1}{2};T)=m_0/2. \tag{3.17}$$

A graph of the generic function $q_s(t;T)$ is shown in Figure 2.

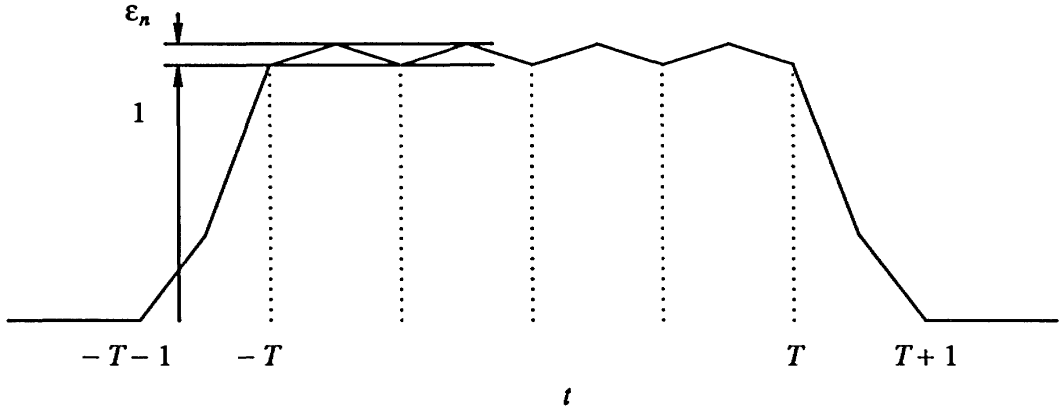

Figure 2 The function $q_s(t;T)$

CONDITIONS FOR $C^*(T)=2T+1$. In view of the inequalities (2.8) and (3.7), if $C(T)=2T+1$ then $C^*(T)=2T+1$ and $2T$ is an integer. Now suppose that $C^*(T)=2T+1$. Then for each $\epsilon>0$ there exist functions $A^*_\epsilon(t;T)$ and $B^*_\epsilon(t;T)$ of the prescribed form such that

$$\int_{-1}^{1} A^*_\epsilon(t;T)\,dt=1 \quad\text{and}\quad \int_{-T}^{T} B^*_\epsilon(t;T)\,dt=2T+1-\epsilon.$$

Then, according to the preceding results,

$$2T+1-\epsilon=\int_{-T}^{T} B^*_\epsilon(t;T)\,dt\le\int_{-\infty}^{\infty} q_s(t;T)B^*_\epsilon(t;T)\,dt$$
$$\le(2T+1)\int_{-1}^{1}(1-|t|)A^*_\epsilon(t;T)\,dt.$$

So first we must have

$$\int_{-1}^{1}(1-|t|)A^*_\epsilon(t;T)\,dt\ge\frac{2T+1-\epsilon}{2T+1}\int_{-1}^{1}A^*_\epsilon(t;T)\,dt, \tag{3.18}$$

which *requires that* $A_\epsilon^*(t;T)\,dt$ *tend* (weak-star) *in* $(-1,1)$ *to a point mass at the origin as* $\epsilon\to 0$. Next we must have

$$(3.19)\qquad 0<\int_{-\infty}^{\infty}[q_s(t;T)-\chi_T(t)]B_\epsilon^*(t;T)\,dt<\epsilon,$$

where $\chi_T(t)$ is the characteristic function of the interval $(-T,T)$ and $q_s(t;T)$ is the "saw-back" function defined in (3.13). So in the interval $(-T,T)$, $B_\epsilon^*(t;T)$ must be concentrated near the zeros of the difference $q_s(t;T)-\chi_T(t)$, these being the points $\{k-T\}$, $k=0,1,\dots,2T$. Note that this difference divided by $2\epsilon_n$ is precisely the distance of the point t from the set $\{k-T\}$ for $-T<t<T$. So if $E_T(h)$ denotes the set of points in $(-T,T)$ a distance h or more from the set $\{k-T\}$, then we have

$$(3.20)\qquad \epsilon>\int_{-T}^{T}[q_s(t;T)-1]B_\epsilon^*(t;T)\,dt>2\epsilon_n h\int_{E_T(h)}B_\epsilon^*(t;T)\,dt.$$

Since ϵ_n just depends on T, the integral over $E_T(h)$ must tend to zero with ϵ for any positive h. Now denote by $\overline{E_T}(h)$ the set of points in $(-T,T)$ a distance less than h from the set $\{k-T\}$. For small h, the set $\overline{E_T}(h)$ consists of $2T+1$ small intervals, the outer ones being of length h and the others being of length $2h$. For any positive h, the integral of $B_\epsilon^*(t;T)$ over these $2T+1$ intervals must tend to $2T+1$ as ϵ tends to zero. Furthermore, since

$$C^*(h)\le M(h)=1+O(h),$$

the integral of $B_\epsilon^*(t;T)$ *over each of the small intervals must tend to* 1 *as* ϵ *tends to zero.*

In addition, there is the *outside gap condition* which follows from replacing $q_s(t;T)$ in (3.19) by the function $q_0(t;T)$: *For every positive* h, *the integrals*

$$\int_{-T-1+h}^{-T}B_\epsilon^*(t;T)\,dt\quad and\quad \int_{T}^{T+1-h}B_\epsilon^*(t;T)\,dt$$

must tend to zero with ϵ.

The conditions for $C(T)=2T+1$ require the functions to have much in common with the periodic pulse trains described in the Introduction (when the period is 1), with the important differences that in $B(t)$ the masses of the pulses centered on $-T$ and T must somehow be included in the interval $(-T,T)$, while in $A(t)$ the masses of the pulses centered on -1 and 1 must somehow be excluded from the interval $(-1,1)$.

4. Improved upper bounds for small T. The upper bound

$$C^*(T)\le M(T)=\frac{1}{1-T},\quad 0<T<\frac{1}{2}$$

can be significantly improved for small T by taking in (2.4)

$$(4.1)\qquad q(t)=\frac{p(t)}{p(T)}\quad (p\in PD\,1),$$

provided

$$0 < p(T) \le p(t) \le p(0) = 1, \quad -T \le t \le T. \tag{4.2}$$

Then one solution of (2.4)–(2.5) is just a point mass at the origin, giving

$$C^*(T) \le \frac{1}{p(T)}. \tag{4.3}$$

This solution will in fact be a minimal-mass solution if one other condition is satisfied.

THEOREM 2. *If, in addition to* (4.1)–(4.2), *the function* $p(t)$ *is concave on* $(-2T, 2T)$, *then the corresponding solution of* (2.4)–(2.5) *is a minimal-mass solution for that choice of* p; *that is,*

$$M(T; p) = \frac{1}{p(T)}.$$

Proof. Equality is obtained in

$$\int_{-T}^{T} q(x)\, d\mu_0(x; T) = \int_{-T}^{T} q_0(x; T)\, d\mu(x) \le \int_{-T}^{T} d\mu(x),$$

where

$$q_0(t; T) = \int_{-T}^{T} p(t-x)\, d\mu_0(x; T) = \frac{1}{2p(T)}[p(t-T) + p(t+T)]$$

is even and concave over $(-T, T)$ and therefore satisfies

$$q_0(t; T) \le 1, \quad -T \le t \le T,$$

with equality holding for $t = 0$. (Cf. §2: the packing problem, the lemma, and (2.16).) □

What we would like next, of course, for use in (4.3) is the maximum over $PD\,1$ of $p(T)$, subject to the monotonicity constraint (4.2).

The least upper bound for $|p(T)|$ without the constraint, given by Boas and Kac [3], is

$$|p(T)| \le \cos\left(\frac{\pi}{n+1}\right), \quad \frac{1}{n} \le T < \frac{1}{n-1} \quad (n = 1, 2, \dots). \tag{4.4}$$

This upper bound can in fact be achieved with the monotonicity constraint for $T = 1/n$, $n = 2, 3, \dots$, a fact we state here without proof. An example for $T = 1/3$ is the piecewise-linear function with nodes at the multiples of $1/3$, taking the values

$$p(0) = 1, \quad p(1/3) = \sqrt{1/2}, \quad p(2/3) = 1/4, \quad p(1) = 0, \tag{4.5}$$

which gives the best inequality of the form (4.3); namely,

$$C^*(1/3) \le \sqrt{2}, \tag{4.6}$$

where $M(T)$ gives 1.5 for an upper bound.

The modification of the Kac–Boas inequality, under the monotonicity constraint, is an interesting problem for future consideration. A good estimate for small T is provided by the function given for $|t| \le 1$ by

$$p(t)=(1-|t|)\cos(\pi t)+\frac{1}{\pi}\left|\sin(\pi t)\right|. \tag{4.7}$$

This is the function in $PD1$ (the self-convolution of the restriction of $\cos(\pi t)$ to $(-\frac{1}{2},\frac{1}{2})$) which minimizes $|p''(0)|$ and therefore may be said to be maximally flat at the origin, with the behavior

$$p(t)=1-\frac{\pi^2 t^2}{2}+O(|t|^3), \quad t\to 0. \tag{4.8}$$

This function, being decreasing in $(0,1)$, gives

$$C^*(T)\le 1+\frac{\pi^2 T^2}{2}+O(T^3), \quad T\to 0, \tag{4.9}$$

and this, according to the Kac–Boas inequality, is the best that (4.3) can give as $T\to 0$.

5. Lower bounds by construction. It is helpful at this point to view Montgomery's problem from another perspective by defining

$$g(x)=\frac{1}{2\pi}\int_{-T}^{T} e^{ixt}B(t)\,dt=\sum_1^N b_k\frac{\sin T(x-\lambda_k)}{\pi(x-\lambda_k)}, \tag{5.1}$$

$$f(x)=\frac{1}{2\pi}\int_{-1}^{1} e^{ixt}A(t)\,dt=\sum_1^N a_k\frac{\sin(x-\lambda_k)}{\pi(x-\lambda_k)}. \tag{5.2}$$

Now we would like the make the ratio

$$R(T;g,f)=\frac{\int_{-\infty}^{\infty}|g(x)|^2\,dx}{\int_{-\infty}^{\infty}|f(x)|^2\,dx}\le C(T) \tag{5.3}$$

large, with $a_k>0$ and $|b_k|\le a_k$, taking N as large as we please and somehow choosing the set of frequencies $\Lambda=\{\lambda_k\}$.

It is easy to see that the ratio can be increased if $|b_k|$ is actually less than a_k by taking $b_k=e^{i\theta_k}a_k$ for some real θ_k. Intuitively, $g(x)$ should be real-valued. However, there are simple cases ($N=3$) with $\{a_k\}$ and Λ *fixed* where the ratio is not maximized for real b_k. It is perhaps true, when *all* the parameters are varied, that $g(x)$ may as well be real-valued, but this has not been proved. The usual L^2-arguments involving eigenfunctions of certain operators cannot be applied here because of the positivity constraint on the coefficients $\{a_k\}$. The ideas are useful, however, in constructions.

With Λ and N fixed, the calculus of variations shows the necessary conditions for a maximum to be

$$\begin{gathered} |g(\lambda_k)|=R(T;g,f)f(\lambda_k), \\ b_k\bar{g}(\lambda_k)=a_k|g(\lambda_k)|, \quad k=1,2,\dots,N. \end{gathered} \tag{5.4}$$

When Λ and N are unrestricted, the first of these conditions becomes

$$|g(\lambda)|\le R(T;g,f)f(\lambda), \quad -\infty<\lambda<\infty, \tag{5.5}$$

with equality only for $\lambda \in \Lambda$. This is not very instructive, except for showing that

$$f(x) \geq 0, \quad -\infty < x < \infty, \tag{5.6}$$

is a necessary condition for a maximum. Recall that $f(x)$ is the Fourier transform of the restriction of $A(t)$ to $(-1,1)$; so (5.6) is consistent with that restriction being concentrated around the origin.

Although we expect that the maximum is not attainable for any T, the variational conditions provide some guidance in constructing examples.

THE CASE $T=1$. All the new lower bounds for $C(T)$ stem from the construction for this special case. In this case we partition Λ into disjoint sets, Λ_1 and Λ_2, and define

$$f(x) = f_1(x) + f_2(x), \tag{5.7}$$

$$g(x) = f_1(x) - f_2(x), \tag{5.8}$$

where

$$f_i(x) = \sum_{\Lambda_i} a_k \frac{\sin(x-\lambda_k)}{\pi(x-\lambda_k)}, \quad a_k > 0 \ \ (i=1,2). \tag{5.9}$$

That is, we take

$$b_k = \begin{cases} a_k & \text{if } \lambda_k \in \Lambda_1, \\ -a_k & \text{if } \lambda_k \in \Lambda_2. \end{cases} \tag{5.10}$$

Now both f and g are real and the problem is symmetric in f_1 and f_2.

Abbreviating the previous ratio by R, we have

$$R = 1 - \frac{4\int_{-\infty}^{\infty} f_1(x) f_2(x)\,dx}{\int_{-\infty}^{\infty} [f_1(x)+f_2(x)]^2\,dx}. \tag{5.11}$$

Clearly, we want

$$\int_{-\infty}^{\infty} f_1(x) f_2(x)\,dx < 0. \tag{5.12}$$

We may suppose that

$$\int_{-\infty}^{\infty} [f_1(x)]^2\,dx = 1. \tag{5.13}$$

Then if we replace $f_2(x)$ in (5.11) by $mf_2(x)$, where $m>0$, we find that the best value of the multiplier is $m=1$ when

$$\int_{-\infty}^{\infty} [f_2(x)]^2\,dx = 1, \tag{5.14}$$

which is to be expected from the symmetry, and which we suppose to be true. Thus we have two functions of the form (5.9), of unit norm in a special subspace of L^2 where the reproducing kernel is

$$K(x,t) = \frac{\sin(x-t)}{\pi(x-t)}. \tag{5.15}$$

We wish to minimize the dot product (f_1, f_2), which, owing to the special form $\sum a_k K(x, \lambda_k)$ of the functions f_1 and f_2, may be written

(5.16) $$\int_{-\infty}^{\infty} f_1(x) f_2(x)\, dx = \sum_{\Lambda_1} a_k f_2(\lambda_k) = \sum_{\Lambda_2} a_k f_1(\lambda_k).$$

Here, $a_k > 0$ and

(5.17) $$\sum_{\Lambda_1} a_k f_1(\lambda_k) = \sum_{\Lambda_2} a_k f_2(\lambda_k) = 1.$$

Now if we could make

(5.18) $$\begin{aligned} f_2(\lambda_k) &= -\rho f_1(\lambda_k), \quad \lambda_k \in \Lambda_1, \\ f_1(\lambda_k) &= -\rho f_2(\lambda_k), \quad \lambda_k \in \Lambda_2 \end{aligned}$$

we would have $(f_1, f_2) = -\rho$, giving

(5.19) $$R = \frac{1+\rho}{1-\rho}.$$

The upper bound we have for R is 3, and we know from the conditions for $C(T) = 2T+1$ that 3 cannot actually be attained. Therefore (theorem)

(5.20) $$(f_1, f_2) > -\tfrac{1}{2}.$$

The challenge here is to show that (f_1, f_2) may be arbitrarily close to $-\frac{1}{2}$, and thereby establish that $C(1) = 3$.

It is easy to falter at this point after a few unsuccessful tries; indeed, it was difficult at first to find an example where ρ exceeded $\frac{1}{3}$, giving $C(1) > 2$. This example resulted from taking Λ_1 to consist of a single point, the origin, so that

$$f_1(x) = \sqrt{\pi}\, \frac{\sin x}{\pi x},$$

and then taking Λ_2 to consist of the points $\pm(3\pi/2 + 2n\pi)$, $n = 0, 1, \ldots$. So Λ_2 almost coincides with the set of points where f_1 takes its local minima. Furthermore, the points in Λ_2 fall on a lattice where the translates of $(\sin x)/x$ are orthogonal, which simplifies the construction wherein f_2 is made proportional to $-f_1$ on this set, and scaled so as to have unit norm. This construction gives

$$\rho^2 = \frac{\dfrac{1}{3^2} + \dfrac{1}{7^2} + \dfrac{1}{11^2} + \cdots}{1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \cdots} > \frac{1}{3^2},$$

or

$$\rho \approx .358849668.$$

However, it may be shown that the condition (5.5) for a maximum is not satisfied, and therefore this ρ gives

$$C(1) > 2.11939\ldots.$$

It is clear that the set Λ_2 (and f_2) can be perturbed to obtain a local maximum, but the resulting value of ρ (which can be closely estimated) will be only slightly larger. What is needed is a *drastic* change in order for ρ to be significantly increased. In this example the expected symmetry in f_1 and f_2 is not found; f_1 offers to f_2 all the negative values $f_1(\lambda_k)$, $\lambda_k \in \Lambda_2$, while f_2 offers to f_1 only the negative value $f_2(0)$. There must be a symmetric scheme in which the functions can offer more substantial negative values to each other.

The choice of Λ_1 and Λ_2 is, of course, crucial. There is not much guidance here, but for the purpose of simplifying the construction we will require that the points in Λ_1 fall on one lattice while the points in Λ_2 fall on another lattice, the lattice points being spaced by a multiple of π. In order for the functions to interfere strongly one might expect that the points in Λ_1 and Λ_2 should interlace, so that any point in one set would not be distant from some point in the other set, but under scrutiny this arrangement appears to be at odds with the objective of making the function belonging to one set be strongly negative on the other set. It is somewhat surprising that, in an arrangement which works, the points in the two sets fall on opposite half-lines. In this arrangement we take

$$f_1(x) = \sum_{k=0}^{\infty} a_k \frac{\sin(x-2k\pi)}{\pi(x-2k\pi)}, \tag{5.21}$$

$$f_2(x) = f_1(-\theta - x), \tag{5.22}$$

where θ and $\{a_k\}$ are to be chosen. The nominal value of θ is $3\pi/2$. In contrast to the previous example, f_1 is now offering negative values only on the left, but these are being reinforced by the additional terms in the sum.

What we would like here is

$$f_2(2n\pi) = -\rho f_1(2n\pi), \quad n = 0, 1, 2, \ldots \tag{5.23}$$

or

$$\frac{-\sin\theta}{2\pi} \sum_{k=0}^{\infty} \frac{a_k}{n+k+\lambda} = \rho a_n, \quad n = 0, 1, 2, \ldots, \tag{5.24}$$

where

$$\lambda = \frac{\theta}{2\pi}, \quad a_n > 0,$$

and ρ is close to $\frac{1}{2}$. We know from (5.20) that we cannot have $\rho = \frac{1}{2}$ with $\sum a_k^2 < \infty$ and $a_k > 0$.

The relation in (5.24) suggests the integral identity,

$$\frac{1}{\pi} \int_0^{\infty} \frac{t^{-\nu}\, dt}{x+t} = \frac{x^{-\nu}}{\sin \pi\nu}, \quad x > 0 \quad (0 < \nu < 1), \tag{5.25}$$

where the integral could be closely approximated by a sum for $x = n + \lambda$. In fact, we can obtain a discrete analog of (5.25) from the integral for the beta function,

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

Setting $y = 1 - \nu$, we have

$$(1-t)^{-\nu} = \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} t^k,$$

and then

$$\sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} \frac{1}{(x+k)} = \Gamma(1-\nu) \frac{\Gamma(x)}{\Gamma(x+1-\nu)} \quad (x>0,\ \nu<1). \tag{5.26}$$

For the special values of x, $x=n+\nu$, this relation may be written as

$$\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} \frac{1}{(n+k+\nu)} = \frac{1}{\sin \pi\nu} \frac{(\nu)_n}{n!}, \quad n=0,1,\dots. \tag{5.27}$$

Here $0<\nu<1$ and

$$\frac{(\nu)_n}{n!} = \frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(n+1)} \sim \frac{1}{\Gamma(\nu) n^{1-\nu}}, \quad n \to \infty. \tag{5.28}$$

(So actually we would replace ν by $1-\nu$ to copy the asymptotic behavior in (5.25).)

Although (5.27) reveals an exact solution of (5.24) for $\theta = 2\pi\nu$, it is not what we want. We must have $\sin\theta$ close to -1 and ν slightly less than $\frac{1}{2}$ in order to have ρ close to $\frac{1}{2}$; it is not important that (5.24) be exactly satisfied as ν tends to $\frac{1}{2}$. If we take

$$f_1(x) = c_\nu \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} \frac{\sin(x-2k\pi)}{\pi(x-2k\pi)} \tag{5.29}$$

and

$$f_2(x) = f_1(-3\pi/2 - x) \tag{5.30}$$

then

$$f_2(2n\pi) = -\frac{c_\nu}{2\pi} \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} \frac{1}{(n+k+\frac{3}{4})} = -\rho_n f_1(2n\pi), \tag{5.31}$$

where

$$\rho_n = \frac{1}{2\sin\pi\nu} \frac{\Gamma(n+\frac{3}{4})\Gamma(n+1)}{\Gamma(n+\nu)\Gamma(n-\nu+\frac{7}{4})} \sim \frac{1}{2\sin\pi\nu}, \quad n\to\infty. \tag{5.32}$$

The effective ρ_n will be very close to $\frac{1}{2}$ if we take $\nu = \frac{1}{2} - \epsilon$, where ϵ is a sufficiently small positive number, for with

$$a_k = c_\nu \frac{(\nu)_k}{k!} \quad \text{and} \quad \sum_{k=0}^{\infty} a_k^2 = \pi \tag{5.33}$$

the multiplier c_ν must tend to 0 as ν tends to $\frac{1}{2}$, since

$$\sum_{k=0}^{\infty} \frac{(\nu)_k(\nu)_k}{k!\,k!} = {}_2F_1(\nu,\nu;1;1) = \frac{\Gamma(1-2\nu)}{\Gamma(1-\nu)\Gamma(1-\nu)}. \tag{5.34}$$

Thus for any fixed m the first m values of ρ_n become insignificant as ν tends to $\frac{1}{2}$, with the result that the dot product (f_1, f_2) tends to $-\frac{1}{2}$. In fact, we can express the dot product as the ratio of two hypergeometric series and find the effective ρ_n,

$$\bar{\rho} = -\sum_{n=0}^{\infty} f_1(2n\pi) f_2(2n\pi) = \frac{\{\Gamma(1-\nu)\}^2}{2\pi} \frac{\Gamma(\lambda)}{\Gamma(\lambda+1-2\nu)} \quad (\nu<\tfrac{1}{2}), \tag{5.35}$$

where $\lambda = \frac{3}{4}$. Therefore, if δ is a given small positive number then a corresponding small positive number ϵ can be found such that $\bar{\rho} > \frac{1}{2} - \delta$ for $\nu = \frac{1}{2} - \epsilon$. Finally, N can be found such that $\sum_N^\infty a_k^2 < \delta$. Hence, we have the result

$$C(1) = C^*(1) = 3. \tag{5.36}$$

A graph of the limit function $f_1(x)/f_1(0)$ for $\nu = \frac{1}{2}$ is shown in Figure 3.

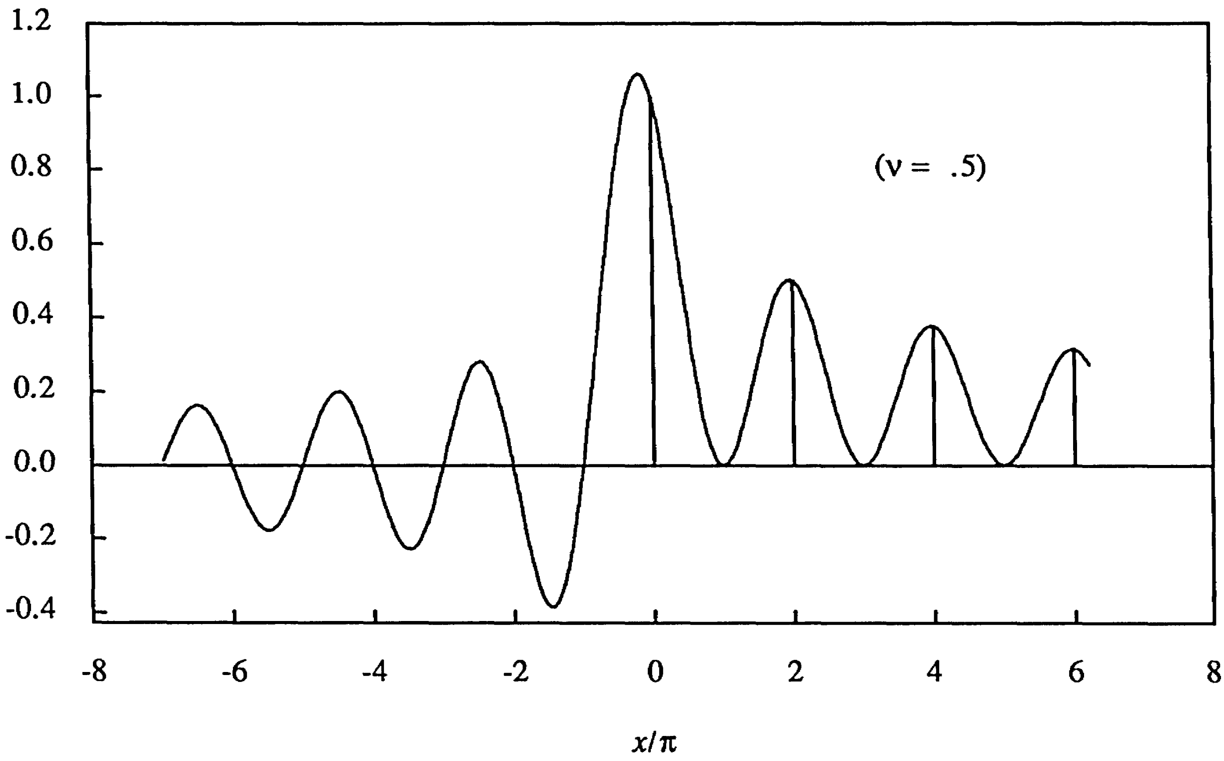

Figure 3 The limit function $f_1(x)/f_1(0)$

When f_1 and f_2 are shifted to the right by $\lambda\pi = 3\pi/4$ and the normalization ignored, the construction here is equivalent to taking

$$A(t) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \cos(\lambda\pi + 2n\pi)t, \tag{5.37}$$

$$B(t) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \sin(\lambda\pi + 2n\pi)t \tag{5.38}$$

and letting ν approach $\frac{1}{2}$ from below. The same result is obtained if the lower limit in the sum is replaced by any positive integer, or if any integer is added to λ. The period of these functions is 8 and

$$B(t) = A(t+2). \tag{5.39}$$

We will see in the next section that this same pair of functions serves to establish the result

$$C(T) = C^*(T) = 2T+1, \quad T = 1, 3, 5, \ldots, \tag{5.40}$$

except that $B(t)$ is replaced by $A(t)$ in case $T=3,7,11,\dots$, conforming (incidentally) to the constraint in Montgomery's first problem.

THE CASE $2T=1,2,3,\dots$. We are led from the construction for the case $T=1$ to consider the function

$$F_\nu(t)=(1-e^{i2\pi t})^{-\nu}=\frac{e^{i\nu\phi(t)}}{|2\sin\pi t|^\nu}\quad(0<\nu<1), \tag{5.41}$$

where

$$\begin{aligned}&\phi(t)=(1-2t)\pi/2,\quad 0<t<1,\\ &\phi(0)=0,\\ &\phi(t)=\phi(t+1).\end{aligned} \tag{5.42}$$

Although we are concerned here only with the case $\nu\approx\frac{1}{2}$, let us note that for $0<\nu<1$ the function is locally integrable and has the Fourier series

$$F_\nu(t)=\sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}e^{i2n\pi t}, \tag{5.43}$$

which converges unless t is an integer. Since the Fourier coefficients are positive, the function $F_\nu(t)$ may be said to be *positive definite*. Usually, a positive definite function is understood to be finite at the origin, and therefore, being the Fourier transform of a finite positive measure, is a continuous function. However, there is no real reason for excluding discontinuous positive-definite functions, such as $F_\nu(t)$. Actually, we are required in Montgomery's problem to approximate such discontinuous functions with finite sums, but we can rely on the partial sums of the series ($\nu<\frac{1}{2}$) to converge in mean-square to the limit function, which we take as the nucleus of the construction. Knowing the end result of the previous section, we can now give a simple independent explanation of how the construction works.

The phase discontinuity $\nu\pi$ in $\nu\phi(t)$ at the integers is essential to the construction. It allows us, in effect, to create *asymmetric* approximate delta functions at the integers by multiplying $F_\nu(t)$ by an exponential, taking the real (or imaginary) part, and letting ν tend to $\frac{1}{2}$. Thus, the square of the function

$$G_\nu(t;\lambda,\theta)=\operatorname{Re}F_\nu(t)e^{i(\lambda\pi t+\theta)}=\frac{g_\nu(t;\lambda,\theta)}{|2\sin\pi t|^\nu}, \tag{5.44}$$

where

$$g_\nu(t;\lambda,\theta)=\cos\{\lambda\pi t+\theta+\nu\phi(t)\}, \tag{5.45}$$

will, with suitable normalization, behave as $\nu\to\frac{1}{2}$ like a sequence of asymmetric delta functions situated at the integers, with the mass to the left at the point $t=n$ tending to

$$\mu(n-)=g_\nu^2(n-;\lambda,\theta)=\cos^2(\lambda n\pi+\theta-\nu\pi/2), \tag{5.46}$$

and the mass to the right tending to

$$\mu(n+)=g_\nu^2(n+;\lambda,\theta)=\cos^2(\lambda n\pi+\theta+\nu\pi/2). \tag{5.47}$$

Note that the sum of the right and left masses tends to 1 as ν tends to $\frac{1}{2}$. Thus with unit masses at $t=1, 2, \ldots, n-1$, we have

$$\lim_{\nu \to 1/2} \frac{\int_0^n G_\nu^2(t;\lambda,\theta)\,dt}{\int_0^1 |2\sin \pi t|^{-2\nu}\,dt} = \mu(0+) + (n-1) + \mu(n-), \tag{5.48}$$

where

$$\mu(0+) = \cos^2(\theta + \pi/4), \quad \mu(n-) = \cos^2(\lambda n\pi + \theta - \pi/4).$$

This heuristic result can be verified from the exact formula

$$\int_0^n G_\nu^2(t;\lambda,\theta)\,dt = \frac{\Gamma(1-2\nu)}{2\{\Gamma(1-\nu)\}^2}\{n + \gamma\cos(n\lambda\pi + 2\theta)\sin(n\lambda\pi)\}, \tag{5.49}$$

where

$$\gamma = \gamma(\lambda, \nu) = \frac{\{\Gamma(1-2\nu)\}^2}{\pi}\,\frac{\Gamma(\lambda)}{\Gamma(\lambda + 1 - 2\nu)} \quad \text{and} \quad \nu < \tfrac{1}{2}.$$

The Fourier integral,

$$\frac{1}{\pi}\int_0^\pi \frac{e^{i2xt}\,dt}{(2\sin t)^{2\nu}} = \frac{\Gamma(1-2\nu)e^{i\pi x}}{\Gamma(1-\nu-x)\Gamma(1-\nu+x)}, \quad \nu < \tfrac{1}{2}, \tag{5.50}$$

is the basic formula, which may be obtained by integrating the analytic function $z^\alpha(1-z^2)^\beta$ around a semicircular path connecting the points $z=-1$ and $z=+1$, expressing the integral along the real axis in terms of the beta function and taking the appropriate values of α and β.

Now, by choice of λ and θ, both of the limiting end-point masses $\mu(0+)$ and $\mu(n-)$ in (5.48) can be made equal to 1, giving a total mass of $n+1$ in the open interval $(0, n)$. This is the result we need for one integral, but there is another integral to consider.

For $2T = n$ an integer, we set

$$B(t-T) = G_\nu(t;\lambda,\theta), \tag{5.51}$$

$$A(t) = G_\nu(t;\lambda,0), \tag{5.52}$$

where $\nu < \frac{1}{2}$, $A(t)$ is positive definite, and the coefficient constraints are satisfied. Then we choose λ and θ to maximize the ratio

$$\frac{\int_{-n/2}^{n/2} B^2(t)\,dt}{\int_{-1}^{1} A^2(t)\,dt} = \frac{n + \gamma\cos(n\lambda\pi + 2\theta)\sin(n\lambda\pi)}{2 + \gamma\sin(2\lambda\pi)} \tag{5.53}$$

as $\nu \to \frac{1}{2}$. Clearly θ will be chosen so that

$$\cos(n\lambda\pi + 2\theta)\sin(n\lambda\pi) = |\sin(n\lambda\pi)|. \tag{5.54}$$

Then we have

$$C\left(\frac{n}{2}\right) \geq \max_\lambda \frac{n + |\sin(n\lambda\pi)|}{2 + \sin(2\lambda\pi)}. \tag{5.55}$$

In order for the maximum in (5.55) to be $n+1$, it is readily seen that $\lambda - \frac{3}{4}$ must be some (any) integer, say 0, and that $n/2 = T$ must be an odd integer. Therefore,

$$C(T) = C^*(T) = 2T+1, \quad T = 1, 3, 5, \dots . \tag{5.56}$$

For other integral values of $2T$, it is not possible to choose λ and θ so as to satisfy the conditions for $C(T) = 2T+1$. If λ does not differ from $\frac{3}{4}$ by an integer, then $\cos\{\lambda\pi t + \nu\phi(t)\}$ will not vanish for $t = 1-$ and $\nu = \frac{1}{2}$, as it must in order for

$$\int_{-1}^{1} A^2(t)\,dt \to \int_{-1}^{1} (1-|t|)A^2(t)\,dt$$

as $\nu \to \frac{1}{2}$, that is, in order to exclude the masses at the end points of the interval. When this condition is satisfied, the other end-point conditions [i.e., $\mu(0+) = \mu(n-) = 1$ in (5.48)] required for

$$\int_{-T}^{T} |B(t)|^2\,dt \to (2T+1)\int_{-1}^{1} A^2(t)\,dt$$

cannot be met except for $T = 1, 3, 5, \dots$; otherwise a compromise must be made.

A graph of selected ordinates for the limit function

$$A(t) = \mathrm{Re}\,\frac{e^{i(3/4)\pi t}}{(1-e^{i2\pi t})^{1/2}}, \tag{5.57}$$

for the case $T = 1, 3, 5, \dots$, is shown in Figure 4.

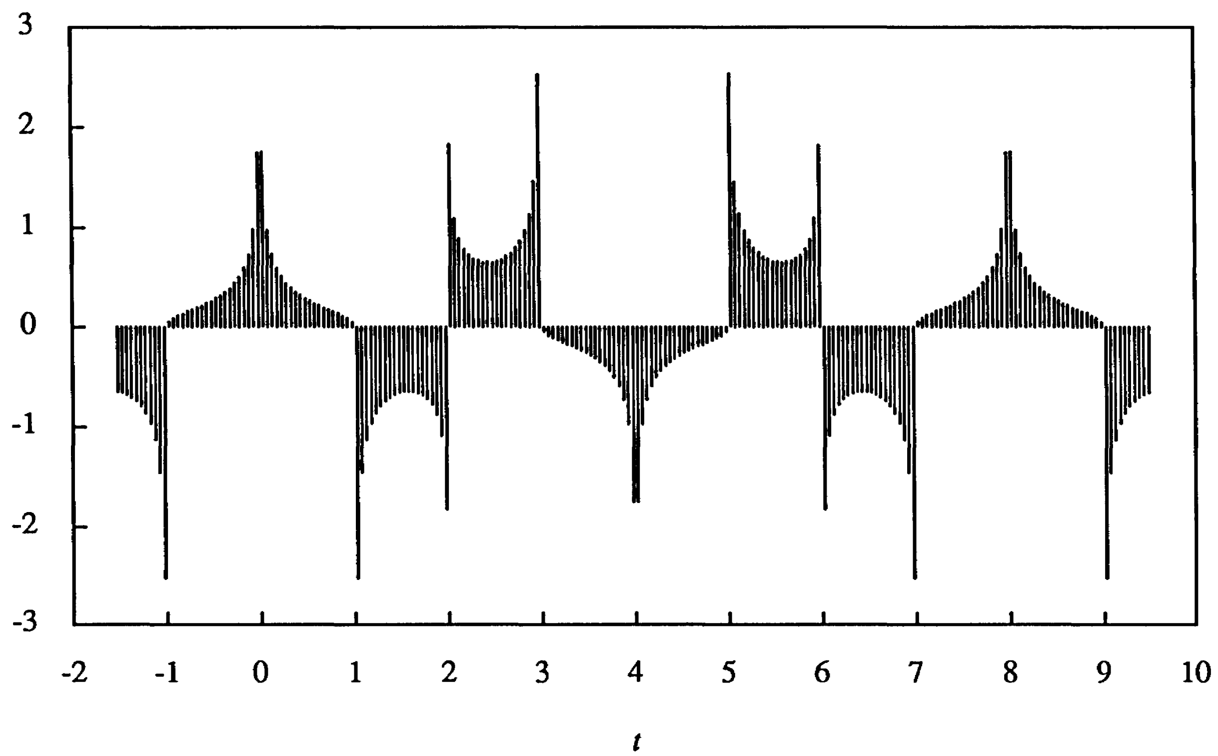

Figure 4 Ordinates of the limit function $A(t)$; $T = 1, 3, 5, \dots$

The conspicuous feature is the asymmetry of the singularities at the odd integers. The idea is to take $B(t)$ to be a translate of $A(t)$ such that one of the stronger singularities is turned inward at each end of the interval $(-T, T)$. Thus we take

$$B(t)=A(t+2),\quad T=1,5,9,\ldots, \tag{5.58}$$
$$B(t)=A(t),\quad T=3,7,11,\ldots.$$

The construction here certainly favors the case in which T is an odd integer, and raises the question as to whether or not this case is intrinsically more favorable than the others. For the other cases we have the following lower bounds for $C(T)$.

In (5.55), for $n/2=T=2m$, set $2\lambda\pi=3\pi/2+\epsilon$. Then

$$C(2m)\geq \max_{\epsilon} \frac{4m+|\sin(2m\epsilon)|}{2-\cos\epsilon}, \tag{5.59}$$

which gives, for $\epsilon=\pi/(4m)$,

$$C(2m)>\frac{4m+1}{2-\cos(\pi/4m)}\sim 4m+1-\frac{\pi^2}{8m}. \tag{5.60}$$

In case $2T=2m+1=n$, it is convenient to set

$$2\lambda\pi=-\frac{\pi}{2}+\frac{2\alpha}{2m+1}.$$

Then

$$C(m+\tfrac{1}{2})\geq \max_{\alpha} \frac{2m+1+|\sin\{(2m+1)\pi/4-\alpha\}|}{2-\cos(2\alpha/(2m+1))}. \tag{5.61}$$

Here α should be approximately $\pm\pi/4$, the sign depending on m, so as to make $|\sin\{xx\}|$ close to 1. There is a choice of sign with $\alpha=\pm\pi/4$ which gives $|\sin\{xx\}|=1$, and therefore

$$C(m+\tfrac{1}{2})>\frac{2m+2}{2-\cos(\pi/(4m+2))}\sim 2m+2-\frac{\pi^2}{16m}. \tag{5.62}$$

Note the difference in the asymptotic lower bounds when expressed in terms of T:

$$C(T)>\sim 2T+1-\frac{\pi^2}{4T},\quad T=2,4,6,\ldots, \tag{5.63}$$
$$C(T)>\sim 2T+1-\frac{\pi^2}{16T},\quad T=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots.$$

The maximum in (5.55) gives the lower bounds for $C(T)$ shown in Table 1.

In conclusion, some comments on discontinuous positive-definite functions seem in order. We are familiar with the inequality $|F(t)|\leq F(0)$ for continuous positive-definite functions, where equality can hold for some $t\neq 0$ if and only if $F(t)$ is periodic. Once $F(0)$ is allowed to be infinite, we may see unfamiliar behavior in $|F(t)|$. For example, we could have

$$|F(t+a)|>|F(t)|,\quad 0<t<\epsilon. \tag{5.64}$$

We see this in Figure 4 for $a=1,5,9,\ldots$. We could even have the two-sided inequality

$$|F(t+a)|>|F(t)|\quad -\epsilon<t<0 \text{ and } 0<t<\epsilon. \tag{5.65}$$

Table 1 Lower bounds for $C(T)$

T	*Lower Bound*	T	*Lower Bound*
0.5	1.739712	5.5	11.913011
1.0	3	6.0	12.680572
1.5	3.808848	6.5	13.923650
2.0	4.378802	7.0	15
2.5	5.851906	7.5	15.931991
3.0	7	8.0	16.745037
3.5	7.879757	8.5	17.938700
4.0	8.574933	9.0	19
4.5	9.898996	9.5	19.944212
5.0	11	10.0	20.753259

We see this in the example of $F(t) = G_\nu(t; \frac{1}{2}, 0)$, $\frac{1}{2} < \nu < 1$ [cf. (5.44)], where $|F(t)|$ is even about the point $t=1$ as well as the origin and

$$\frac{|F(t+1)|}{|F(t)|} = |\tan\{\pi t/2 + \nu\phi(t)\}|, \tag{5.66}$$

which is approximately $\tan(\nu\pi/2)$ for small $|t|$.

A discontinuous positive-definite function $F(t)$, when convolved with a continuous positive-definite function that vanishes outside $(-\epsilon, \epsilon)$, must always give a continuous positive-definite resultant $F_\epsilon(t)$. Thus the inequality in (5.65) implies that $|F(t)|$ is not a positive-definite function, since the resultant would be larger for $t=a$ than for $t=0$. On the other hand, $|F(t)|^2$ is always positive definite, provided this function is also integrable over an interval containing the origin. So (5.65), which cannot hold if $F(0)$ is finite, also implies that F is not locally L^2.

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Appendix. Here we establish the inequality (1.2) concerning the interference over $(-\tau, \tau)$ of exponentials with positive coefficients. We may assume $\tau = 1$. Then the result is equivalent to

$$\pi \int_{-\infty}^{\infty} [f(x)]^2\, dt \geq \frac{1}{2} \sum_{-\infty}^{\infty} a_k^2, \tag{A.1}$$

where

$$f(x) = \sum_{-\infty}^{\infty} a_k \frac{\sin(x-\lambda_k)}{\pi(x-\lambda_k)}, \quad a_k > 0. \tag{A.2}$$

This result is closely related to, but not equivalent to, the inequality established in Section 5,

$$\int_{-\infty}^{\infty} f_1(x) f_2(x)\, dx > -\frac{1}{2}, \tag{A.3}$$

where f_1 and f_2 are any two functions of the form (A.2), each of norm 1 in L^2.

We suspect that the inequality in (A.1) is also strict, but this has not been proved. Recall that the strict inequality in (A.3) comes from detailed arguments showing that the supremum $C(1)=3$ cannot actually be attained. Similar arguments are lacking for the inequality (A.1), which is obtained from the upper bound $C(T)\le 2T+1$, letting $T\to\infty$.

As in Section 5, set

$$g(x)=\sum_{-\infty}^{\infty} b_k \frac{\sin T(x-\lambda_k)}{\pi(x-\lambda_k)}, \quad |b_k|\le a_k \tag{A.4}$$

and

$$R(T;g,f)=\frac{\int_{-\infty}^{\infty}|g(x)|^2\,dx}{\int_{-\infty}^{\infty}[f(x)]^2\,dx}. \tag{A.5}$$

Now we can always take $|b_k|=a_k$ and, by choosing the sign of b_k (in perhaps a non-optimal way), make

$$\pi\int_{-\infty}^{\infty}|g(x)|^2\,dx\ge T\sum_{-\infty}^{\infty}a_k^2. \tag{A.6}$$

To see that this is possible, relabel the a_k and λ_k so that $a_k=0$ for $k<0$, and $a_0\ge a_1\ge a_2\cdots$. Then define

$$g_n(x)=\sum_{k=0}^{n} b_k \frac{\sin T(x-\lambda_k)}{\pi(x-\lambda_k)}, \tag{A.7}$$

where $b_0=a_0$ and for $n\ge 1$ take $b_n=\pm a_n$, where the sign is chosen to give

$$\pi\int_{-\infty}^{\infty}[g_n(x)]^2\,dx\ge\pi\int_{-\infty}^{\infty}[g_{n-1}(x)]^2\,dx+Ta_n^2, \tag{A.8}$$

and hence (A.6).

Now define $\xi(f)$ by

$$\int_{-\infty}^{\infty}[f(x)]^2\,dx=\xi(f)\sum_{-\infty}^{\infty}a_k^2. \tag{A.9}$$

Then from (A.6) and (A.9) we have

$$R(T;g,f)\ge\frac{T}{\xi(f)}. \tag{A.10}$$

Then, since

$$R(T;g,f)\le C(T)\le 2T+1, \quad T>0, \tag{A.11}$$

we must have

$$\xi(f)\ge\tfrac{1}{2} \tag{A.12}$$

in order for (A.11) to be satisfied as $T\to\infty$. Furthermore, $\frac{1}{2}$ is the greatest lower bound for $\xi(f)$, since we have given a construction in Section 5, $f(x)=f_1(x)+f_2(x)$, which allows $\xi(f)$ to be arbitrarily close to $\frac{1}{2}$. In this construction we had $\lambda_k=2k\pi$ for $k\ge 0$, and $\lambda_k=2k\pi+\pi/2$ for $k<0$. It would be of interest to have a characterization of sets $\Lambda=\{\lambda_k\}$ which allow $\xi(f)$ to be arbitrarily close to $\frac{1}{2}$. We only know of one more essentially different example.

In this example, the set Λ is a lattice in which the spacing between points is slightly less than 2π. This would seem to be an unlikely set for the desired result, in view of the fact that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(x-\lambda_j)}{(x-\lambda_j)}\frac{\sin(x-\lambda_k)}{(x-\lambda_k)}\,dx=\frac{\sin(\lambda_j-\lambda_k)}{(\lambda_j-\lambda_k)}$$

is practically zero for $\lambda_j \neq \lambda_k$. The example is a simple one which perhaps has not been recognized as an example of extreme interference.

Suppose $f_\epsilon(x)$ is a positive continuous function of L^2 whose Fourier transform vanishes outside $[-\epsilon,\epsilon]$. Then

$$\text{(A.13)} \qquad f_\epsilon(x)=\int_{-\infty}^{\infty} f_\epsilon(t)\frac{\sin\alpha(x-t)}{\pi(x-t)}\,dt, \quad \alpha\geq\epsilon.$$

Now for fixed x and α, the Fourier transform of the function of t,

$$g_\beta(t)=f_\epsilon(t)\frac{\sin\alpha(x-t)}{\pi(x-t)},$$

vanishes outside $(-\beta,\beta)$, where $\beta=\alpha+\epsilon$. Then, by the Poisson sum formula,

$$\text{(A.14)} \qquad \tau\sum_{-\infty}^{\infty} g_\beta(k\tau)=\int_{-\infty}^{\infty} g_\beta(t)\,dt, \quad \tau\leq\frac{2\pi}{\beta}.$$

Thus,

$$\text{(A.15)} \qquad f_\epsilon(x)=\tau\sum_{-\infty}^{\infty} f_\epsilon(k\tau)\frac{\sin\alpha(x-k\tau)}{\pi(x-k\tau)}$$

and

$$\text{(A.16)} \qquad \int_{-\infty}^{\infty}[f_\epsilon(x)]^2\,dx=\tau\sum_{-\infty}^{\infty}[f_\epsilon(k\tau)]^2.$$

Now suppose $\alpha=1>\epsilon>0$ and $\tau=2\pi/(1+\epsilon)$. Then

$$\text{(A.17)} \qquad f_\epsilon(x)=\sum_{-\infty}^{\infty} a_k\frac{\sin(x-k\tau)}{\pi(x-k\tau)},$$

where

$$f_\epsilon(k\tau)=\frac{1+\epsilon}{2\pi}a_k.$$

Hence,

$$\text{(A.18)} \qquad \pi\int_{-\infty}^{\infty}[f_\epsilon(x)]^2\,dx=\frac{1+\epsilon}{2}\sum_{-\infty}^{\infty} a_k^2.$$

So here we have another example of extreme interference in the sense of (A.1). It is rather remarkable that for very small ϵ the sum in (A.17) can represent a function that is essentially constant over long intervals, where it appears that τ is roughly two times larger than it should be. This is also a simple example of the fact that a given function $f(x)$, especially a positive function, may have countless representations of the form (A.2). The particular representation (A.17) of $f_\epsilon(x)$ is naturally interpreted as the Fourier transform of the restriction to $(-1,1)$ of a periodic pulse train of period $1+\epsilon$, as described in the Introduction.

Finally, the example (A.17) shows that it is not enough in Montgomery's problem to find a set Λ allowing extreme interference in the sense of (A.1), for in the case $\lambda_k = 2k\pi/(1+\epsilon)$, the integral

$$\pi \int_{-\infty}^{\infty} \left| \sum b_k \frac{\sin T(x-\lambda_k)}{\pi(x-\lambda_k)} \right|^2 dx,$$

which is a nondecreasing function of T, is equal to $T\sum |b_k|^2$ for $T=1+\epsilon$. Thus, although the lattice allows the extreme interference (A.18), we have

$$R(T; g, f_\epsilon) \leq 2, \quad T \leq 1+\epsilon. \tag{A.19}$$

We have seen in the Introduction that equality may hold in (A.19) for T as small as $\frac{1}{2}+\epsilon$. But then the ratio cannot be made larger until T exceeds $1+\epsilon$. Thus it does not follow from

$$\int_{-\infty}^{\infty} [f(x)]^2\, dx = \frac{1+\delta}{2} \sum_{-\infty}^{\infty} a_k^2 \quad (\delta \geq 0), \tag{A.20}$$

where $f(x)$ is of the form (A.2), that there is a decomposition

$$f(x) = af_1(x) + bf_2(x), \quad a, b > 0, \tag{A.21}$$

where f_1 and f_2 are functions of the same form of norm 1 in L^2 with (f_1, f_2) arbitrarily close to $-\frac{1}{2}$. For if there were, we could with $f = f_1 + f_2$ and $g = f_1 - f_2$ make $R(T; g, f)$ arbitrarily close to 3 for $T=1$, but for $\lambda_k = 2k\pi/(1+\epsilon)$ we have

$$R(1; g, f) \leq \frac{(1+\epsilon)\sum |b_k|^2}{\xi(f)\sum a_k^2} \leq \frac{2(1+\epsilon)}{(1+\delta)} \leq 2(1+\epsilon). \tag{A.22}$$

Thus the extreme values of $\xi(f)$ and the normalized dot product (f_1, f_2) are, in general, different measures of the interference properties of a set $\{\lambda_k\}$.

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