GREEN'S THEOREM AND BALAYAGE

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1. Introduction. For Q a cube in $\mathbf{R}^d$ with sides parallel to the coordinate axes, let $|Q|$ denote Q's Lebesgue measure. For $\phi \in L^1_{\text{loc}}(\mathbf{R}^d)$, define

$$\phi_Q \equiv \frac{1}{|Q|} \int_Q \phi \, dx.$$

We say that ϕ is in BMO if

$$\|\phi\|_* \equiv \sup_{Q \subset \mathbf{R}^d} \frac{1}{|Q|} \int_Q |\phi - \phi_Q| \, dx < \infty.$$

For Q as above we set $\hat{Q} = \{(t, y) \in \mathbf{R}^{d+1}_+ : t \in Q,\ 0 < y < \ell(Q)\}$, where $\ell(Q)$ is the sidelength of Q. We say that μ, a Borel measure on $\mathbf{R}^{d+1}_+$, is a *Carleson measure* if

$$\|\mu\|_C \equiv \sup_{Q \subset \mathbf{R}^d} \frac{|\mu|(\hat{Q})}{|Q|} < \infty.$$

There is an intimate connection between the space BMO and the family of Carleson measures. Roughly speaking, a Carleson measure is a conformally invariant finite measure, while a BMO function is a conformally invariant L^1 function. This connection is made more explicit through the following fact. Let $K \in L^1(\mathbf{R}^d)$ satisfy $\int K = 1$, $|K(x)| \le (1+|x|)^{-d-1}$, $|\nabla K(x)| \le (1+|x|)^{-d-2}$. For $y > 0$ let $K_y(x) = y^{-d} K(x/y)$. Consider the function

$$S_{\mu,K} \equiv \int_{\mathbf{R}^{d+1}_+} K_y(x-t) \, d\mu(t, y), \tag{1}$$

where μ is a Borel measure on $\mathbf{R}^{d+1}_+$. It is easy to see that if μ is finite then the integral in (1)—called the *sweep* or *balayage* of μ with respect to K—converges absolutely for a.e. $x \in \mathbf{R}^d$, and $\|S_{\mu,K}\|_1 \le C(d) \|\mu\|$.

More is true if μ is a Carleson measure. In that case, $S_{\mu,K} \in \text{BMO}$ and

$$\|S_{\mu,K}\|_* \le C(d) \|\mu\|_C. \tag{2}$$

The proof of (2) is quite easy. What is more remarkable (and also true) is that (2) has a converse [1; 2; 3].

THEOREM A. *Let $K \in L^1(\mathbf{R}^d)$ satisfy $\int K = 1$, $|K(x)| \le (1+|x|)^{-d-1}$. Let $\phi \in$ BMO have compact support. There exist $g \in L^\infty(\mathbf{R}^d)$ and a finite Carleson measure μ such that $\phi(x) = g(x) + S_{\mu,K}(x)$, where $\|g\|_\infty + \|\mu\|_C \le C(d) \|\phi\|_*$.*

The proofs in [1; 2; 3] work by an iteration argument. One builds a $\tilde{g}$ and a $\tilde{\mu}$ for which $\tilde{\phi} \equiv \tilde{g} + S_{\tilde{\mu},K}$ is close to ϕ in BMO, and then one repeats the argument on $\phi - \tilde{\phi}$. One does this infinitely often, obtaining g and μ in the limit. The effect

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of this iteration is to produce a g and a μ which do not, in general, have compact supports. Also, the construction of g and μ is not very explicit.

In this note, we give a new proof of Theorem A for the case of $K_y =$ the Poisson kernel. The proof uses the Poisson kernel's semigroup property, and Green's theorem. Green's theorem gives us formulas which let us avoid the iteration. They also make our construction much more explicit. In addition, we are able to obtain a g (but not, alas, a μ) with compact support.

We prove the following.

THEOREM B. *Let $\phi \in BMO$,* $\operatorname{supp}\phi \subseteq \{x=(x_1,\dots,x_d): |x_i|\le 1,\ i=1,\dots,d\}$. *Let P_y be the Poisson kernel. There exist a $g \in L^\infty(\mathbf{R}^d)$ and a finite Carleson measure μ such that $\phi = g + S_{\mu,P}$ with $\|g\|_\infty + \|\mu\|_C \le C(d)\|\phi\|_*$ and* $\operatorname{supp} g \subseteq \{x=(x_1,\dots,x_d): |x_i|\le 2,\ i=1,\dots,d\}$.

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2. Proof of Theorem B. Let $\phi(x,y) \equiv P_y * \phi(x)$.

We will use two facts about BMO.

(1) Let $Q(x,y)$ denote the cube with center x and sidelength y. Then:

$$|\phi(x,y) - \phi_{Q(x,y)}| \le C(d)\|\phi\|_*.$$

(2) Let ∇ denote the full gradient in $\mathbf{R}^{d+1}_+$. We have:

$$|y\nabla\phi(x,y)| \le C(d)\|\phi\|_*.$$

For proofs of these see [4].

Henceforth we shall assume that $\|\phi\|_* \le 1$. For $Q \subset \mathbf{R}^d$ a cube, let $z_Q = (x_Q, \ell(Q))$. where x_Q is the center of Q (z_Q is the center of the top face of $\hat{Q}$). Let $Q_0 = \{x=(x_1,\dots,x_d): |x_i|\le 2,\ i=1,\dots,d\}$. We define *generations*, $\mathbf{G}_k$, of subcubes of Q_0 as follows:

$$\mathbf{G}_0 = \{Q_0\}$$

$$\mathbf{G}_{k+1} = \{Q' \subset Q \in \mathbf{G}_k : Q' \text{ maximal dyadic such that } |\phi(z_Q) - \phi(z_{Q'})| > A\} \quad k \ge 0,$$

where A is a large constant to be chosen later. By fact (1), $|\phi(z_Q)-\phi(z_{Q'})|>A$ implies that $|\phi_Q - \phi_{Q'}| > A/2$ if A is large enough. For sufficiently large A we have also that

$$\sum_{\substack{Q' \subset Q \in \mathbf{G}_k \\ Q' \in \mathbf{G}_{k+1}}} |Q'| \le \frac{1}{2}|Q|$$

for each k. We shall henceforth assume that A is large enough. For $x \in \mathbf{R}^d$ let $u_x(t,y) = P_y(t-x)$. By Green's theorem,

$$\phi = 2\int_{\mathbf{R}^{d+1}_+} y\nabla\phi \cdot \nabla u_x \, dt\, dy \tag{3}$$

as a distribution.

We are now going to cut up integral (3). Let $E_1=\hat{Q}_0$ and let $E_2=\mathbf{R}_+^{d+1}\setminus\hat{Q}_0$. Then:

$$\begin{aligned}\phi&=2\int_{E_1} y\nabla\phi\cdot\nabla u_x\,dt\,dy+2\int_{E_2} y\nabla\phi\cdot\nabla u_x\,dt\,dy\\ &=(I)+(II).\end{aligned}$$

We will deal with (I) first.

For $Q\in\mathbf{G}_k$, let

$$\Sigma_Q=\hat{Q}\ \Big\backslash \bigcup_{\substack{Q'\subset Q\\ Q'\in\mathbf{G}_{k+1}}}\hat{Q}'.$$

Clearly,

$$\int_{E_1} y\nabla\phi\cdot\nabla u_x\,dt\,dy=\sum_{k=0}^{\infty}\sum_{Q\in\mathbf{G}_k}\int_{\Sigma_Q}\text{————}\,dt\,dy. \tag{4}$$

We now apply Green's theorem to each of the summands in (4). We get:

$$\begin{aligned}&\int_{\Sigma_Q} y\nabla\phi\cdot\nabla u_x\,dt\,dy\\ &\quad=\int_{\partial\Sigma_Q}\left(y\left(\frac{\partial\phi}{\partial\nu}u_x(t,y)+(\phi-\phi(z_Q))\frac{\partial u_x}{\partial\nu}\right)-(\phi-\phi(z_Q))u_x\frac{\partial y}{\partial\nu}\right)d\sigma_Q,\end{aligned}$$

where $d\sigma_Q$ is d-dimensional surface measure on $\partial\Sigma_Q$ and $\partial/\partial\nu$ denotes differentiation in the outward normal direction (outward relative to Σ_Q; $\partial\Sigma_Q$ is just smooth enough to let us do this). Let $\partial\Sigma_Q=B_0\cup B_+$, where $B_0=\partial\Sigma_Q\cap\mathbf{R}^d$ and $B_+=\partial\Sigma_Q\setminus B_0$. We write:

$$\begin{aligned}&\int_{\partial\Sigma_Q}\left(y\left(\frac{\partial\phi}{\partial\nu}u_x(t,y)+(\phi-\phi(z_Q))\frac{\partial u_x}{\partial\nu}\right)-(\phi-\phi(z_Q))u_x\frac{\partial y}{\partial\nu}\right)d\sigma_Q\\ &\quad=\int_{B_0}\text{————}\,d\sigma_Q+\int_{B_+}\text{————}\,d\sigma_Q\\ &\quad=(i)_Q+(ii)_Q.\end{aligned}$$

As a distribution, $(i)_Q$ is equal to

$$(\phi(x)-\phi(z_Q))\chi_{B_0}(x)\equiv g_Q(x).$$

Since $x\in B_0$ implies $x\notin\bigcup_{Q'\in\mathbf{G}_{k+1}}Q'$, we must have $|g_Q(x)|\le A$ a.e. The supports of the different g_Q's are easily seen to be disjoint, and so we may set

$$g\equiv\sum_{Q\in\bigcup\mathbf{G}_k}g_Q$$

with $\|g\|_\infty\le A$ and $\operatorname{supp}g\subseteq\{x=(x_1,\dots,x_d)\colon |x_i|\le 2,\ i=1,\dots,d\}$.

We will be finished with (I) once we show that

$$\sum_Q (ii)_Q=S_{\mu,P} \tag{5}$$

for some μ with $\|\mu\|_C\le C(d)$. We proceed to do this now. Write:

$$(ii)_Q = \int_{\partial\Sigma_Q \cap \{y>0\}} \left(y\left(\frac{\partial\phi}{\partial\nu} u_x(t,y) + (\phi - \phi(z_Q)) \frac{\partial u_x}{\partial\nu}\right) - (\phi - \phi(z_Q)) u_x \frac{\partial y}{\partial\nu}\right) d\sigma_Q.$$

By fact (2) and the way we chose the Q's,

$$\left| y\frac{\partial\phi}{\partial\nu} - (\phi - \phi(z_Q)) \frac{\partial y}{\partial\nu} \right| \le C(d)$$

when $(t,y) \in \partial\Sigma_Q \cap \{y>0\}$. Hence,

$$\int_{\partial\Sigma_Q \cap \{y>0\}} \left(y\frac{\partial\phi}{\partial\nu} - (\phi - \phi(z_Q)) \frac{\partial y}{\partial\nu}\right) u_x(t,y)\, d\sigma_Q$$
$$= \int_{\partial\Sigma_Q \cap \{y>0\}} P_y(x-t)\, h_Q(t,y)\, d\sigma_Q$$

for some $h_Q(t,y)$ with $|h_Q| \le C(d)$. It is well-known that

$$(\dagger\dagger) \qquad \left\| \sum_{Q \in \bigcup \mathbf{G}_k} d\sigma_Q \right\|_C \le C(d)$$

(see [4]), from which it follows that

$$\left\| \sum_{Q \in \bigcup \mathbf{G}_k} h_Q(t,y)\, d\sigma_Q \right\|_C \le C(d).$$

Thus, to obtain (5), we only need to estimate the integrals

$$\int_{\partial\Sigma_Q \cap \{y>0\}} y(\phi - \phi(z_Q)) \frac{\partial u_x}{\partial\nu}\, d\sigma_Q.$$

For this we need a lemma.

LEMMA. *Let* $\{z_i\} = \{(x_i, y_i)\} \subset \mathbf{R}_+^{d+1}$ *be points and let* δ_{z_i} = *the Dirac mass at* z_i. *Assume that*

$$(6) \qquad \left\| \sum_i y_i^d \delta_{z_i} \right\|_C \le 1.$$

Let ds_i *denote d-dimensional Lebesgue measure on the hyperplane* $\{y = y_i\} \subset \mathbf{R}_+^{d+1}$ *and let* $\Phi(x) = (1+|x|)^{-d-1}$. *Set*

$$\mu(x,y) = \sum_i y_i^d \Phi_{y_i}(x - x_i)\, ds_i.$$

Then $\|\mu\|_C \le C(d)$.

Proof of Lemma. Let $Q_\#$ be a cube. What we need to show is that

$$(7) \qquad \sum_{y_i \le \ell(Q_\#)} \int_{Q_\#} y_i^d \Phi_{y_i}(x - x_i)\, dx \le C(d) |Q_\#|.$$

For $k = 0, 1, 2, 3, \ldots$, let $Q_{\#,k}$ denote the cube concentric with $Q_\#$ and with side-length 2^k times as big. We set:

$$S_0 = \sum_{\substack{y_i \le \ell(Q_\#) \\ x_i \in \overline{Q_\#}}} \int_{Q_\#} y_i^d \Phi_{y_i}(x - x_i)\, dx;$$

$$S_k = \sum_{\substack{y_i \le \ell(Q_\#) \\ x_i \in \overline{Q_{\#,k}} \setminus \overline{Q_{\#,k-1}}}} \int_{Q_\#} y_i^d \Phi_{y_i}(x - x_i)\, dx \quad k > 0.$$

Clearly,

$$S_0 \le C(d)\,|Q_\#| \quad \text{and} \quad S_1 \le C(d)\,|Q_\#|$$

by (6).

If $x \in \overline{Q_\#}$, $x_i \in \overline{Q_{\#,k}} \setminus \overline{Q_{\#,k-1}}$, $k \ge 2$, and $y_i \le \ell(Q_\#)$, then

$$\Phi_{y_i}(x - x_i) \le \frac{C(d)\,\ell(Q_\#)}{(2^k \ell(Q_\#))^{d+1}}.$$

Thus,

$$\begin{aligned} S_k &\le \frac{C(d)\,\ell(Q_\#)}{(2^k \ell(Q_\#))^{d+1}} \times |Q_\#| \times \left[\sum_{\substack{y_i \le \ell(Q_\#) \\ x_i \in \overline{Q_{\#,k}} \setminus \overline{Q_{\#,k-1}}}} y_i^d \right] \\ &\le C(d)\, 2^{-k} \| Q_\# |, \end{aligned}$$

and now summing on k yields (7). □

We will now finish our treatment of (II) by showing that every integral

$$\int_{\partial\Sigma_Q \cap \{y > 0\}} y(\phi - \phi(z_Q))\, \frac{\partial u_x}{\partial \nu}\, d\sigma_Q$$

can be written in the form

$$\int P_y(x - t)\, d\rho_Q(t, y),$$

where

$$d\rho_Q = \sum y_{i,Q}^d (\Psi^{i,Q})_{y_{i,Q}}(x_i - t)\, ds_{i,Q}(t)$$

with

(8) $$|\Psi^{i,Q}(x)| \le C(d)\,\Phi(x)$$

and the $z_{i,Q} = (x_{i,Q}, y_{i,Q})$ satisfy

$$\left\| \sum_{i,Q} y_{i,Q}^d \delta_{z_{i,Q}} \right\|_C \le C(d).$$

Write

$$\partial\Sigma_Q \cap \{y > 0\} = \bigcup_i E_{i,Q},$$

where each $E_{i,Q}$ is of the form

$$E_{i,Q} = \begin{cases} \{(x, y) : x \in \partial Q_i,\ \ell(Q_i) \le y \le 2\ell(Q_i)\} \cap \partial\Sigma_Q & \text{or} \\ \{(x, y) : x \in \bar{Q}_i,\ y = \ell(Q_i)\} \end{cases}$$

for some dyadic cube Q_i. These $E_{i,Q}$ make a tiling of $\partial\Sigma_Q \cap \{y>0\}$, with the size of the tiles going to zero as $y \to 0$. Let $(x_{i,Q}, \tilde{y}_{i,Q})$ be the centroid of $E_{i,Q}$ (in $\mathbf{R}_+^{d+1}$) and let $\xi_{i,Q} = (x_{i,Q}, \frac{1}{2}\tilde{y}_{i,Q}) \equiv (x_{i,Q}, y_{i,Q})$.

It is easy to see that, for any cube Q^*,

$$\int_{\hat{Q}^*} \sum_i \tilde{y}_{i,Q}^d \delta_{\xi_{i,Q}} \le \int_{4\hat{Q}^*} d\sigma_Q,$$

where $4Q^*$ is the quadruple (the double of the double) of Q^*. Therefore, by (††),

$$\|\sum y_{i,Q}^d \delta_{\xi_{i,Q}}\|_C \le C(d).$$

Now, if $(t, y) \in E_{i,Q}$, then

$$y\frac{\partial u_x}{\partial \nu}(t, y) = \int P_{y_{i,Q}}(x-s)\Psi_{y-y_{i,Q}}^{i,Q}(t, y)(s-x_{i,Q})\, ds,$$

where the $\Psi^{i,Q}(t, y)(x)$ satisfy (8) *uniformly* in (t, y). (This is because $y - y_{i,Q} \ge \frac{1}{3}y_{i,Q}$ and $|t - x_{i,Q}| \le Cy_{i,Q}$ for $(t, y) \in E_{i,Q}$.) Therefore, inequality (8) and fact (2) imply that

$$\int_{E_{i,Q}} y(\phi - \phi(z_Q))\frac{\partial u_x}{\partial \nu}\, d\sigma_Q = y_{i,Q}^d \int P_{y_{i,Q}}(x-s)\Lambda_{y_{i,Q}}^{i,Q}(s-x_{i,Q})\, ds_{i,Q},$$

where

$$\Lambda_{y_{i,Q}}^{i,Q}(x) = C(E_{i,Q})\left(\int_{E_{i,Q}} d\sigma_Q\right)^{-1} \int_{E_{i,Q}} (\phi(t, y) - \phi(z_Q))\Psi_{y-y_{i,Q}}^{i,Q}(t, y)(x)\, d\sigma_Q$$

and $C(E_{i,Q})$ is either $(\frac{4}{3})^d$ or 2^d, depending on where $E_{i,Q}$ sits. We obviously have

$$|\Lambda^{i,Q}(x)| \le C(d)\Phi(x),$$

and this, with the lemma, finishes off (I).

Now for (II). Fortunately this is easy. Recall that (II) equals

$$\int_{\mathbf{R}_+^{d+1}\setminus \hat{Q}_0} y\nabla\phi \cdot \nabla u_x\, dt\, dy.$$

By Green's theorem, we may write this as:

$$\int_{\partial\hat{Q}_0 \cap \mathbf{R}_+^{d+1}} \left(y\left(\frac{\partial \phi}{\partial \nu}u_x(t, y) + (\phi - \phi(z_{Q_0})\frac{\partial u_x}{\partial \nu}\right) - (\phi - \phi(z_{Q_0}))u_x\frac{\partial y}{\partial \nu}\right) d\sigma_Q,$$

where $d\sigma_{Q_0}$ is d-dimensional surface measure on $\partial\hat{Q}_0$ and $\partial/\partial\nu$ now denotes the normal derivative *into* $\hat{Q}_0$. Since $\partial\hat{Q}_0 \cap \mathbf{R}_+^{d+1}$ is away from the support of ϕ, and $\|\phi\|_1 \le C(d)$, we have

$$|\phi - \phi(z_{Q_0})| \le C(d) \quad \text{and} \quad |y\nabla\phi| \le C(d)$$

on $\partial\hat{Q}_0 \cap \mathbf{R}_+^{d+1}$. So, since $d\sigma_{Q_0}$ is obviously a Carleson measure (with a norm $\le C(d)$) these terms present no problem. We now handle the other term by cutting $\partial\hat{Q}_0$ into tiles, just as we did for (I), and this finishes the proof. □

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