LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS

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1. Introduction. Let L be a Lie group with finitely many components, K a maximal compact subgroup of L, and Λ a lattice of L. Λ acts properly discontinuously on the contractible manifold $K\backslash L$. The isotropy subgroups are finite and the orbit space $K\backslash L/\Lambda$ is an orbifold. If Λ is torsion-free, then the action of Λ is free and the orbit space is a manifold. The purpose of this article is to prove a structure theorem for $K\backslash L/\Lambda$; it roughly says that either it is a Riemannian orbifold of nonpositive sectional curvature or it Seifert fibers over such an orbifold. We do this if L satisfies the following extra condition (*):

(*) the center of $MR\backslash L_0$ is finite, where L_0 is the identity component of L, R is the radical of L, and M is the Lie subgroup of L_0 which corresponds to the sum of the compact simple factors of the semi-simple semi-direct summand of a Levi decomposition of the Lie algebra of L_0.

Without condition (*), our construction still produces a Seifert fibration

$$K\backslash L/\Lambda \to O^m$$

over an orbifold O^m of dimension $m>0$. The condition (*) is used to show that O^m has non-positive sectional curvature. If L is amenable, then (*) is satisfied and it is not a restriction at all. The precise statement of the main theorem is:

THEOREM 1. *Let L be a non-compact Lie group with finitely many components satisfying (*), K a maximal compact subgroup of L, and Λ a lattice of L. Then there is an orbifold Seifert fibration*

$$K\backslash L/\Lambda \to O^m,$$

where O^m is a Riemannian orbifold of dimension $m>0$ and of nonpositive sectional curvature. If L is amenable, O^m can be chosen to be flat.

REMARKS. (1) Condition (*) is unnecessary if L is connected and Λ is uniform. See §4. (2) O^m is in general not a manifold, even when $K\backslash L/\Lambda$ is a manifold. (3) If Λ is only a discrete subgroup of L, our construction may not produce a Seifert fibration. We heavily use the lattice property of Λ. (4) Some special cases of Theorem 1 have been known; see Farrell and Hsiang [5; 6] and Quinn [12].

To begin the construction, we first choose a connected closed normal subgroup S of L. Then KS is closed, and we have a fiber bundle

$$K\backslash KS \to K/L \to KS\backslash L.$$

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L acts on $K\backslash L$ by right multiplication. L acts also on $KS\backslash L$ by right multiplication; let N denote the kernel of this action, that is, $N=\{g\in L: KSxg=KSx$ for all $x\in L\}$. The action of N on K/L leaves all fibers invariant; in other words, we have a family of right N-spaces parameterized over $KS\backslash L$.

LEMMA 2. *The right N-spaces $K\backslash KSx$ $(x\in L)$ are equivariantly diffeomorphic.*

Proof. Since K is compact, $K\backslash L$ has an L-invariant Riemannian metric. Fix such a Riemannian metric. Pick two distinct fibers $K\backslash KSx$ and $K\backslash KSy$ $(x, y\in L)$. It suffices to construct an N-equivariant diffeomorphism from $K\backslash KSx$ onto $K\backslash KSy$ when they are sufficiently close to each other, because $KS\backslash L$ is connected.

Fix a point p of $K\backslash KSx$ and let d be the distance between p and $K\backslash KSy$. $K\backslash L$ is complete and $K\backslash KSy$ is closed; therefore, d is positive and can be achieved as the length of a geodesic γ connecting p and a point q of $K\backslash KSy$. S is contained in N and acts transitively on each fiber. The action of an element s of S sends γ to a geodesic $\gamma\cdot s$ of the same length d connecting $p\cdot s$ and $q\cdot s$. Thus the distance from a point of $K\backslash KSx$ to $K\backslash KSy$ is independent of the choice of the point, and γ is one of the shortest geodesics connecting $K\backslash KSx$ and $K\backslash KSy$. Therefore γ is perpendicular to $K\backslash KSx$ at p. Let $(T_p(K\backslash KSx))^{\perp}$ denote the orthogonal complement of the tangent space $T_p(K\backslash KSx)$ of $K\backslash KSx$ at p in the tangent space of $K\backslash L$ at p. As the exponential map Exp is a diffeomorphism near the origin, any fiber $K\backslash KSz$ that meets $\mathrm{Exp}(V)$ meets $\mathrm{Exp}(V)$ exactly once, where V is a sufficiently small neighborhood in $(T_p(K\backslash KSx))^{\perp}$ of the origin. This implies that γ is the unique geodesic of length d connecting p and $K\backslash KSy$, as long as $K\backslash KSy$ is sufficiently close to $K\backslash KSx$. Let us suppose that this is the case. Then the correspondence $p\cdot x\mapsto q\cdot s$ $(s\in S)$ defines a diffeomorphism $K\backslash KSx\to K\backslash KSy$, which is obviously N-equivariant because it sends a point in $K\backslash KSx$ to the unique point closest to it in $K\backslash KSy$ and N acts on $K\backslash L$ by isometries. □

REMARK. The N-equivariant diffeomorphism above defines a local trivialization of the fiber bundle $K\backslash L\to KS\backslash L$ so that the action of N on $K\backslash L$ is locally a product of the action of N on a fiber and the action of a trivial group on the base.

If Λ is a lattice of L, the action of L on $K\backslash L$ restricts to an action of Λ on $K\backslash L$. $\Pi=\Lambda\cap N$ is a normal subgroup of Λ which leaves the fibers invariant. By Lemma 2, we have a fiber bundle:

$$K\backslash KS/\Pi\to K\backslash L/\Pi\to KS\backslash L.$$

The quotient group $\Gamma=\Lambda/\Pi$ acts on $K\backslash L/\Pi$ and $KS\backslash L$ such that $(K\backslash L/\Pi)/\Gamma = K\backslash L/\Lambda$ and $(KS\backslash L)/\Gamma = KS\backslash L/\Lambda$; the fiber bundle map induces a map:

$$q: K\backslash L/\Lambda\to (KS\backslash L)/\Gamma.$$

Note that $KS\backslash L$ can be naturally identified with $(S\backslash KS)\backslash(S\backslash L)$, which has an $(S\backslash L)$-invariant (and hence L-invariant) Riemannian metric. Thus Γ can be thought of as a subgroup of the group $I(KS\backslash L)$ of all the isometries of $KS\backslash L$ with respect to this Riemannian metric.

Suppose that Γ is discrete in $I(KS\backslash L)$. Then the isotropy subgroup Γ_v of Γ at $v \in KS\backslash L$ is finite for each v, and the inverse image $q^{-1}([v])$ of the orbit $[v] \in (KS\backslash L)/\Gamma$ is $((K\backslash KSx)/\Pi)/\Gamma_v$, where $v = KSx$ $(x \in L)$. Thus a "fiber" of q is homeomorphic to a quotient of the "general fiber" $K\backslash KS/\Pi$ by an action of a finite group; that is, q is a Seifert fibration [3].

In the following three sections, we choose S and show that Γ is discrete and that $O^m = KS\backslash L/\Lambda$ has nonpositive sectional curvature.

In §5, we will compute the Wall groups (tensored with $\mathbf{Z}[1/2]$) of virtually poly-cyclic groups in terms of certain homology theory (Theorem 6). A virtually poly-cyclic group Λ can be embedded discretely and cocompactly in an amenable Lie group L, and we use the Seifert structure of $K\backslash L/\Lambda$ over a flat orbifold $\mathbf{R}^m/\Gamma$ given by Theorem 1 for the computation. Such a structure is called a fibering apparatus for Λ in [3]. Recently, Farrell and Jones succeeded in computing the Wall groups (without $\otimes\mathbf{Z}[1/2]$) of torsion-free virtually poly-cyclic groups [7] using new ideas (e.g., pseudo-fibering apparatuses in place of fibering apparatuses and a foliated metric control theorem in place of the Chapman–Ferry–Quinn control theory).

As in our previous papers [16; 17], the rational computation of Wall groups mentioned above implies the so-called Novikov conjecture for virtually poly-cyclic groups (Corollary 7), which is originally due to Cappell [2]. More recently, Kasparov proved this corollary for arbitrary discrete subgroups of a connected Lie group [9]. Also, the author has been informed by the referee that a rational split injectivity result like Theorem 6 was proven by Ferry and Weinberger for discrete subgroups of semisimple Lie groups or amenable Lie groups.

2. Non-amenable case. Recall that a Lie group L with finitely many components is *amenable* if and only if L/R is compact, where R denotes the radical (=the unique maximal connected normal solvable subgroup) of L. See Milnor [10]. In this section we handle the case when L is *not* amenable. We use R as S, following [6]; that is, we are going to show that

$$K\backslash L/\Lambda \to KR\backslash L/\Lambda$$

is a Seifert fibration with the desired property. As in the previous section, identify $KR\backslash L$ with $(R\backslash KR)\backslash(R\backslash L) = \mathbf{R}^m$ $(m > 0)$. $R\backslash L$ is a non-compact semi-simple Lie group, and $R\backslash KR$ is a maximal compact subgroup of $R\backslash L$. Using condition $(*)$, the Killing form, and the Cartan decomposition, one introduces an $(R\backslash L)$-invariant (and hence L-invariant) Riemannian metric g on $\mathbf{R}^m$ with non-positive sectional curvature. In fact, any $(R\backslash L)$-invariant Riemannian metric on $\mathbf{R}^m$ has nonpositive sectional curvature. See Helgason [8]. Thus we have a homomorphism $\Phi: L \to I(\mathbf{R}^m, g)$. Let Γ denote the image $\Phi(\Lambda)$ of Λ. To prove the theorem, it suffices to show that Γ is discrete in $I(\mathbf{R}^m, g)$. Let τ denote the natural projection $L \to R\backslash L$. If the image $\gamma(\Lambda)$ of Λ in $R\backslash L$ is discrete, then Γ is obviously discrete. Unfortunately $\gamma(\Lambda)$ may not be discrete in general. We remedy this situation as follows.

Let L_0 denote the identity component of L. $\Lambda \cap L_0$ is a subgroup of Λ with finite index. Therefore it suffices to show that $\Phi(\Lambda \cap L_0)$ is discrete in $I(\mathbf{R}^m, g)$. As $(R\backslash(K\cap L_0)R)\backslash(R\backslash L_0)$ can be naturally identified with $\mathbf{R}^m$, we may assume from the beginning that L is connected.

Now there is a semi-simple Lie subgroup S of L such that $L = SR$ and such that $S \cap R$ is discrete (Levi decomposition). Let $\sigma: S \to \mathrm{Aut}(R)$ denote the action of S on R. A sufficient condition for $\gamma(\Lambda)$ to be discrete in $R\backslash L$ is that the identity component $(\ker\sigma)_0$ of the kernel of σ has no compact factors (Raghunathan [13, p. 150]). Let C denote the unique maximal compact normal subgroup of $(\ker\sigma)_0$. It is a characteristic subgroup of $(\ker\sigma)_0$, and hence it is normal in $\ker\sigma$ and in S. On the other hand, C commutes with elements of R. Therefore C is normal in L. Let $\pi: L \to L/C$ denote the natural projection, and let $L' = \pi(L)$, $S' = \pi(S)$, $R' = \pi(R)$, $\Lambda' = \pi(\Lambda)$, $K' = \pi(K)$. Then S' is semi-simple, R' is the radical of L', Λ' is a lattice of L', and K' is a maximal compact subgroup of L'. Let $\sigma': S' \to \mathrm{Aut}\, R'$ denote the action of S' on R'. Then it is easily observed that $\ker\sigma' = (\ker\sigma)/C$, since $C \cap R$ is finite. So the identity component of $\ker\sigma'$ has no compact factors, and this implies that the image Λ'' of Λ' in $R'\backslash L'$ is discrete. Thus the action of Λ on $\mathbf{R}^m$ factors through a properly discontinuous action of Λ'' on $K'R'\backslash L = KR\backslash L$. Therefore, Γ is discrete in $I(\mathbf{R}^m, g)$. This completes the proof of Theorem 1 when L is not amenable.

REMARK. Let $q: K\backslash L/\Lambda \to KR\backslash L/\Lambda$ be the Seifert fibration constructed above. Then the "fiber" $q^{-1}(KRx\Lambda)$ over the point $KRx\Lambda \in KR\backslash L/\Lambda$ $(x \in L)$ is homeomorphic to

$$(x^{-1}Kx)\backslash(x^{-1}KRx)/(x^{-1}KRx \cap \Lambda).$$

It is easily observed that $x^{-1}KRx \cap \Lambda$ is a uniform lattice (= discrete cocompact subgroup) of $x^{-1}KRx$. In particular, we have

COROLLARY 3. *Let L be a Lie group with finitely many components, K a maximal compact subgroup of L, R the radical of L, and Λ a lattice of L. Then $KR \cap \Lambda$ is a uniform lattice of KR.*

3. Amenable case. Now let us assume that L is non-compact and amenable. Let K be a maximal compact subgroup and R the radical of L as before. Since L is amenable, $L = KR$.

We define a sequence $N^{(j)}$ $(j \geq -1)$ of closed characteristic subgroups of L as follows:

(1) $N^{(-1)}$ is the radical R;
(2) $N^{(0)}$ is the nil-radical (i.e., the maximal connected normal nilpotent subgroup) of L;
(3) $N^{(j)}$ is the commutator subgroup $[N^{(j-1)}, N^{(j-1)}]$ of $N^{(j-1)}$ for $j > 0$.

It may not be so obvious that $N^{(j)}$'s are closed when $j > 0$; in general, the commutator subgroup of a Lie group may not be closed. This will be observed later, and we continue the construction. There exists an integer k such that $N^{(k)} = \{1\}$. Consider the following sequence:

$$L = KN^{(-1)} \supset KN^{(0)} \supset KN^{(1)} \supset \cdots \supset KN^{(k)} = K.$$

There exists an integer $i \geq 0$ such that

$$L = KN^{(-1)} = KN^{(0)} = \cdots = KN^{(i-1)} \neq KN^{(i)},$$

because L is non-compact. Let us write $M = N^{(i-1)}$ and $N = N^{(i)}$. We introduce a flat L-invariant Riemannian metric on $KN\backslash L$.

Let us study the action of L on $KN\backslash L$ defined by right multiplication. An element ky of $KM = L$ ($k \in K$, $y \in M$) acts on an element KNx ($x \in M$) of $KN\backslash L$ as follows:

$$\begin{aligned} KNx \cdot (ky) &= KNxky \\ &= KN(k^{-1}xk)y. \end{aligned}$$

Note that we have $[M, M] \subset N$; we identify the coset space $KN\backslash L$ with the simply-connected abelian Lie group $(K \cap M)N\backslash M = \mathbf{R}^m$ ($m > 0$). Now the induced action of L on $\mathbf{R}^m$ is:

$$(K \cap M)Nx \cdot (ky) = (K \cap M)N(k^{-1}xk)y.$$

The following are easily observed: (1) this action, when restricted to K, defines an homomorphism $\alpha: K \to \mathrm{Aut}(\mathbf{R}^m)$ and its image $\alpha(K)$ lies in the orthogonal group $O(m)$ with respect to some inner product of $\mathbf{R}^m$; and (2) if $k \in K \cap M$, then $(K \cap M)Nx \cdot k = (K \cap M)Nk^{-1}xk = (K \cap M)N(k^{-1}xkx^{-1})x = (K \cap M)Nx$ for $x \in M$, and so $K \cap M$ acts trivially on $\mathbf{R}^m$. Let $\beta: M \to (K \cap M)N\backslash M$ denote the natural projection. We now define a map $\Phi: L = KM \to \alpha(K) \ltimes ((K \cap M)N\backslash M) \subset O(m) \ltimes \mathbf{R}^m = I(\mathbf{R}^m)$ by sending ky ($k \in K$, $y \in M$) to $(\alpha(k), \beta(y)) \in O(m) \ltimes \mathbf{R}^m$. This is a well-defined homomorphism. Here $\ltimes$ denotes the obvious semi-direct product. Let Γ denote the image of Λ by Φ in $I(\mathbf{R}^m)$.

It remains to observe that $N^{(j)}$'s are closed and that Γ is a discrete subgroup of $I(\mathbf{R}^m)$. To do this we use the following lemma:

LEMMA 4. *If N is a connected nilpotent Lie group and Δ is a discrete cocompact subgroup of N, then the commutator subgroup $[N, N]$ is closed in N and $\Delta \cap [N, N]$ is cocompact in $[N, N]$.*

Proof. This is well known if N is simply connected, so consider the universal cover $p: U \to N$ of N; it can be identified with the natural projection $U \to U/\Pi$, where Π is the kernel of p. To see that $[N, N]$ is closed in N, it suffices to show that $N/[N, N]$ is Hausdorff. As $p^{-1}([N, N]) = \Pi[U, U]$, we have homeomorphisms:

$$\begin{aligned} N/[N, N] &\cong U/\Pi[U, U] \\ &\cong (U/[U, U])/(\Pi[U, U]/[U, U]). \end{aligned}$$

Here $U/[U, U]$ is a Lie group, because U is simply-connected and hence its commutator subgroup $[U, U]$ is closed. Note that the preimage $p^{-1}(\Delta)$ of Δ is discrete and cocompact in U. Since U is simply-connected, $p^{-1}(\Delta) \cap [U, U]$ is cocompact in $[U, U]$. Therefore, the image $p^{-1}(\Delta)[U, U]/[U, U]$ of $p^{-1}(\Delta)$ by the projection $U \to U/[U, U]$ is discrete. As $\Pi \subset p^{-1}(\Delta)$, $\Pi[U, U]/[U, U]$ is also discrete and hence closed in $U/[U, U]$. Therefore $(U/[U, U])/(\Pi[U, U]/[U, U])$ is Hausdorff. This proves the first statement as observed above.

Since we have homeomorphisms:

$$[N, N]/\Delta \cap [N, N] \cong \Pi[U, U]/p^{-1}(\Delta) \cap \Pi[U, U]$$
$$\cong [U, U]/p^{-1}(\Delta) \cap [U, U],$$

the second statement is obvious. □

Now we prove

LEMMA 5. *$N^{(j)}$'s are closed subgroups of L, and Γ is a crystallographic subgroup of $I(\mathbf{R}^m)$.*

Proof. If $\Lambda \cap R$ is cocompact in $R = N^{(-1)}$, then $\Lambda \cap N^{(0)}$ is a discrete cocompact subgroup of $N^{(0)}$ and we can apply Lemma 4 to prove that $N^{(j)}$'s are closed for $j \geq 1$. Unfortunately, $\Lambda \cap R$ may not be cocompact in R, in general. To remedy this situation we introduce a quotient Lie group L' of L as in the previous section. We may assume that L is connected. We have Levi decomposition $L = SR$, where S is a connected semi-simple (and hence compact) subgroup, R is the radical as above, and the intersection $S \cap R$ is finite. Let $\sigma: S \to \mathrm{Aut}(R)$ denote the action of S on R. The identity component $(\ker \sigma)_0$ of $\ker \sigma$ is a connected compact normal subgroup of L, because it commutes with elements of R. In particular, $(\ker \sigma)_0 \subset \ker \alpha \subset K$. Let $\pi: L \to L/(\ker \sigma)_0$ be the natural map. Now define: $L' = L/(\ker \sigma)_0$, $\Lambda' = \pi(\Lambda)$, $K' = \pi(K)$, $S' = \pi(S)$, $R' = \pi(R)$. Then Λ' is a lattice of L', K' is a maximal compact subgroup of L', S' is a semi-simple subgroup of L', R' is the radical of L', and the action $\sigma': S' \to \mathrm{Aut}(R')$ of S' on R' is almost faithful (i.e., $\ker \sigma'$ is finite).

Let us define a sequence $N'^{(j)}$ $(j \geq -1)$ of characteristic subgroups of L' by:

(1) $N'^{(-1)} = R'$;
(2) $N'^{(0)} =$ the nil-radical of L';
(3) $N'^{(j)} = [N'^{(j-1)}, N'^{(j-1)}]$ for $j \geq 1$.

Then $\Lambda' \cap R'$ and $\Lambda' \cap N'^{(0)}$ are cocompact in R' and $N'^{(0)}$ respectively. By successively using Lemma 4, we know that all $N'^{(j)}$'s are closed. Note that $\pi \mid R$: $R \to R'$ is a finite covering map; this implies that $N^{(j)}$ is the identity component of $(\pi \mid R)^{-1}(N'^{(j)})$ for each j. Therefore $N^{(j)}$'s are closed in L.

Next, we show that Γ is a discrete cocompact subgroup of $I(\mathbf{R}^m)$. Note that we have

$$L' = K'N'^{(-1)} = K'N'^{(0)} = \cdots = K'N'^{(i-1)} \neq K'N'^{(i)}$$

for the same i and that $K'N' \backslash K'M' = KN \backslash KM$, where $M' = N'^{(i-1)}$ and $N' = N'^{(i)}$. $\Lambda' \cap N'^{(j)}$ is cocompact in $N'^{(j)}$ for all j. In particular, $\Lambda' \cap M'$ is cocompact in M'. So the image of Λ' in $M' \backslash L'$ is discrete; furthermore, it is finite, because $M' \backslash L'$ is compact. Looking at the diagram:

$$\begin{array}{ccccc}
 & & \Lambda & \dashrightarrow & \text{finite} \\
 & & \cap & & \cap \\
\pi^{-1}(M') & \longrightarrow & L & \longrightarrow & \pi^{-1}(M')\backslash L \\
\downarrow & & \downarrow \pi & & \downarrow \cong \\
M' & \longrightarrow & L' & \longrightarrow & M'\backslash L' \\
 & & \cup & & \cup \\
 & & \Lambda' & \longrightarrow & \text{finite,}
\end{array}$$

we know that $\Lambda \cap \pi^{-1}(M')$ has a finite index in Λ. So it suffices to show that the image $\Phi(\Lambda \cap \pi^{-1}(M'))$ is a discrete cocompact subgroup of $I(\mathbf{R}^m)$. As $\ker \sigma \subset \ker \alpha$, Φ sends elements in $\pi^{-1}(M') = (\ker \sigma)_0 M$ to elements in $\mathbf{R}^m \subset I(\mathbf{R}^m)$. Now consider the following commutative diagram:

$$\begin{array}{ccccc}
 & L & \xrightarrow{\quad\Phi\quad} & O(m) \ltimes \mathbf{R}^m & \\
 & \cup & & \cup & \\
\Lambda \cap \pi^{-1}(M') \subset & \pi^{-1}(M') = (\ker \sigma)_0 M & \xrightarrow{\Phi} & \mathbf{R}^m = (K \cap M)\backslash M & \\
\downarrow & \downarrow \pi & & \downarrow (\pi \mid M)_* & \\
\Lambda' \cap M' \subset & M' & \xrightarrow{\quad\Phi\quad} & (K' \cap M')N'\backslash M', &
\end{array}$$

where Φ' is the natural map and $(\pi \mid M)_*$ is the map induced by the restriction of π to M, $\pi \mid M \colon M \to M'$. $K \cap M$ and $K' \cap M'$ are maximal compact subgroups of M and M', respectively, and $\pi(K \cap M) = K' \cap M'$; therefore, $(\pi \mid M)^{-1}(K' \cap M') = K \cap M$. Using this, it is easily verified that $(\pi \mid M)^{-1}((K' \cap M')N') = (K \cap M)N$. Therefore $(\pi \mid M)_*$ is an isomorphism. Since $(\Lambda' \cap M') \cap N' = \Lambda' \cap N'$ is cocompact in N', $(\Lambda' \cap M') \cap (K' \cap M')N'$ is cocompact in $(K' \cap M')N'$; thus $\Phi'(\Lambda' \cap M')$ is a discrete cocompact subgroup of $(K' \cap M')N'\backslash M'$. Therefore $\Phi(\Lambda \cap \pi^{-1}(M'))$ is a discrete cocompact subgroup of $\mathbf{R}^m$ (and hence in $I(\mathbf{R}^m)$). This completes the proof of Lemma 5. □

Thus $K\backslash L/\Lambda \to KN\backslash L/\Lambda$ is a desired Seifert fibration as observed in the first section. This completes the proof of Theorem 1. □

REMARK. A fiber of the Seifert fibration above has the form $K\backslash KNx\Lambda/\Lambda$, and is homeomorphic to

$$(x^{-1}Kx)\backslash(x^{-1}KNx)/(x^{-1}KNx \cap \Lambda).$$

If Λ is a lattice of L (which is automatically uniform), then $x^{-1}KNx \cap \Lambda$ is a uniform lattice of $x^{-1}KNx$.

4. Uniform lattices of connected Lie groups. Let us assume that L is connected and not amenable, and observe that the condition (*) can be deleted if Λ is a uniform lattice of L. Without (*), we may not have nonpositive sectional curvature on $KR\backslash L/\Lambda$, so we replace KR by some larger closed subgroup in the following way.

Let $r \oplus s$ be a Levi decomposition of the Lie algebra of L into the radical r and a semi-simple subalgebra s, and let $k \oplus p$ be a Cartan decomposition of the sum of the non-compact factors of s, where k is a maximal compactly embedded subalgebra. Also let H be the subgroup of L corresponding to the sum of r, the compact factors of s, and k. Then H is closed and $H\backslash L$ has an L-invariant Riemannian metric with nonpositive sectional curvature ([8, p. 252]). If the dimension of $H\backslash L$ is 0 then $p=0$ and therefore k is compactly embedded in itself, that is, k is compact. By assumption, k must be 0; that is, L is amenable. So $\dim(H\backslash L)>0$ as long as L is not amenable. The argument in §2 together with the density theorem of Borel [13, 5.17] implies that the image $\Gamma\subset I(H\backslash L)$ of Λ is discrete. Thus we get a map $K\backslash L/\Lambda\to H\backslash L/\Lambda$ onto an orbifold with the desired curvature.

Unfortunately, H is not in the form of KS (with S normal) and we can not apply Lemma 2 to prove that this is a Seifert fibration. More precisely, we fail to prove that the distance from each point of $K\backslash Hx$ to $K\backslash Hy$ is constant as in Lemma 2.

However, if we assume that Λ is a uniform lattice then a point inverse of $K\backslash L/\Lambda\to H\backslash L/\Lambda$ is compact. Therefore $\Pi=N\cap\Lambda$ acts cocompactly on each fiber $K\backslash Hx$, where N is the largest normal subgroup of L in H; so the distance from each point of $K\backslash Hx$ to $K\backslash Hy$ is bounded and, for any $\epsilon>0$, the fibers $K\backslash Hy$ sufficiently close to $K\backslash Hx$ are contained in the ϵ-neighborhood of $K\backslash Hx$. On the other hand, there exists an $\epsilon>0$ such that the intersections of $\operatorname{Exp}(V)$ and fibers $K\backslash Hy$ are transverse, where V is the ϵ-neighborhood of the origin of $T_p(K\backslash Hx)^{\perp}$, for each $p\in K\backslash Hx$, because $x^{-1}Hx$ acts transitively on the fiber $K\backslash Hx$. (See the proof of Lemma 2 for the notation.) Then the map which sends a point $p\in K\backslash Hx$ to the unique intersection point of $K\backslash Hy$ and $\operatorname{Exp}(V)$ is the desired equivariant diffeomorphism if the fibers are sufficiently close. The inverse map sends a point $q\in K\backslash Hy$ to the unique closest point in $K\backslash Hx$ from q. Therefore $K\backslash L/\Pi\to H\backslash L$ is a fiber bundle, and $K\backslash L/\Lambda\to H\backslash L/\Lambda$ is a Seifert fibration.

THEOREM 1′. *Theorem 1 holds true without condition* (*) *if* Λ *is a uniform lattice of a connected Lie group.*

5. A rational computation of Wall's L-groups. Let L be an amenable Lie group with finitely many components, K a maximal compact subgroup of L, and Λ a uniform lattice of L. Such a discrete group Λ is virtually poly-cyclic [10]. Conversely, any virtually poly-cyclic group can be embedded discretely and cocompactly in some amenable Lie group [1]. In this section we compute rationally the L-groups of Λ in terms of certain generalized homology of $K\backslash L/\Lambda$.

K/L is diffeomorphic to some Euclidean space $\mathbf{R}^n$, and the isotropy subgroup $\Lambda_y=x^{-1}Kx\cap\Lambda$ of Λ at $y=Kx$ $(x\in L)$ is finite. The action of Λ on $\mathbf{R}^n$ is free if Λ is torsion-free; in general, $\mathbf{R}^n/\Lambda$ is an orbifold, which is Seifert fibered over some flat orbifold as observed in §3.

Let $W\Lambda$ be a contractible free Λ-complex, and let p denote the projection

$$(\mathbf{R}^n\times W\Lambda)/\Lambda\to\mathbf{R}^n/\Lambda,$$

where Λ acts on $\mathbf{R}^n\times W\Lambda$ diagonally. The preimage $p^{-1}([y])$ of an orbit $[y]\in\mathbf{R}^n/\Lambda$ by p is homeomorphic to $W\Lambda/\Lambda_y$, and p is a sort of Seifert fibration (it is called a "stratified system of fibrations" in [11]).

Let $L^{-\infty}(\Lambda)$ denote the limit of Ranicki's lower L-groups $L^{(-j)}(\mathbf{Z}L)$ [16]. Modulo 2-torsion, it coincides with Wall's surgery obstruction group. We have a functor $\mathbf{L}^{-\infty}(-)$ from the category of spaces to the category of Ω-spectra such that the homotopy group of $\mathbf{L}^{-\infty}(X)$ is equal to $L_*^{-\infty}(\pi_1 X)$. Applying $\mathbf{L}^{-\infty}(-)$ to each fiber of p, we obtain a sheaf of spectra, denoted $\mathbf{L}^{-\infty}(p)$. Quinn defines the homology group $H_*(\mathbf{R}^n/\Lambda; \mathbf{L}^{-\infty}(p))$ of $\mathbf{R}^n/\Lambda$ with coefficients $\mathbf{L}^{-\infty}(p)$. See [11] and [16]. The following is a rational computation of $L_*^{-\infty}(\Lambda)$ in terms of this homology.

THEOREM 6. *Let* Λ *be as above. Then there is a natural isomorphism*

$$H_*(\mathbf{R}^n/\Lambda; \mathbf{L}^{-\infty}(p)) \otimes \mathbf{Z}[1/2] \to L_*^{-\infty}(\Lambda) \otimes \mathbf{Z}[1/2].$$

The map is induced by the following map between stratified systems of fibrations:

$$\begin{array}{ccc} (\mathbf{R}^n \times W\Lambda)/\Lambda & \xrightarrow{\text{id.}} & (\mathbf{R}^n \times W\Lambda)/\Lambda \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathbf{R}^n/\Lambda & \longrightarrow & \text{pt.} \end{array}$$

Note that $(\mathbf{R}^n \times W\Lambda)/\Lambda = B\Lambda$ is a classifying space for Λ and that

$$H_*(\text{pt.}; \mathbf{L}^{-\infty}(B\Lambda \to \text{pt.})) = L_*^{-\infty}(\Lambda)$$

[16].

It is to be noted that Theorem 6 says that the $\mathbf{L}^{-\infty}(p)$ coefficient homology of $\mathbf{R}^n/\Lambda$ is independent (modulo 2-torsion) of the action of Λ on $\mathbf{R}^n$. It is conceivable that the orbifold $\mathbf{R}^n/\Lambda$ has a certain topological rigidity. In fact, Connolly and Kosniewski [4] have proven a geometric analogue of Theorem 6 for certain crystallographic groups: suppose Γ is a crystallographic group $\subset I(E^n)$ with odd order holonomy group G, has no gaps of dimension 2, and has no fixed sets of dimension 3 or 4. Suppose further that Γ acts locally smoothly and properly discontinuously on a topological manifold M, such that M/Γ is compact and M^H is contractible for each finite subgroup H of Γ. Then any equivariant homotopy equivalence $M \to E^n$ with trivial topological G-Whitehead torsion (in the sense of Steinberger and West [14]) is equivariantly homotopic to a homeomorphism. The author has been informed by the referee that Weinberger has constructed a counterexample to this when Γ has a holonomy group of order 2.

See also Takeuchi [15] for the topological rigidity of certain sufficiently large 3-dimensional orbifolds.

Proof of Theorem 6. The proof is by induction on the dimension n of $K\backslash L$. Let $q: \mathbf{R}^n/\Lambda \to \mathbf{R}^m/\Gamma$ denote the Seifert fibration constructed in §3. Modulo 2-torsion, we have

$$\begin{aligned} H_*(\mathbf{R}^n/\Lambda; \mathbf{L}^{-\infty}(p)) &\cong H_*\Big(\mathbf{R}^m/\Gamma; \bigcup_{w \in \mathbf{R}^m/\Gamma} \mathbf{H}(q^{-1}(w)); \mathbf{L}^{-\infty}(p \mid q^{-1}(w))\Big) \\ &\cong H_*\Big(\mathbf{R}^m/\Gamma; \bigcup_{w} \mathbf{L}^{-\infty}((qp)^{-1}(w))\Big) \\ &= H_*(\mathbf{R}^m/\Gamma; \mathbf{L}^{-\infty}(qp)) \end{aligned}$$

by induction hypothesis, where $\mathbf{H}$ denotes the homology theory spectrum [16]. In [17] we proved that $H_*(\mathbf{R}^m/\Gamma; \mathbf{L}^{-\infty}(qp)) \otimes \mathbf{Z}[1/2]$ is naturally isomorphic to $L_*^{-\infty}(\Lambda)$. The key ingredients of the proof are the classification of crystallographic groups by Farrell and Hsiang and the controlled L-theory. Actually the proof is only a slight modification of that of the main theorem of [16]. This completes the proof of Theorem 6. □

COROLLARY 7 (Novikov Conjecture). *Let* Λ *be as above. Then the assembly map*

$$H_*(B\Lambda; \mathbf{L}^{-\infty}(1)) \to L_*^{-\infty}(\Lambda)$$

is rationally split injective.

Corollary 7 was also obtained in our earlier paper [17], but Theorem 6 above is more general than the main theorem in [17].

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