HILBERT TRANSFORM IN THE COMPLEX PLANE AND AREA INEQUALITIES FOR CERTAIN QUADRATIC DIFFERENTIALS

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Introduction. In this paper we are concerned with a singular integral of Calderon–Zygmund type defined for functions of one complex variable as

$$(Tf)(z) = -\frac{1}{\pi}\iint \frac{f(\zeta)\,d\mu(\zeta)}{(z-\zeta)^2},$$

where $d\mu(\zeta)$ denotes the Lebesgue measure in $\mathbf{C}$. This integral is known as the Hilbert transform in the complex plane or the Ahlfors–Beurling transform. Its Fourier multiplier is

$$m(\xi) = \frac{(Tf)\hat{}(\xi)}{f\hat{}(\xi)} = \frac{\xi}{\bar{\xi}}, \quad \xi \in \mathbf{C}-\{0\}.$$

In particular, T is a unitary operator in $L^2(\mathbf{C})$ and it changes the complex derivatives $\partial/\partial\bar{z}$ and $\partial/\partial z$; in symbols,

$$T\circ\frac{\partial}{\partial\bar{z}} = \frac{\partial}{\partial z}. \tag{1}$$

This fundamental property of the Hilbert transform has led to various applications to the plane quasiconformal mappings and the theory of systems of partial differential equations in complex variables.

We are specifically concerned with the Hilbert transform of the characteristic function χ_E of a measurable subset E in the unit disk $\mathbf{B} = \{z : |z| < 1\}$. A weak (1.1)-type inequality shows that

$$\iint_{\mathbf{B}} |T\chi_E(z)|\,d\mu(z) \le A|E|\log\frac{\pi}{|E|} + C|E|, \tag{2}$$

where $|E|$ stands for the Lebesgue measure of E. The constants A and C are independent of the set E.

In 1966, Gehring and Reich [2] recognized that the best possible constant A in (2) is strictly related to the degree of regularity of a quasiconformal mapping. This constant is expected to be equal to one. Reich [12; 13] succeeded in proving that $A \le 17$.

In this paper we wish to treat various cases where we have reached $A = 1$ in (2). The main results, however, concern certain sharp estimates for hyperelliptic differentials which are of independent interest. Other papers concerned with sharp inequalities for the Hilbert transform are [11], [3], and [4].

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Among all operators of Calderon–Zygmund type the Hilbert transform is very unique since its kernel $K(z) = -\pi^{-1}z^{-2}$ is homogeneous of degree -2 and holomorphic in $\mathbf{C}-\{0\}$. The theory of analytic functions is therefore an effective tool for treating exact estimates for T. The unitary property and Hölder's inequality imply at once that

$$\iint_E |T\chi_E(z)|\, d\mu(z) \leq |E|,$$

for every $E \subset \mathbf{C}$. What remains to be established is then the best constant A in a somewhat weaker inequality

$$\iint_{\mathbf{B}-E} |T\chi_E(z)|\, d\mu(z) \leq A|E| \log \frac{\pi}{|E|} + C|E|, \tag{3}$$

where E may be compact. The function $T\chi_E(z)$, denoted for brevity by

$$T_E(z) = -\frac{1}{\pi}\iint_E \frac{d\mu(z)}{(z-t)^2},$$

is then holomorphic in $\mathbf{C}-E$ and has the following expansion at infinity:

$$T_E(z) = -\frac{|E|}{\pi z^2} + \text{higher powers of } \frac{1}{z}. \tag{4}$$

It is sufficiently general to limit ourselves to sets E which are unions of a finite number of mutually disjoint closed disks, $B_j = B(z_j, r_j) \subset \mathbf{B}$, $j = 1, \dots, n$; that is,

$$E = \bigcup_{j=1}^{n} \overline{B(z_j, r_j)}. \tag{5}$$

In this case $T_E(z)$ computes explicitly. For $z \in \mathbf{C}-E$ it coincides with a rational function $R(z)$ with double poles at z_j and negative leading coefficients $-r_j^2$, $j = 1, \dots, n$,

$$T_E(z) = R(z) = -\sum_{j=1}^{n} \frac{r_j^2}{(z-z_j)^2}. \tag{6}$$

Certain configurations of such disks, including infinite families, call for considering the trajectory structure of the quadratic differential $R(z)\, dz^2$. In this way we are led to the Jenkins–Spencer theory of the hyperelliptic integrals [9].

We achieve constants $A = 1$ and $C = 0$ in (3), whenever $T_E(z)\, dz^2$ has simple trajectory structure (see Section 4, Theorem 1).

In general, $T_E(z)\, dz^2$ need not be such. Careful examination of examples suggests the study of an extremal metric problem in the circular region $\mathbf{B}-E$. We solve this problem for general multiply connected domains and give sharp estimates of the minimum area integral (Theorem 2). This result generalizes Carleman's theorem on the module of a doubly connected domain. We exploit it to construct a number of rational quadratic differentials $Q_\Gamma(z)\, dz^2$ with closed trajectory structure associated with a given free family $\Gamma = \{\Gamma_1, \dots, \Gamma_N\}$ of homotopy classes of Jordan curves in $\dot{\mathbf{C}} = \mathbf{C}-\{z_1, \dots, z_n\}$. More precisely, $Q_\Gamma(z)\, dz^2$ has the form

$$Q_\Gamma(z) = \sum_{j=1}^{n} \frac{-r_j^2}{(z-z_j)^2} + \sum_{j=1}^{n} \frac{a_j}{z-z_j}, \tag{7}$$

where the complex coefficients $a_j = a_j(\Gamma)$, $j = 1, \ldots, n$, are subject to the conditions

$$\sum_{j=1}^{n} a_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} a_j z_j = 0. \tag{8}$$

The first-order term

$$\phi_\Gamma(z) = \sum_{j=1}^{n} \frac{a_j}{z-z_j} \tag{9}$$

has therefore finite L^1-norm on the complex plane. Moreover, if a trajectory γ encloses precisely the poles $z_{j_1}, \ldots, z_{j_s}$, then its Q_Γ-length equals

$$\ell(\gamma) = 2\pi\sqrt{r_{j_1}^2 + \cdots + r_{j_s}^2}. \tag{10}$$

Such a quadratic differential is unique for each admissible family Γ (see Theorem 3).

We have the following inequalities:

$$\iint_{\mathbf{B}-E} |Q_\Gamma(z)|\, d\mu(z) \le |E| \log \frac{\pi}{|E|} \tag{11}$$

for all Q_Γ, which imply an estimate for the Hilbert transform $T_E(z)$; and

$$\iint_{\mathbf{B}-E} |T_E(z)|\, d\mu(z) \le |E| \log \frac{\pi}{|E|} + \inf_\Phi \|\Phi\|_{L^1(\mathbf{C})}, \tag{12}$$

where the functions Φ run through the convex hull of the first-order terms ϕ_Γ defined by (7), (8), and (9).

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1. Notation and basic facts. The following concavity inequality for the function $\log x$ will be frequently used:

$$x \log \frac{a}{x} + y \log \frac{b}{y} \le (x+y) \log \frac{a+b}{x+y}, \tag{13}$$

for all positive x, y, a, and b.

The Hilbert transform of the characteristic function of a circular annulus $\Delta = \{z : r < |z-a| < R\}$ has an explicit expression

$$T_\Delta(z) = \begin{cases} 0 & \text{if } |z-a| \le r, \text{ the inner complement of } \Delta, \\ \dfrac{r^2}{(z-a)^2} & \text{if } r < |z-a| < R, \\ -\dfrac{|\Delta|}{\pi(z-a)^2} & \text{if } |z-a| \ge R, \text{ the outer complement of } \Delta. \end{cases} \tag{14}$$

In particular, for an Euclidean disk $B=B(a,R)$,

$$T_B(z)=\begin{cases} 0 & \text{if } z\in B, \\ -\dfrac{|B|}{\pi(z-a)^2} & \text{if } z\in \mathbf{C}-B. \end{cases} \tag{15}$$

If $w=w(z)$ is Lipschitz continuous in $\mathbf{C}$ and satisfies the growth condition $w(z)=O(|z|^{-1})$ at infinity, then

$$T\left(\frac{\partial w}{\partial \bar z}\right)=\frac{\partial w}{\partial z} \tag{16}$$

almost everywhere in $\mathbf{C}$. Integration by parts leads then to the following integral identities for the Hilbert transform:

$$\begin{aligned} &\iint Tg(z)\overline{Th(z)}\,d\mu(z)=\iint g(z)\overline{h(z)}\,d\mu(z), \\ &\iint g(z)Th(z)\,d\mu(z)=\iint h(z)Tg(z)\,d\mu(z). \end{aligned} \tag{17}$$

Throughout this paper we use notation which should be clear from the context or can be found in the referred papers.

If γ is a Jordan curve in $\mathbf{C}$ then the bounded component of $\hat{\mathbf{C}}-\gamma$ is called the inner domain of γ while the unbounded component is called the outer domain; γ is the boundary of each component. The components of $\hat{\mathbf{C}}-\gamma$ are referred to as circle domains. A set or a point in the complex plane is said to be enclosed by γ if it lies in the inner domain of γ.

Let Ω be an arbitrary open set in $\mathbf{C}$ and let Γ_0 be a nontrivial homotopy class of Jordan curves in Ω. Then every two nonintersecting curves from Γ_0 bound a ring domain in Ω. Such a ring is said to be of homotopy type Γ_0.

A ring domain Δ determines unique circle domains A and B which are bounded, $\bar A\subset B$ and $\Delta=B-\bar A$. A is called the inner complement of Δ.

The module of Δ is defined as

$$\operatorname{mod}(\Delta)=\frac{1}{2\pi}\log\frac{R}{r}, \tag{18}$$

if $\{z: r<|z|<R\}$ is the circular annulus conformally equivalent to Δ.

CARLEMAN'S THEOREM [1]. *Under the above notation we have*

$$\operatorname{mod}(\Delta)\le\frac{1}{4\pi}\log\frac{|B|}{|A|}. \tag{19}$$

Equality occurs in (19) *only for a circular annulus.*

We shall use basic ideas concerning quadratic differentials and extremal metrics. For fairly complete information about those concepts we refer to the monographs by Jenkins [8] and Strebel [15].

Let $Q(z)\,dz^2$ be a quadratic differential defined on an open set of the Riemann sphere $\hat{\mathbf{C}}$.

If Δ is a ring domain relative to $Q(z)\,dz^2$, then Δ contains no critical points of $Q(z)\,dz^2$. It is swept out by closed trajectories $\gamma\subset\Delta$, all of the same Q-length defined by

$$\ell=\int_\gamma \sqrt{|Q(z)|}\,|dz|=\left|\int_\gamma \sqrt{Q(z)}\,dz\right|.$$

The function

$$f(z)=\exp\frac{2\pi i}{\ell}\int\sqrt{Q(z)}\,dz \tag{20}$$

maps Δ conformally onto a circular annulus of Q-area equal to

$$\iint_\Delta |Q(z)|\,d\mu(z)=\ell^2\operatorname{mod}(\Delta). \tag{21}$$

If D is a circle domain relative to $Q(z)\,dz^2$, then D contains precisely one double pole z_0 of $Q(z)$ and

$$Q(z)=\frac{-r^2}{(z-z_0)^2}+\frac{a}{z-z_0}+\text{regular terms}$$

for some $r>0$. $D-\{z_0\}$ is swept out by closed trajectories separating z_0 from the boundary of D, whose Q-length equals $\ell=2\pi r$. The function (20) extended to have the value zero at z_0 maps D conformally onto an Euclidean disk centered at zero. Every two trajectories, or a trajectory and ∂D, bound a doubly connected domain, say Δ, whose Q-area equals $\ell^2\operatorname{mod}(\Delta)$.

We are mostly interested in holomorphic quadratic differentials which are positive along the boundary curves and have closed trajectories; see [15] for the definitions.

Let S be either the punctured Riemann sphere $\dot{\mathbf{C}}=\mathbf{C}-\{z_1,\dots,z_n\}$ or an n-connected domain bounded by piecewise analytic Jordan curves $C_1,\dots,C_n$ and C.

A family $\Gamma=\{\Gamma_1,\dots,\Gamma_N\}$ of homotopy classes Γ_ν, $\nu=1,\dots,N$, of Jordan curves in S is called *admissible* if the following conditions are satisfied:

(i) all Γ_ν are distinct and nontrivial,
(ii) there exist curves $\gamma_\nu\in\Gamma_\nu$, $\nu=1,\dots,N$, which do not intersect each other, and
(iii) Γ contains the classes generated by simple loops which are arbitrarily close to the boundary components of S.

Such a family Γ admits at most $2n-1$ homotopy classes, and it can always be completed to a maximal family with precisely $N=2n-1$ classes.

A quadratic differential $Q(z)\,dz^2$ on S with closed trajectories is said to be of homotopy type $\Gamma=\{\Gamma_1,\dots,\Gamma_N\}$ if its closed trajectories belong to $\bigcup\Gamma_\nu$. For a given class $\Gamma_\nu\in\Gamma$ the existence of a trajectory in Γ_ν is, however, not always guaranteed.

We shall use tacitly the global structure theorem due to Jenkins and Spencer, in its simplest cases when only closed trajectories are present [7; 9]. We also appeal to Jenkins' [6] module theory for the existence and properties of the extremal metric.

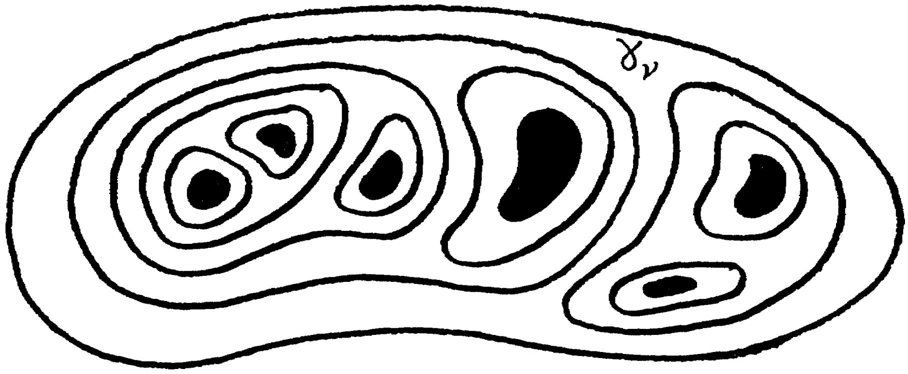

Figure 1

2. Preliminary examples. Let us consider first the best possible bound for $T_E(z)$ from below.

EXAMPLE 1. Let E be an arbitrary simply connected continuum with analytic boundary. Let f map the exterior of E conformally onto the unit disk $\mathbf{B}$, $f(\infty) = 0$. $\mathbf{C} - E$ is therefore swept out by the level curves of f each of which bounds a simply connected domain. Let Ω denote one such domain with boundary $\partial\Omega = \{z : |f(z)| = \rho\}$ for some $\rho < 1$. The ring domain $\Delta = \Omega - E$ is mapped by f onto the circular annulus $\{\xi : \rho < |\xi| < 1\}$.

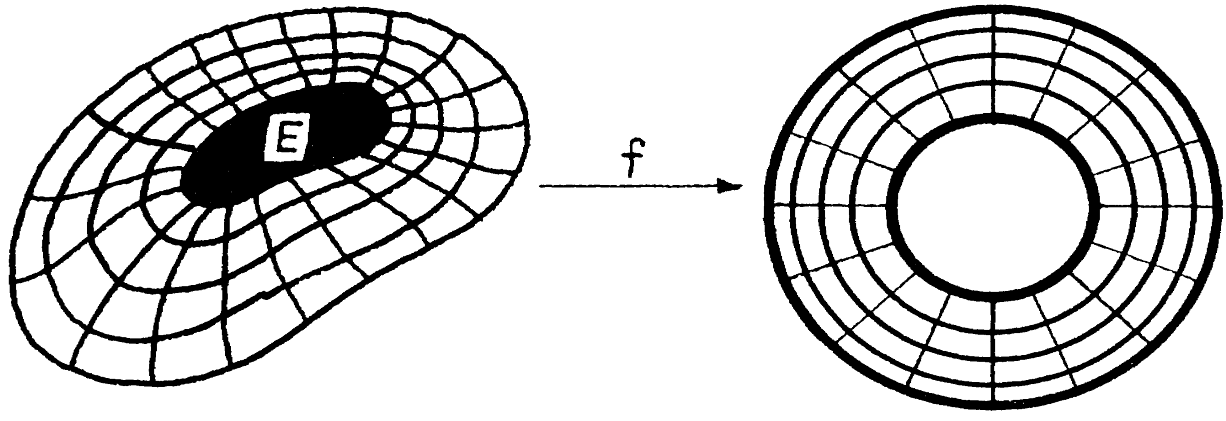

Figure 2

PROPOSITION 1. *We have*

$$\iint_{\Omega - E} |T_E(z)|\, d\mu(z) \geq 4\pi |E| \operatorname{mod}(\Delta) = 2|E| \log \frac{C(\Omega)}{C(E)}, \tag{22}$$

where $C(E)$ and $C(\Omega)$ stand for the conformal capacity of E and Ω respectively.

Equality occurs in (22) when $T_E(z)\, dz^2$ defines a positive quadratic differential on $\mathbf{C} - E$ with closed trajectories which are the level curves of f and have

T_E-length ℓ such that $\ell^2 = 4\pi|E|$. Moreover, with a suitable choice of branch of $\int \sqrt{T_E(z)}\, dz$ in $\mathbf{C}-E$ the conformal map f is expressed by

$$f(z) = \exp \frac{2\pi i}{\ell} \int \sqrt{T_E(z)}\, dz.$$

This happens if E is a disk.

Note that by Carleman's theorem

$$4\pi|E| \operatorname{mod}(\Delta) \leq |E| \log \frac{|\Omega|}{|E|}.$$

We also have $\sqrt{|E|/\pi} \leq C(E)$.

Proof. We examine a Lipschitz continuous function w defined piecewise on $\mathbf{C}$ by

$$w(z) = \begin{cases} 2z \log(1/\rho) & \text{in } E, \\ (2f/f') \log(|f|/\rho) + 2z \log(1/\rho) & \text{in } \Omega - E, \\ (2f/f') \log(1/\rho) + 2z \log(1/\rho) & \text{in } \mathbf{C} - \Omega. \end{cases}$$

It is verified at once that $w(z) = O(|z|^{-1})$ at infinity. Thus by (16) $T(\partial w/\partial \bar{z}) = \partial w/\partial z$. The complex derivatives of w are found to be

$$\omega(z) = \frac{\partial w}{\partial \bar{z}} = \frac{f\bar{f}'}{\bar{f}f'} \chi_{\Omega-E}(z)$$

for all $z \in \mathbf{C}$ while the z-derivative of w is constant on E equal to $2\log(1/\rho)$. We then have $T\omega(z) = 2\log(1/\rho)$ in E. Hence, using integral identity (17), we immediately come to estimate (22). Indeed,

$$\begin{aligned} 2|E| \log \frac{1}{\rho} &= \iint_E T\omega(z)\, d\mu(z) = \iint \chi_E(z) T\omega(z)\, d\mu(z) = \iint \omega(z) T_E(z)\, d\mu(z) \\ &= \iint_{\Omega-E} \omega(z) T_E(z)\, d\mu(z) \leq \iint_{\Omega-E} |T_E(z)|\, d\mu(z). \end{aligned}$$

To reach equality here we should have

$$\frac{\bar{f}f'}{f\bar{f}'} = \frac{T_E(z)}{|T_E(z)|} \quad \text{for } z \in \Omega - E,$$

or alternatively

$$\left(\frac{f}{f'}\right)^2 T_E(z) = \overline{\left[\left(\frac{f}{f'}\right)^2 T_E(z)\right]} \quad \text{for } z \in \Omega - E.$$

This equation, in view of analyticity of f and T_E in $\mathbf{C}-E$, extends to hold in $\mathbf{C}-E$, and consequently both sides equal a real constant. This constant is identified to be $-(1/\pi)|E|$ by the asymptotic formula (4). Hence

$$T_E(z) = -\frac{|E|}{\pi}\left(\frac{f'}{f}\right)^2$$

and

$$\ell^2 = \left(\int_\gamma \sqrt{T_E(z)}\,dz\right)^2 = -\frac{|E|}{\pi}\left(\int_\gamma \frac{df}{f}\right)^2 = 4\pi|E|,$$

$$\iint_\Delta |T_E(z)|\,d\mu(z) = \ell^2 \operatorname{mod}(\Delta) = 4\pi|E| \operatorname{mod}(\Delta),$$

$$f(z) = \exp\frac{2\pi i}{\ell}\int \sqrt{T_E(z)}\,dz.$$

That equality occurs in (22) when E is a disk, say the unit disk, follows from the explicit formulas $f(z)=z^{-1}$ and $T_E(z)=-z^{-2}$ in $\mathbf{C}-E$. □

The next example provides a sharp upper bound for $T_E(z)$ in the simplest case.

EXAMPLE 2. *Let $E=B(a,r)$ be a disk embedded into a measurable set Ω. Then*

$$\iint_{\Omega-E} |T_E(z)|\,d\mu(z) \le |E| \log\frac{|\Omega|}{|E|} \tag{23}$$

with equality occurring only for $\Omega = B(a,R)$, where $\pi R^2 = |\Omega|$.

Proof. By (15) we obtain

$$\iint_{\Omega-E} |T_E(z)|\,d\mu(z) = \iint_{r<|z-a|,\, z\in\Omega} \frac{r^2\,d\mu(z)}{|z-a|^2} \le \iint_{r<|z-a|<R} \frac{r^2\,d\mu(z)}{|z-a|^2}$$
$$= 2\pi r^2 \log\frac{R}{r} = |E|\log\frac{|\Omega|}{|E|}.$$

This proves (23). □

A set $E\subset\mathbf{C}$ is called circularly symmetric if $z\in E$ implies $ze^{i\theta}\in E$, for all $\theta\in[0,2\pi)$; see [12].

EXAMPLE 3. *If $E\subset\mathbf{B}$ is measurable and circularly symmetric then*

$$\iint_{\mathbf{B}-E} |T_E(z)|\,d\mu(z) \le |E| \log\frac{\pi}{|E|}. \tag{24}$$

Proof. By a standard approximation argument the problem reduces to the case where E is the union of a disjoint family of closed rings $E_1,\dots,E_n$ and one disk E_0 centered at zero:

$$T_E(z) = \sum_{j=0}^{n} T_{E_j}(z). \tag{25}$$

Then $\mathbf{B}-E$ is the union of disjoint open rings $\Delta_1,\dots,\Delta_n$. Let Δ be one of those rings and U its inner complement. It is readily seen from formulas (14) and (15) that for each $z\in\Delta$ the nonzero terms in (25) come from those E_j which lie in U, and that the value of $T_{E_j}(z)$ in Δ equals $-\pi^{-1}z^{-2}|E_j|$. Hence, for $z\in\Delta$ we have

$$T_E(z) = \sum_{E_j\subset U} T_{E_j}(z) = -\frac{1}{\pi z^2}\sum_{E_j\subset U}|E_j| = -\frac{|E\cap U|}{\pi z^2} = -\frac{A_\nu}{\pi z^2},$$

where A_ν, $\nu=1,\dots,n$, denotes the area of that part of E which lies in the inner complement of Δ_ν, $\nu=1,\dots,n$. Integrating and summing over all rings Δ_ν we arrive at a weighted sum of moduli

(26) $$\iint_{\mathbf{B}-E} |T_E(z)|\, d\mu(z) = 4\pi \sum_{\nu=1}^{n} A_\nu \operatorname{mod}(\Delta_\nu) \le |E| \log \frac{\pi}{|E|}.$$

The latter estimate can be shown by an induction with respect to the number of rings Δ_ν $(\nu=1,\dots,n)$ and concavity inequality (13). We postpone the proof of this estimate until Section 5, Lemma 2, where the weighted sums of moduli are treated in greater generality. □

It has to be noted that $T_E(z)\,dz^2$ defines a positive quadratic differential on $\mathbf{B}-E$. The rings Δ_ν $(\nu=1,\dots,n)$ are its characteristic domains, and the T_E-length of a trajectory $\gamma\subset\Delta_\nu$ satisfies $\ell^2=4\pi A_\nu$ for $\nu=1,\dots,n$.

Now we proceed to two basic examples.

EXAMPLE 4. *Let us decompose the unit disk* $\mathbf{B}$ *into infinitely many nonoverlapping closed disks* $B_k=\overline{B(z_k,r_k)}\subset\mathbf{B}$, $k=1,2,\dots$. *Fix a parameter* $t\in(0,1)$, *then shrink each* B_k *by the factor* t. *Denote by* $B_k(t)=\overline{B(z_k,tr_k)}$, $k=1,\dots,$ *and*

$$E=\bigcup_{k=1}^{\infty} B_k(t).$$

Clearly, $\sum r_k^2=1$ *and* $|E|=\pi t^2$. *We then have*

(27) $$\iint_{\mathbf{B}-E} |T_E(z)|\, d\mu(z) = |E| \log \frac{\pi}{|E|}.$$

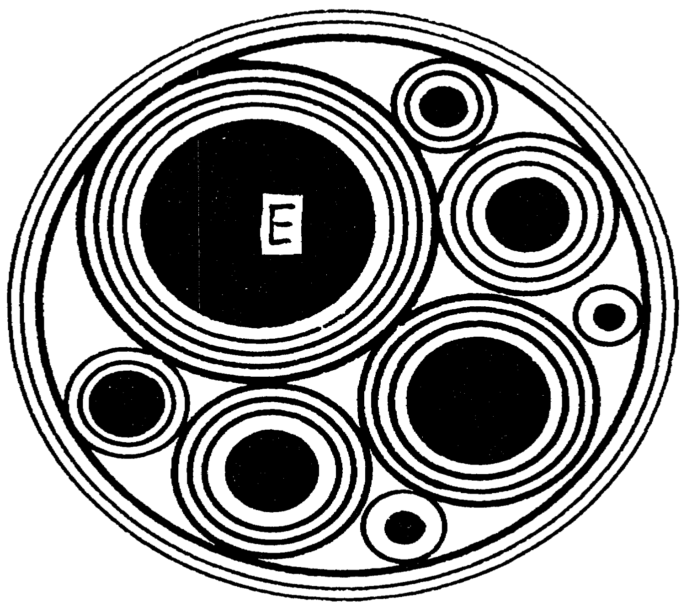

Figure 3

Proof. Notice that $\mathbf{B}-E$ is the union of a countable family of circular annuli $\Delta_k = B_k - B_k(t)$, $k=1,2,\ldots$. An elementary computation shows that

$$T_{B_k(t)}(z) = -\frac{t^2 r_k^2 \chi_{\Delta_k}(z)}{(z-z_k)^2} + t^2 T_{B_k}(z)$$

for every $z \in \mathbf{C}$ and $k=1,2,\ldots$. Summing over all indices $k=1,2,\ldots$ we obtain

$$T_E(z) = -t^2 \sum_{k=1}^{\infty} \frac{r_k^2 \chi_{\Delta_k}(z)}{(z-z_k)^2} + t^2 T_{\mathbf{B}}(z) \tag{28}$$

for all $z \in \mathbf{C}$. Finally, since $T_{\mathbf{B}}(z)$ vanishes for $z \in \mathbf{B}$ we compute

$$\begin{aligned}\iint_{\mathbf{B}-E} |T_E(z)|\, d\mu(z) &= t^2 \sum_{k=1}^{\infty} r_k^2 \iint_{\Delta_k} \frac{d\mu(z)}{|z-z_k|^2} \\ &= 2\pi t^2 \log \frac{1}{t} \sum_{k=1}^{\infty} r_k^2 = |E| \log \frac{\pi}{|E|}.\end{aligned}$$

EXAMPLE 5. (We continue the construction of the previous example, performing a disk decomposition to each $B_k(t)$, $k=1,2,\ldots$.) *Let $B_k(t)$ be the union of nonoverlapping disks, say, $B_{kl} = \overline{B(z_{kl}, r_{kl})}$, $k, l = 1,2,\ldots$, $B_{kl} \subset B_k(t)$ for all $l = 1,2,\ldots$, and*

$$B_k(t) = \bigcup_{l=1}^{\infty} B_{kl}.$$

Choose a parameter $s \in (0,1)$ and shrink each B_{kl} by the factor s. Denote by $B_{kl}(s) = \overline{B(z_{kl}, s r_{kl})}$, $k, l = 1,2,\ldots$, and put

$$F = \bigcup_{k,l=1}^{\infty} B_{kl}(s).$$

Clearly $\sum_{k,l=1}^{\infty} r_{kl}^2 = t^2$ and $|F| = \pi t^2 s^2$. We then have

$$\iint_{\mathbf{B}-F} |T_F(z)|\, d\mu(z) = |F| \log \frac{\pi}{|F|}. \tag{29}$$

Proof. In this case the set $\mathbf{B}-F$ is built of two types of circular annuli:

$$\Delta_k = B_k - B_k(t), \quad k=1,2,\ldots,$$

and

$$\Delta_{kl} = B_{kl} - B_{kl}(s), \quad k,l=1,2,\ldots.$$

Elementary computations show that

$$T_F(z) = -s^2 \sum_{k,l=1}^{\infty} \frac{r_{kl}^2 \chi_{\Delta_{kl}}(z)}{(z-z_{kl})^2} - s^2 t^2 \sum_{k=1}^{\infty} \frac{r_k^2 \chi_{\Delta_k}(z)}{(z-z_k)^2} + s^2 t^2 (\chi_{\mathbf{B}}(z) - 1) z^{-2}. \tag{30}$$

Now the integration of $T_F(z)$ over the rings Δ_k and Δ_{kl}, $k, l = 1,2,\ldots$, is much the same as that in Example 4. The following interpretation is however of particular interest for what follows.

Let Δ be one of the rings Δ_k or Δ_{kl}, $k, l = 1,2,\ldots$. The concentric circles in Δ constitute closed trajectories of $T_F(z)\, dz^2$ which, in view of (30), have T_F-length equal to

$$\ell = \sqrt{4\pi A(\Delta)},$$

where $A(\Delta)$ means the total area of those disks $B_{kl}(s)$ which lay in the inner complement of Δ. More precisely, with obvious notation $\ell_{kl} = 2\pi s r_{kl}$ $(k, l = 1, 2, \dots)$ and $\ell_k = 2\pi st \cdot r_k$ $(k = 1, 2, \dots)$, the length of a trajectory in the unbounded circle domain of $T_F(z)\, dz^2$ is $\ell = \sqrt{4\pi |F|} = 2\pi st$. The moduli of the rings Δ_k and Δ_{kl} are

$$\operatorname{mod}(\Delta_k) = \frac{1}{2\pi} \log \frac{1}{t}, \quad k = 1, 2, \dots,$$

$$\operatorname{mod}(\Delta_{kl}) = \frac{1}{2\pi} \log \frac{1}{s}, \quad k, l = 1, 2, \dots.$$

Finally, as a weighted sum of moduli the T_F-area of $\mathbf{B} - F$ is found to be

$$\begin{aligned}\iint_{\mathbf{B}-F} |T_F(z)|\, d\mu(z) &= \sum \ell^2 \operatorname{mod}(\Delta) \\ &= 4\pi^2 s^2 \sum_{kl=1}^{\infty} r_{kl}^2 \operatorname{mod}(\Delta_{kl}) + 4\pi^2 s^2 t^2 \sum_{k=1}^{\infty} r_k^2 \operatorname{mod}(\Delta_k) \\ &= 2\pi s^2 \log \frac{1}{s} \sum_{kl=1}^{\infty} r_{kl}^2 + 2\pi s^2 t^2 \log \frac{1}{t} \sum_{k=1}^{\infty} r_k^2 \\ &= 2\pi s^2 t^2 \log \frac{1}{s} + 2\pi s^2 t^2 \log \frac{1}{t} = \end{aligned}$$

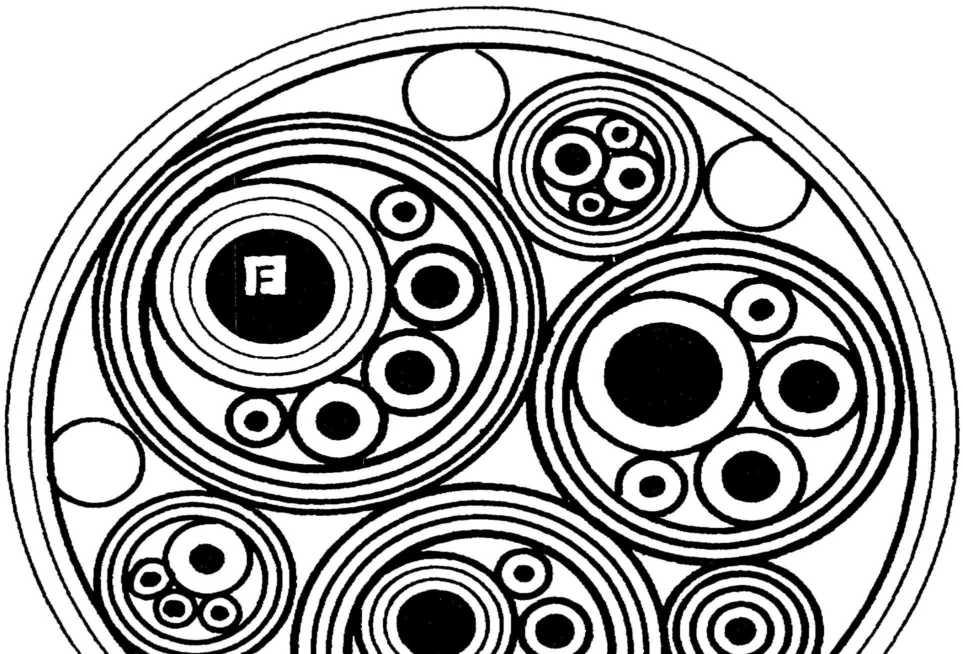

Figure 4

$$= \pi s^2 t^2 \log \frac{\pi}{\pi s^2 t^2} = |F| \log \frac{\pi}{|F|}.$$

This completes the proof of (29). □

3. The reduced norm. From now on E will denote the union of a finite number of disjoint disks $B_j = \overline{B(z_j, r_j)}$ in $\mathbf{B}$, $j = 1, \ldots, n$, $n \geq 2$. Hence, in view of formula (15), $T_E(z)$ takes the form

$$T_E(z) = R(z) = -\sum_{j=1}^{n} \frac{r_j^2}{(z - z_j)^2}, \quad z \in \mathbf{C} - E.$$

Note that the quadratic differential $R(z)\,dz^2$ need not be real along the boundary circles of $\mathbf{B} - E$. The natural domain in which to consider $R(z)\,dz^2$ is the punctured sphere $\dot{\mathbf{C}} = \mathbf{C} - \{z_1, \ldots, z_n\}$. It is possible to eliminate this inconsistency by reducing the problem to its degenerate case. We are going to estimate the R-area of $\mathbf{B} - E$ by means of the so-called reduced norm of $R(z)\,dz^2$ on $\dot{\mathbf{C}}$. It is important however to make such reduction for slightly more general quadratic differentials of the form

$$Q(z) = -\sum_{j=1}^{n} \left[\frac{r_j^2}{(z - z_j)^2} - \frac{a_j}{z - z_j} \right], \tag{31}$$

where the complex numbers a_j, $j = 1, \ldots, n$, will be specified later. They are subject to the following conditions

$$\sum_{j=1}^{n} a_j = 0, \qquad \sum_{j=1}^{n} a_j z_j = 0. \tag{32}$$

Thus the function

$$\phi(z) = \sum_{j=1}^{n} \frac{a_j}{z - z_j} \tag{33}$$

is L^1-integrable on the whole plane and Q has the expansion at infinity

$$Q(z) = -\left(\sum r_j^2\right) \frac{1}{z^2} + O(|z|^{-3}). \tag{34}$$

It may be noted that, in view of (32), $\phi \equiv 0$ if $n = 2$.

There are two equivalent definitions of the reduced norm. Both will be useful in succeeding sections. First fix arbitrary numbers $0 < \tau < 1$ and $R > 1$. Let $B(R) = B(0, R)$, $B_j(\tau) = \overline{B(z_j, \tau r_j)}$, $j = 1, \ldots, n$, and

$$E(\tau) = \bigcup_{j=1}^{n} B_j(\tau). \tag{35}$$

Removing the disks $B_j(\tau)$, $j = 1, \ldots, n$ from $B(R)$ we have left circular region

$$S(\tau, R) = B(R) - E(\tau) \tag{36}$$

on which $Q(z)$ has finite L^1-norm. Let us examine the following function:

$$H(\tau, R) = \iint_{S(\tau, R)} |Q(z)|\, d\mu(z) - \pi\left(\sum r_j^2\right) \log \frac{|B(R)|}{|E(\tau)|}. \tag{37}$$

Note that

$$|E(\tau)| = \tau^2 |E| = \pi\tau^2\left(\sum r_j^2\right).$$

A crude estimation gives a uniform bound for $H(\tau, R)$. Indeed, by concavity inequality (13),

$$\begin{aligned}
\iint_{S(\tau, R)} |Q(z)|\, d\mu(z) &\le \|\phi\|_{L^1(\mathbf{C})} + \sum_{j=1}^{n} r_j^2 \iint_{\tau r_j < |z - z_j| < R+1} \frac{d\mu(z)}{|z - z_j|^2} \\
&= \|\phi\|_{L^1(\mathbf{C})} + 2\pi \sum_{j=1}^{n} r_j^2 \log \frac{R+1}{\tau r_j} \\
&\le \|\phi\|_{L^1(\mathbf{C})} + \pi\left(\sum r_j^2\right) \log \frac{n(R+1)^2}{\tau^2 (\sum r_j^2)} \\
&= \|\phi\|_{L^1(\mathbf{C})} + \pi\left(\sum r_j^2\right) \log \frac{4n|B(R)|}{|E(\tau)|}.
\end{aligned}$$

Therefore

$$H(\tau, R) \le \|\phi\|_{L^1(\mathbf{C})} + \pi\left(\sum r_j^2\right) \log 4n,$$

for every $\tau \in (0, 1)$ and $R > 1$.

Next we shall prove that $H(\tau, R)$ increases as $\tau \to 0$ and $R \to \infty$. For this, it is obviously sufficient to establish that $dH/d\tau \le 0$ and $dH/dR \ge 0$. The standard differentiation rule yields

$$\frac{dH}{d\tau} = -\sum_{j=1}^{n} r_j \int_{|z - z_j| = \tau r_j} |Q(z)|\, |dz| + \frac{2\pi \sum r_j^2}{\tau}.$$

But on the circle $|z - z_j| = \tau r_j$, $|Q(z)|$ can be expressed as

$$|Q(z)| = \tau^{-2} \left| 1 - \frac{a_j}{r_j^2}(z - z_j) + \sum_{\nu = 1, \nu \ne j}^{n} \frac{r_\nu^2}{r_j^2} \left(\frac{z - z_j}{z - z_\nu}\right)^2 - \sum_{\nu = 1, \nu \ne j}^{n} \frac{a_\nu (z - z_j)^2}{r_j^2 (z - z_\nu)} \right|.$$

The right-hand side of this identity admits a subharmonic extension inside the disk $B(z_j, r_j)$ with value τ^{-2} at the center. Hence by the mean value inequality

$$r_j \int_{|z - z_j| = \tau r_j} |Q(z)|\, |dz| \ge \frac{2\pi r_j^2}{\tau}.$$

We then have $dH/d\tau \le 0$. The inequality $dH/dR \ge 0$ can be treated analogously.

As is seen from this proof, H is constant if and only if $n = 1$, $a_1 = z_1 = 0$. Hence we immediately conclude the following.

PROPOSITION 2. *Let $Q(z)\, dz^2$ have the form* (31)–(32). *Then its reduced norm is defined by*

$$\langle\langle Q \rangle\rangle = \lim_{\tau \to 0, R \to \infty} H(\tau, R). \tag{38}$$

Moreover we have

$$\iint_{\mathbf{B}-E} |Q(z)|\, d\mu(z) \leq |E| \log \frac{\pi}{|E|} + \langle\langle Q \rangle\rangle \tag{39}$$

with equality occurring here only for

$$Q(z) = -r^2 z^{-2}, \quad r > 0.$$

Before passing to the second definition we make certain observations.

Since $Q(z)\,dz^2$ has double poles $z_1, \ldots, z_n, \infty$, the corresponding coefficients being all real and negative, then by local structure theory $Q(z)\,dz^2$ admits characteristic circle domains

$$D_1, D_2, \ldots, D_n \text{ and } D \tag{40}$$

associated respectively with $z_1, z_2, \ldots, z_n$ and ∞. They are simply connected and bounded by piecewise analytic curves composed of a finite number of critical trajectories and their endpoints, zeros of $Q(z)$. Of course other characteristic domains may be present in the trajectory structure. The Q-length of a trajectory in D_j is equal to

$$\ell_j = 2\pi r_j, \quad j = 1, 2, \ldots, n, \tag{41}$$

while for a trajectory in D its Q-length equals

$$\ell = 2\pi \sqrt{r_1^2 + \cdots + r_n^2}. \tag{42}$$

The functions

$$f_j = \exp \frac{2\pi i}{\ell_j} \int \sqrt{Q(z)}\, dz, \quad j = 1, 2, \ldots, n \tag{43}$$

and

$$f = \exp \frac{2\pi i}{\ell} \int \sqrt{Q(z)}\, dz \tag{44}$$

map those domains conformally onto Euclidean disks centered at zero, $f_j(z_j) = 0$, $j = 1, \ldots, n$, $f(\infty) = 0$. The level curves of f_j $(j = 1, \ldots, n)$ and f constitute trajectories of $Q(z)\,dz^2$ in D_j $(j = 1, \ldots, n)$ and D respectively.

Take now $\tau > 0$ small enough and R sufficiently large to define simply connected regions $F_j(\tau) \subset D_j$ $(j = 1, \ldots, n)$ and $\Omega(R) \supset \mathbf{C} - D$ which are bounded by level curves of f_j $(j = 1, \ldots, n)$ and f, respectively, and have Lebesgue measure precisely equal to

$$|\Omega(R)| = \pi R^2, \quad |F_j(\tau)| = \pi \tau^2 r_j^2, \quad j = 1, \ldots, n. \tag{45}$$

Setting

$$F(\tau) = \bigcup_{j=1}^{n} \overline{F_j(\tau)}, \tag{46}$$

we obtain an n-connected domain $\Omega(R) - F(\tau)$ and the quadratic differential $Q(z)\,dz^2$ on it, positive along the boundary curves and of finite norm.

By analogy with $H(\tau, R)$ we introduce a function

$$G(\tau, R) = \iint_{\Omega(R) - F(\tau)} |Q(z)|\, d\mu(z) - \pi\left(\sum r_j^2\right) \log \frac{|\Omega(R)|}{|F(\tau)|}. \tag{47}$$

This, contrary to $H(\tau, R)$ is defined only for small τ and large R. But its limit is the same. To see this let us estimate the difference

$$|H(\tau,R)-G(\tau,R)| = \iint_{\Omega(R)/B(R)} |Q(z)|\, d\mu(z) + \sum_{j=1}^{n} \iint_{F_j(\tau)/B_j(\tau)} |Q(z)|\, d\mu(z). \tag{48}$$

Let Δ be the narrowest circular annulus, $a(R) < |z| < b(R)$, containing the boundary of $\Omega(R)$, $\gamma(R) = \partial\Omega(R)$. Geometric arguments show that $\Omega(R)/B(R) \subset \Delta$ because $|\Omega(R)| = |B(R)|$, by the assumptions. Now it is not difficult to derive the following estimate

$$\iint_{\Omega(R)/B(R)} |Q(z)|\, d\mu(z) \leq \iint_{\Delta} |Q(z)|\, d\mu(z) \leq \frac{\ell^2}{2\pi} \log \frac{b(R)}{a(R)} + o(1)$$

as R tends to infinity. Since $\gamma(R)$ is a level curve of the conformal map $f: D \to \mathbf{C}$, $f(\infty) = 0$, then the eccentricity of $\gamma(R)$ (i.e., the quotient $b(R)/a(R)$) approaches unity as $R \to \infty$. This implies

$$\lim_{R\to\infty} \iint_{\Omega(R)/B(R)} |Q(z)|\, d\mu(z) = 0.$$

The integrals over $F_j(\tau)/B_j(\tau)$ in (48) can be treated analogously:

$$\lim_{\tau\to 0} \iint_{F_j(\tau)/B_j(\tau)} |Q(z)|\, d\mu(z) = 0, \quad j = 1, \ldots, n.$$

We then conclude

$$G(\tau, R) = H(\tau, R) + o(1) \tag{49}$$

as $\tau \to 0$ and $R \to \infty$.

The alternative definition for the reduced norm then reads as follows:

$$\langle\langle Q \rangle\rangle = \lim_{\tau\to 0,\, R\to\infty} G(\tau, R). \tag{50}$$

This formula has an effective advantage in estimations of the reduced norm for the quadratic differentials with closed trajectories.

4. The circle trajectory structure. Employing the notation from Section 3 we assume here that the circle domains D_j $(j = 1, \ldots, n)$ and D cover the Riemann sphere $\hat{\mathbf{C}}$ up to a set of measure zero. We were concerned with such a case in Example 4 but with infinite number of poles and with $a_j = 0$, $j = 1, 2, \ldots$. Recall that for $n = 2$ the above assumption is not necessary since it follows from the three-pole theorem [5; 10] and from the global structure theorem for hyperelliptic trajectories (see Jenkins and Spencer [9]).

THEOREM 1. *If $Q(z)\, dz^2$ has circle trajectory structure, then*

$$\iint_{\mathbf{B}-E} |Q(z)|\, d\mu(z) \leq |E| \log \frac{\pi}{|E|} \tag{51}$$

with equality occurring only for $n = 1$, $a_1 = 0$, and $z_1 = 0$.

Proof. In view of Proposition 2 it suffices to show that $\langle\langle Q\rangle\rangle \le 0$. Because of the circle trajectory structure of $Q(z)\,dz^2$, the domain $\Omega(R)-F(\tau)$ is decomposed onto characteristic ring domains

$$\Delta_j(\tau)=D_j-F_j(\tau), \quad j=1,\dots,n$$

and

$$\Delta(R)=D\cap\Omega(R).$$

Therefore, by (21) and (41)-(42) we have

$$\begin{aligned}\iint_{\Omega(R)-F(\tau)} |Q(z)|\,d\mu(z) &= \sum_{j=1}^{n} \ell_j^2 \operatorname{mod}\Delta_j(\tau)+\ell^2 \operatorname{mod}\Delta(R)\\ &= 4\pi^2 \sum_{j=1}^{n} r_j^2 \operatorname{mod}\Delta_j(\tau)+4\pi^2(\textstyle\sum r_j^2)\operatorname{mod}\Delta(R)\end{aligned}$$

$$\tau^2\iint_{\Omega(R)-F(\tau)} |Q(z)|\,d\mu(z)=4\pi\sum_{j=1}^{n}|F_j|\operatorname{mod}\Delta_j+4\pi|F|\operatorname{mod}\Delta,$$

where, for notational convenience, we drop the variables τ and R. To estimate the moduli of the ring domains we apply Carleman's theorem; see (19):

$$\operatorname{mod}\Delta_j \le \frac{1}{4\pi}\log\frac{|D_j|}{|F_j|}, \quad j=1,2,\dots,n$$

and

$$\operatorname{mod}\Delta \le \frac{1}{4\pi}\log\frac{|\Omega|}{|\mathbf{C}-D|}.$$

Finally, by concavity inequality (13) we find at once that

$$\begin{aligned}\tau^2\iint_{\Omega(R)-F(\tau)} |Q(z)|\,d\mu(z) &\le \sum_{j=1}^{n}|F_j|\log\frac{|D_j|}{|F_j|}+|F|\log\frac{|\Omega|}{|\mathbf{C}-D|}\\ &\le (\textstyle\sum |F_j|)\log\frac{\sum|D_j|}{\sum|F_j|}+|F|\log\frac{|\Omega|}{|\mathbf{C}-D|}\\ &= |F|\log\frac{|\mathbf{C}-D|}{|F|}+|F|\log\frac{|\Omega|}{|\mathbf{C}-D|}\\ &= |F|\log\frac{|\Omega|}{|F|}=\pi(\tau^2\textstyle\sum r_j^2)\log\frac{|\Omega(R)|}{|F(\tau)|}.\end{aligned}$$

This, together with (47) and (50), implies that $\langle\langle Q\rangle\rangle\le 0$. The equality statement follows from the corresponding statement in Proposition 2. The proof is then complete. □

Now an important remark. Given the negative leading coefficients $-r_j^2$ ($j=1,\dots,n$) associated with the second-order poles z_j ($j=1,\dots,n$), there exists a unique quadratic differential $Q(z)\,dz^2$ of the form (31)-(32) whose trajectory structure consists exactly of circle domains. This can be readily derived from Theorem 23.5 in [15]; see also the remark in [8, p. 123]. We do not go into the details of this fact since a much stronger result will be presented later in Section 6.

In some cases the first-order coefficients $a_1, \ldots, a_n$ of $Q(z)$ vanish. Let us end this section with the following example.

Suppose that z_ν $(\nu = 1, \ldots, n)$ are the nth roots of unity

$$z_\nu = \exp \frac{2\nu\pi i}{n}, \quad \nu = 1, 2, \ldots, n.$$

Consider the rational function

$$R(z) = -\frac{1}{n} \sum_{\nu=1}^{n} (z - z_\nu)^{-2} = -\frac{z^{n-2}(z^n + n - 1)}{(z^n - 1)^2}.$$

$R(z)\,dz^2$ has n zeros of order one at

$$e_\nu = \sqrt[n]{n-1} \exp \frac{(2\nu+1)\pi i}{n}, \quad \nu = 1, \ldots, n$$

and one zero of order $n-2$ at the origin. These are the finite critical points of $R(z)\,dz^2$. It is not difficult to verify that the critical trajectories of $R(z)\,dz^2$ are the line segments connecting the origin with e_ν $(\nu = 1, \ldots, n)$ and Jordan arcs connecting e_ν with $e_{\nu+1}$ $(\nu = 1, \ldots, n)$, $e_{n+1} = e_1$. There are precisely $n+1$ circle characteristic domains relative to $R(z)\,dz^2$.

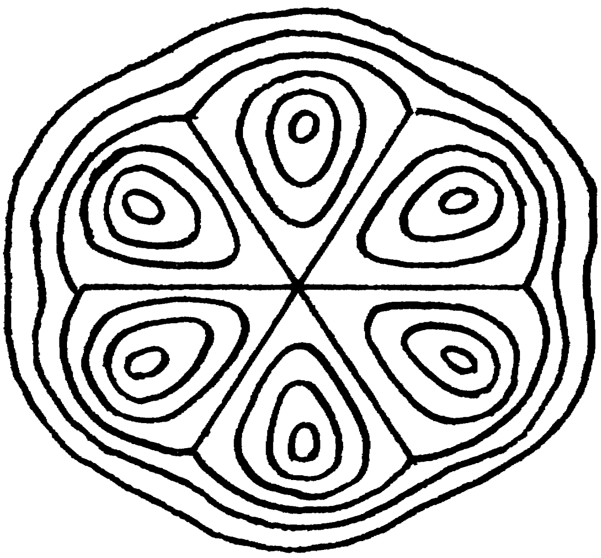

Figure 5

5. Certain extremal metrics and minimum area inequality. The general concept is as follows: take a measure m in $\mathbf{C}$, that is, a nonnegative σ-additive set function defined on Borel subsets of the complex plane. For every Jordan curve $\gamma \subset \mathbf{C}$ we denote by $m[\gamma]$ the m-area of its inner domain. Let Γ be a family of Jordan curves in $\mathbf{C}$. A nonnegative Borel measurable function $\rho(z)$ on $\mathbf{C}$ is said to define a Γ-admissible metric $\rho(z)\,|dz|$ if it satisfies the following "isoperimetric" inequality,

$$\left(\int_\gamma \rho(z)\,|dz|\right)^2 \geq 4\pi m[\gamma], \tag{52}$$

for all locally rectifiable curves $\gamma \in \Gamma$.

EXAMPLES. (a) If m is the Lebesgue measure in $\mathbf{C}$, $dm(z) = d\mu(z)$, then the metric $\rho(z)\,|dz| = |dz|$ is admissible for the family of all Jordan curves in $\mathbf{C}$.

(b) If m is a singular measure concentrated at one point $z_0 \in \mathbf{C}$, $m\{z_0\} = \pi r^2$, then the metric

$$\rho(z)\,|dz| = \frac{r\,|dz|}{|z-z_0|}$$

is admissible for the family of all Jordan curves in $\dot{\mathbf{C}} = \mathbf{C} - \{z_0\}$. If m is concentrated at two points z_1 and z_2, $m\{z_1\} = \pi r_1^2$ and $m\{z_2\} = \pi r_2^2$, then we can prove that

$$\rho(z)\,|dz| = \left(\left|\frac{r_1^2}{(z-z_1)^2} + \frac{r_2^2}{(z-z_2)^2}\right|\right)^{1/2} |dz|$$

is admissible for all Jordan curves in $\dot{\mathbf{C}} = \mathbf{C} - \{z_1, z_2\}$.

Another interesting example is:

(c) Let Ω be simply connected and let f belong to the Hardy space $H^p(\Omega)$, $0 < p < \infty$; then it is known that

$$\left(\int_{\partial D} |f(z)|^{p/2}\,|dz|\right)^2 \geq 4\pi \iint_D |f(z)|^p\,d\mu(z)$$

for every compact subdomain $D \subset \Omega$ bounded by a rectifiable Jordan curve. This shows that if $dm(z) = |f(z)|^p\,d\mu(z)$, then the metric

$$\rho(z)\,|dz| = |f(z)|^{p/2}\,|dz|$$

is admissible for all Jordan curves in Ω.

Now a Γ-admissible metric $\rho(z)\,|dz|$ is called Γ-extremal if it minimizes the following area integral:

$$A_\Gamma(m) = \inf \iint \rho^2(z)\,d\mu(z). \tag{53}$$

It is not difficult to see, by using a convexity argument, that there can be at most one Γ-extremal metric. Such a metric must be supported in $\overline{\bigcup_{\gamma \in \Gamma} \gamma}$. We will appeal to Jenkins' module theory [6] for the existence and properties of certain extremal metrics associated with multiple curve families. We use it here only insofar as it relates to our subject; see Lemma 1 below.

Let S be an n-connected domain bounded by analytic Jordan curves. Denote by E_j $(j = 1, \ldots, n)$ the bounded components of $\mathbf{C} - S$. Put $E = \bigcup_{j=1}^n E_j$; then $\Omega = S \cup E$ becomes a simply connected domain with analytic boundary.

To every component E_j $(j = 1, \ldots, n)$ we assign a positive number m_j which may be viewed as the m-area of E_j; that is,

$$m(E_j) = m_j, \quad j = 1, 2, \ldots, n.$$

Let $\Gamma = \{\Gamma_1, \ldots, \Gamma_N\}$ be a free family of homotopy classes $\Gamma_1, \ldots, \Gamma_N$ of Jordan curves in $S = \Omega - E$ satisfying the conditions (i)–(iii) from Section 1. Then we have the following.

LEMMA 1. *There exists a unique quadratic differential $Q(z)\,dz^2$ holomorphic in S and positive along the boundary curves such that:*

(i) *the Q-length of any curve $\gamma \in \bigcup_{\nu=1}^{N} \Gamma_\nu$ satisfies*

$$\ell^2(\gamma) \geq 4\pi m[\gamma]; \tag{54}$$

(ii) *$Q(z)\,dz^2$ has closed trajectories of homotopy type Γ. For every closed trajectory γ we have equality in* (54).

$Q(z)\,dz^2$ is said to solve the Γ-extremal metric problem on S, because the Γ-extremal metric is given by

$$\rho(z)\,|dz| = \sqrt{|Q(z)|}\,|dz|.$$

THEOREM 2. *Let $E \subset \Omega$, $S = \Omega - E$, and Γ be defined as above. Let $dm(z) = k^2\chi_E(z)\,d\mu(z)$, $k > 0$. Then the minimum area integral satisfies*

$$A_\Gamma(m) \leq k^2|E| \log \frac{|\Omega|}{|E|}. \tag{55}$$

The inequality is sharp. It reduces to Carleman's theorem if $S = \Omega - E$ is doubly connected.

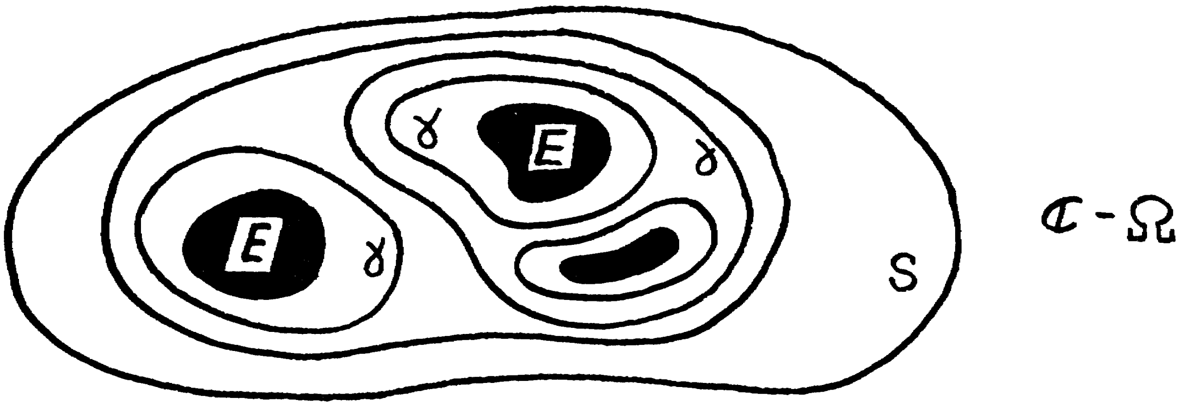

Figure 6

Proof. Let $Q(z)\,dz^2$ solve the Γ-extremal metric problem on S with associated measure $dm = k^2\chi_E\,d\mu$; that is, $m_j = k^2|E_j|$, $j = 1, \dots, n$. By the global structure theorem we have a decomposition

$$\bar{S} = \bigcup_{\nu=1}^{N} \bar{\Delta}_\nu,$$

where Δ_ν ($\nu = 1, \dots, N$) either degenerates to a curve in Γ_ν or it is a ring domain of homotopy type Γ_ν. For every curve $\gamma \in \bigcup_{\nu=1}^{N} \Gamma_\nu$ its Q-length satisfies

$$\ell^2 \geq 4\pi k^2 A(\gamma), \tag{56}$$

where $A(\gamma)$ stands for the total (Lebesgue) area of all components of E which lie in the inner domain of γ.

If Δ_ν does not degenerate, $\nu=1,\dots,N$, then the closed trajectories in Δ_ν have Q-length equal to

$$\ell_\nu=\sqrt{4\pi A_\nu}\,k. \tag{57}$$

Hence the Q-area of Δ_ν is

$$\iint_{\Delta_\nu}|Q(z)|\,d\mu(z)=4\pi k^2 A_\nu \operatorname{mod}(\Delta_\nu),$$

$\nu=1,\dots,N$, where A_ν denotes the total (Lebesgue) area of all components of E which lie in the inner complement of Δ_ν, $\nu=1,\dots,N$. Therefore the minimum area integral equals

$$A_\Gamma(k^2\chi_E\,d\mu)=\iint_{\Omega-E}|Q(z)|\,d\mu(z)=4\pi k^2\sum_{\nu=1}^{N}A_\nu \operatorname{mod}(\Delta_\nu). \tag{58}$$

To end the proof of this theorem we need the following auxiliary result.

LEMMA 2. *Let Ω be an arbitrary open set and E a Lebesgue measurable subset in Ω. Let $\mathfrak{F}=\{\Delta_1,\dots,\Delta_N\}$ be a system of disjoint ring domains $\Delta_\nu\subset\Omega-E$, $\nu=1,\dots,N$. To every ring Δ_ν assign a weight A_ν as*

$$A_\nu=|E\cap U_\nu|,\quad \nu=1,\dots,N,$$

where U_ν denotes the inner complement of Δ_ν. Then we have the following estimate of the weighted sum of moduli:

$$M_{\mathfrak{F}}(E,\Omega)=\sum_{\nu=1}^{N}A_\nu \operatorname{mod}(\Delta_\nu)\le\frac{1}{4\pi}|E|\log\frac{|\Omega|}{|E|} \tag{59}$$

with equality occurring if E and Ω are concentric disks and $\mathfrak{F}$ consists of a single ring $\Delta=\Omega-E$.

We should point out, however, that other cases when equality occurs in (59) are possible.

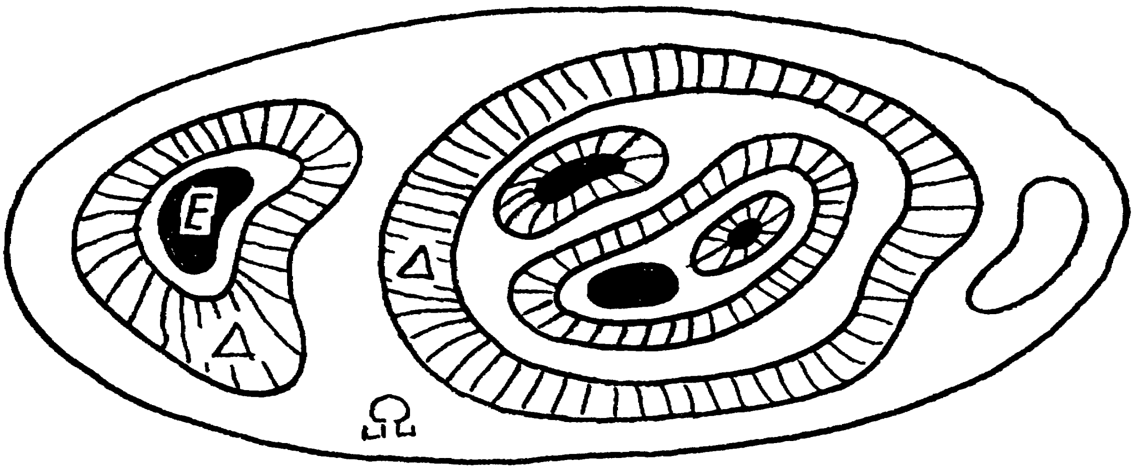

Figure 7

Proof. We perform an induction with respect to the number of rings in $\mathfrak{F}$. If $\mathfrak{F}$ is empty, then (59) holds by trivial means. If not, then there exists at least one maximal ring. A ring in $\mathfrak{F}$ is called maximal if it is not contained in the inner complement of any ring in $\mathfrak{F}$. Let Δ be such a maximal ring. Denote by U its inner complement and set $D=U\cup\Delta$. We then have the following decompositions:

$$\Omega=\Omega'\cup\Delta\cup\Omega'',\quad E=E'\cup E'',\quad \mathfrak{F}=\mathfrak{F}'\cup\{\Delta\}\cup\mathfrak{F}'',$$

where

$$\Omega'=U\cap\Omega,\quad \Omega''=\Omega-D,\quad E'=\Omega'\cap E,\quad E''=\Omega''\cap E,$$

and

$$\mathfrak{F}'=\{\Delta_\nu\cap\Omega';\ \nu=1,\dots,N\},\quad \mathfrak{F}''=\{\Delta_\nu\cap\Omega'';\ \nu=1,\dots,N\}.$$

From these definitions it follows at once that

$$M_{\mathfrak{F}}(E,\Omega)=M_{\mathfrak{F}'}(E',\Omega')+|E'|\operatorname{mod}(\Delta)+M_{\mathfrak{F}''}(E'',\Omega'').$$

Hence by the induction hypothesis and Carleman's theorem applied to the ring $\Delta=D-U$ we obtain

$$M_{\mathfrak{F}}(E,\Omega)\le\frac{|E'|}{4\pi}\log\frac{|\Omega'|}{|E'|}+\frac{|E'|}{4\pi}\log\frac{|D|}{|U|}+\frac{|E''|}{4\pi}\log\frac{|\Omega''|}{|E''|}. \tag{60}$$

It is evident that $D-\Omega=U-\Omega$ and $U\cap\Omega\subset D\cap\Omega$. Thus

$$\frac{|D|}{|U|}=\frac{|D-\Omega|+|D\cap\Omega|}{|U-\Omega|+|U\cap\Omega|}\le\frac{|D\cap\Omega|}{|U\cap\Omega|}=\frac{|D\cap\Omega|}{|\Omega'|}.$$

Insert this into (60) to get

$$M_{\mathfrak{F}}(E,\Omega)\le\frac{|E'|}{4\pi}\log\frac{|D\cap\Omega|}{|E'|}+\frac{|E''|}{4\pi}\log\frac{|\Omega-D|}{|E''|}.$$

Finally, by concavity inequality (13), we obtain

$$M_{\mathfrak{F}}(E,\Omega)\le\frac{|E'|+|E''|}{4\pi}\log\frac{|D\cap\Omega|+|\Omega-D|}{|E'|+|E''|}=\frac{|E|}{4\pi}\log\frac{|\Omega|}{|E|},$$

completing the proof of the lemma.

By this lemma and (58) we also end the proof of Theorem 2. □

6. The complete system of the quadratic differentials $Q_\Gamma(z)\,dz^2$.

THEOREM 3. *Take distinct points z_j ($j=1,\dots,n$, $n\ge2$) in the complex plane and positive numbers r_j assigned to every z_j ($j=1,\dots,n$). Let $\Gamma=\{\Gamma_1,\dots,\Gamma_N\}$ be an admissible family of homotopy classes Γ_ν ($\nu=1,\dots,N$) of Jordan curves in $\dot{\mathbf{C}}=\mathbf{C}-\{z_1,\dots,z_n\}$. There exists exactly one rational quadratic differential $Q_\Gamma(z)\,dz^2$ of the form*

$$Q_\Gamma(z)=\sum_{j=1}^{n}\frac{-r_j^2}{(z-z_j)^2}+\sum_{j=1}^{n}\frac{a_j}{z-z_j}, \tag{61}$$

where

$$\sum_{j=1}^{n} a_j = 0, \qquad \sum_{j=1}^{n} a_j z_j = 0 \tag{62}$$

and such that:

(i) $Q_\Gamma(z)\, dz^2$ *has closed trajectory structure of homotopy type* Γ;

(ii) *for each* $\gamma \in \bigcup_{\nu=1}^{N} \Gamma_\nu$ *its* Q_Γ*-length satisfies*

$$\ell(\gamma) \geq 2\pi \sqrt{r_{j_1}^2 + \cdots + r_{j_k}^2} \tag{63}$$

if γ *encloses precisely the poles* $z_{j_1}, \ldots, z_{j_k}$;

(iii) *inequality* (63) *converts into equality whenever* γ *is a closed trajectory of* $Q_\Gamma(z)\, dz^2$.

Furthermore, Q_Γ *has negative reduced norm*

$$\langle\langle Q_\Gamma \rangle\rangle < 0. \tag{64}$$

The case when Γ consists of precisely $n+1$ homotopy classes, generated by small circles around $z_1, \ldots, z_n$ and one large circle centered at the origin, can be treated by studying the surface of reduced moduli; see Strebel [15, Thm. 23.5] and [14]. This is the case that admits circle trajectory structure we discussed in Section 4. Presumably, the method of reduced moduli allows for our generalization. However, we did not find any explicit formulation of such a result. Our proof exploits essentially the area inequality (55) for the extremal metric and is based on an exhaustion method.

Proof. For notational convenience we assume that $|z_j| < 1$ $(j = 1, \ldots, n)$ and r_j are small enough that the disks $B_j = \overline{B(z_j, r_j)}$ $(j = 1, \ldots, n)$ are mutually disjoint and contained in the unit disk $\mathbf{B}$. Let m be the singular measure concentrated at $\{z_1, \ldots, z_n\}$ such that $m\{z_j\} = \pi r_j^2$, $j = 1, \ldots, n$.

We begin with defining an increasing exhaustion of $\dot{\mathbf{C}} = \mathbf{C} - \{z_1, \ldots, z_n\}$ by circular regions. For this fix $0 < \tau \leq 1 \leq R$ and remove the disks $B_j(\tau) = \overline{B(z_j, \tau r_j)}$ $(j = 1, \ldots, n)$ from the disk $B(R) = B(0, R)$. Put

$$E = E(\tau) = \bigcup_{j=1}^{n} B_j(\tau) \quad \text{and} \quad S = S(\tau, R) = B(R) - E(\tau).$$

Clearly, $S(\tau', R') \subset S(\tau, R)$ if $\tau \leq \tau'$ and $R' \leq R$ (see Figure 8);

$$\dot{\mathbf{C}} = \bigcup_{\tau < 1 < R} S(\tau, R).$$

Next we solve the Γ-extremal metric problem on $S = S(\tau, R)$, where we understand that Γ is restricted to those curves which lie in S. According to Lemma 1, we have a positive quadratic differential $Q_{\tau,R}(z)\, dz^2$ holomorphic on S with closed trajectories and such that the following condition holds: If $\gamma \in \bigcup_{\nu=1}^{N} \Gamma_\nu$ encloses exactly the points $z_{j_1}, \ldots, z_{j_k}$ then its $Q_{\tau,R}$-length satisfies

$$\ell(\gamma) = \int_\gamma |Q_{\tau,R}(z)|^{1/2}\, |dz| \geq 2\pi \sqrt{r_{j_1}^2 + \cdots + r_{j_k}^2}. \tag{65}$$

Equality holds here if γ happens to be a closed trajectory of $Q_{\tau,R}(z)\, dz^2$.

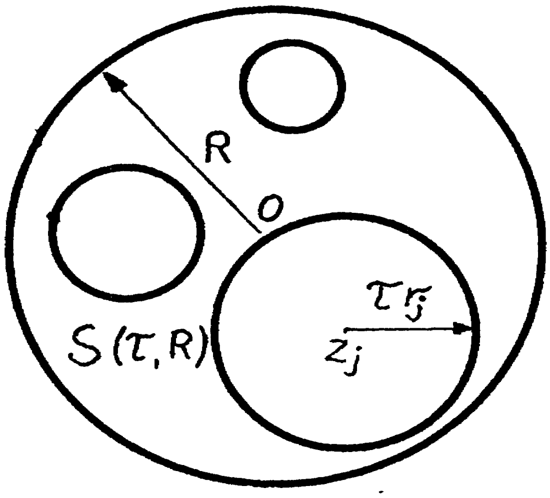

Figure 8

The m-area of any component $B_j(\tau) \subset \mathbf{C} - S$ $(j = 1, \ldots, n)$ is the same as its $\tau^{-2}\chi_{E(\tau)}\, d\mu$-area. Therefore, by Theorem 2,

$$\begin{aligned}\iint_{S(\tau,R)} |Q_{\tau,R}(z)|\, d\mu(z) &\le \tau^{-2}|E(\tau)| \log \frac{|B(R)|}{|E(\tau)|} \\ &= \pi(\textstyle\sum r_j^2) \log \dfrac{R^2}{\tau^2 \sum r_j^2}.\end{aligned} \tag{66}$$

We need, however, a slightly more general estimate. Namely,

$$\iint_{S(\tau',R')} |Q_{\tau,R}(z)|\, d\mu(z) \le \tau^{-2}|E(\tau)| \log \frac{|B(R')|}{|E(\tau')|}, \tag{67}$$

provided $\tau < \tau'$ and $R' < R$. To show this, consider the circular annuli

$$\Delta = \{z : R' < |z| < R\} \quad \text{and} \quad \Delta_j = \{z : \tau r_j < |z - z_j| < \tau' r_j\}, \quad j = 1, \ldots, n.$$

By Hölder's inequality,

$$\begin{aligned}\iint_{\Delta} |Q_{\tau,R}(z)|\, d\mu(z) &\ge \frac{1}{2\pi} \int_{R'}^{R} \left(\int_{|z|=t} \sqrt{|Q_{\tau,R}(z)|}\, |dz| \right)^2 \frac{dt}{t} \\ &\ge 2\pi(\textstyle\sum r_j^2) \displaystyle\int_{R'}^{R} \frac{dt}{t} = 2\pi(\textstyle\sum r_j^2) \log \dfrac{R}{R'}.\end{aligned}$$

Here we used the fact that the $Q_{\tau,R}$-length of every circle $\{z : |z| = t\}$, $R' < t < R$, is at least $2\pi\sqrt{\sum r_j^2}$; see (65).

Similar arguments apply to the annuli Δ_j $(j = 1, \ldots, n)$, yielding

$$\iint_{\Delta_j} |Q_{\tau,R}(z)|\, d\mu(z) \ge 2\pi r_j^2 \log \frac{\tau'}{\tau}, \quad j = 1, \ldots, n,$$

where the $Q_{\tau,R}$-length of every circle $\{z : |z - z_j| = t\}$, $\tau r_j < t < \tau' r_j$, is at least $2\pi r_j$, $j = 1, \ldots, n$.

Combining these estimates in view of (66) we obtain the inequality (67). This inequality shows in particular that for a fixed region $S(\tau', R')$, $\tau' \leq 1 \leq R'$, the family $\{Q_{\tau,R}\}_{\tau<\tau',R'<R}$ is uniformly bounded in $L^1(S(\tau', R'))$. One can therefore select sequences $\tau_i \to 0$ and $R_i \to \infty$ $(i = 1, 2, \ldots)$ so that the functions $Q_i(z) = Q_{\tau_i,R_i}(z)$ converge uniformly on every $S(\tau', R')$ to a function

$$Q_\Gamma(z) = \lim_{i\to\infty} Q_i(z), \tag{68}$$

which is holomorphic in $\dot{\mathbf{C}} = \mathbf{C} - \{z_1, \ldots, z_n\}$ and satisfies

$$\begin{aligned}\iint_{S(\tau,R)} |Q_\Gamma(z)|\, d\mu(z) &\leq \pi\left(\sum r_j^2\right) \log \frac{|B(R)|}{|E(\tau)|} \\ &= \pi\left(\sum r_j^2\right) \log \frac{R^2}{\tau^2 \sum r_j^2}\end{aligned} \tag{69}$$

for all $0 < \tau \leq 1 \leq R$.

We shall prove that $Q_\Gamma(z)\, dz^2$ defines the quadratic differential stated in Theorem 3. The length inequality (63) follows readily from (65). Also, the reduced norm of Q_Γ is negative by (69) and the definitions in (37)–(38). It is not evident however that $Q_\Gamma(z)$ takes the form (61)–(62).

Let this function have expansion at z_j

$$Q_\Gamma(z) = \sum_{\nu=3}^{\infty} \frac{c_\nu}{(z-z_j)^\nu} - \frac{\omega_j^2}{(z-z_j)^2} + \frac{a_j}{z-z_j} + \cdots$$

for $0 < |z - z_j| < r_j$, $j = 1, \ldots, n$. Direct computation gives the formulas for the coefficients c_ν:

$$c_\nu = \frac{(\nu-1)(\tau r_j)^{2\nu-2}}{\pi(1-\tau^{2\nu-2})} \iint_{\tau r_j<|z-z_j|<r_j} (\bar{z} - \bar{z}_j)^{-\nu} Q_\Gamma(z)\, d\mu(z)$$

for $0 < \tau < 1$ and $\nu = 3, 4, \ldots$. Hence by (69)

$$|c_\nu| \leq \frac{(\nu-1)(\tau r_j)^{\nu-2}}{\pi(1-\tau^{2\nu-2})} \iint_{\mathbf{B}-E(\tau)} |Q_\Gamma(z)|\, d\mu(z) = O\left(\tau^{\nu-2} \log \frac{1}{\tau}\right).$$

Letting τ tend to zero we conclude that $c_\nu = 0$ for $\nu = 3, 4, \ldots$. Therefore $Q_\Gamma(z)$ reduces to the form

$$Q_\Gamma(z) = \sum_{j=1}^{n} \frac{-\omega_j^2}{(z-z_j)^2} + \sum_{j=1}^{n} \frac{a_j}{z-z_j} + \text{regular term}.$$

In much the same way we prove that the expansion at infinity must start from the z^{-2} term

$$Q_\Gamma(z) = \frac{-\omega^2}{z^2} + \text{higher powers of } \frac{1}{z}.$$

These formulas imply

$$Q_\Gamma(z) = \sum_{j=1}^{n} \frac{-\omega_j^2}{(z-z_j)^2} + \sum_{j=1}^{n} \frac{a_j}{z-z_j}, \tag{70}$$

where

$$\sum_{j=1}^{n} a_j = 0 \quad \text{and} \quad \omega^2 = \sum_{j=1}^{n} (\omega_j^2 - a_j z_j). \tag{71}$$

We find from this that

$$\iint_{S(\tau,R)} |Q_\Gamma(z)|\, d\mu(z) \geq 2\pi|\omega|^2 \log R + 2\pi\left(\sum |\omega_j|^2\right) \log \frac{1}{\tau} + O(1)$$

as $\tau \to 0$ and $R \to \infty$. And by (69),

$$|\omega|^2 \leq \sum_{j=1}^{n} r_j^2 \quad \text{and} \quad \sum_{j=1}^{n} |\omega_j|^2 \leq \sum_{j=1}^{n} r_j^2.$$

On the other hand, because of the length inequality (63) which we have already established, we obtain

$$|\omega_j|^2 = \lim_{\epsilon \to 0} \frac{\epsilon}{2\pi} \int_{|z-z_j|=\epsilon} |Q_\Gamma(z)|\, |dz| \geq \lim_{\epsilon \to 0} \left(\frac{1}{2\pi} \int_{|z-z_j|=\epsilon} \sqrt{|Q_\Gamma(z)|}\, |dz| \right)^2 \geq r_j^2,$$

$j = 1, \ldots, n$, and

$$|\omega|^2 = \lim_{R \to \infty} \frac{R}{2\pi} \int_{|z|=R} |Q_\Gamma(z)|\, |dz| \geq \lim_{R \to \infty} \left(\frac{1}{2\pi} \int_{|z|=R} \sqrt{|Q_\Gamma(z)|}\, |dz| \right)^2 \geq r_1^2 + \cdots + r_n^2.$$

Those inequalities are possible only if

$$|\omega|^2 = r_1^2 + \cdots + r_n^2 \quad \text{and} \quad |\omega_j|^2 = r_j^2, \quad j = 1, \ldots, n.$$

It remains to prove that all ω_j $(j = 1, \ldots, n)$ and ω are real. The following interesting fact gives a clue to achieving this.

LEMMA 3. *Let S be a bordered Riemann surface and let $Q(z)\, dz^2$ be a positive holomorphic quadratic differential on S with closed trajectories. Let γ be an arbitrary closed analytic curve on S such that there is a continuous branch of $\sqrt{Q(z)}$ along γ. Then the integral $\int_\gamma \sqrt{Q(z)}\, dz$ is real.*

This fact follows from the global structure theorem and Cauchy's formula. The integral along γ can be reduced to the sum of integrals along critical trajectories and boundary curves. All of them are real by the definition (see Figure 9). It may be noted that the above property characterizes holomorphic and positive quadratic differentials on S which have closed trajectories.

The leading coefficients $-\omega_j^2$ $(j = 1, \ldots, n)$ and $-\omega^2$ do not vanish, so for sufficiently small ϵ and large R there are continuous branches of $\sqrt{Q_\Gamma(z)}$ along the circles $|z - z_j| = \epsilon$ $(j = 1, \ldots, n)$ and $|z| = R$, respectively. The same is true for all $Q_k(z)$, provided k is large enough. Cauchy's formula gives

$$\omega_j = \pm \frac{1}{2\pi} \int_{|z-z_j|=\epsilon} \sqrt{Q_\Gamma(z)}\, dz = \pm \frac{1}{2\pi} \lim_{k \to \infty} \int_{|z-z_j|=\epsilon} \sqrt{Q_k(z)}\, dz \in \mathbf{R},$$

for $j = 1, 2, \ldots, n$, and

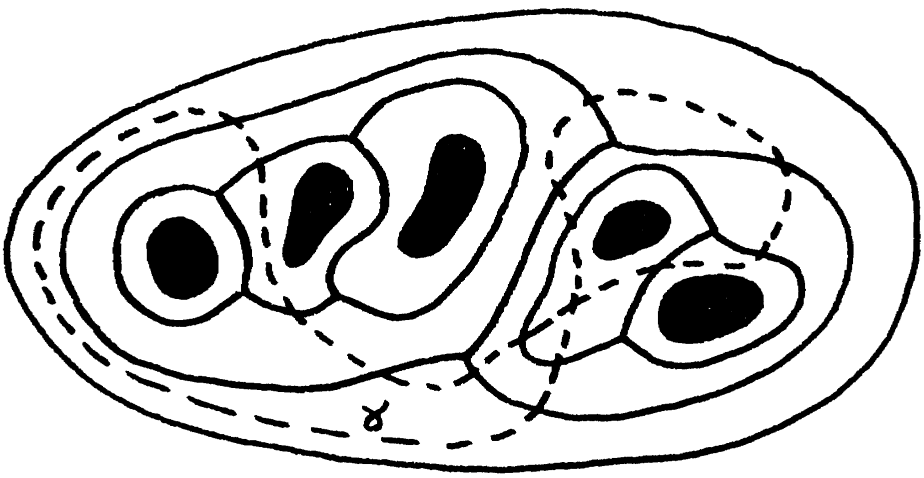

Figure 9

$$\omega = \pm\frac{1}{2\pi}\int_{|z|=R}\sqrt{Q_\Gamma(z)}\,dz = \pm\frac{1}{2\pi}\lim_{k\to\infty}\int_{|z|=R}\sqrt{Q_k(z)}\,dz \in \mathbf{R}.$$

We then conclude that $\omega_j^2 = r_j^2$ $(j=1,\dots,n)$ and $\omega^2 = r_1^2+\cdots+r_n^2$, so $\sum a_j z_j = 0$ by (71). This proves formulas (61)–(62).

Now by local structure theory $Q_\Gamma(z)\,dz^2$ has circle domains around double poles $z_1,\dots,z_n$ and ∞. Outside these circle domains the convergence of $Q_i(z)$ to $Q_\Gamma(z)$ is uniform. That $Q_\Gamma(z)\,dz^2$ has closed trajectory structure of homotopy type Γ follows by general compactness principles; see [15]. Also the length equality (iii) follows from that. To prove the uniqueness statement assume to the contrary that there are two quadratic differentials $Q'_\Gamma(z)\,dz^2$ and $Q''_\Gamma(z)\,dz^2$ with the properties described in Theorem 3. Employing the notation from Section 3, formulas (45)–(46), we have defined n-connected domains $S' = \Omega'(R) - F'(\tau)$ and $S'' = \Omega''(R) - F''(\tau)$. The quadratic differentials $Q'_\Gamma(z)\,dz^2$ and $Q''_\Gamma(z)\,dz^2$ are positive on S' and S'', so they define Γ-extremal metrics $\rho_{\prime}(z)\,|dz|$ and $\rho_{\prime\prime}(z)\,|dz|$ on S' and S'' respectively. The metric

$$\begin{aligned}\rho(z)\,|dz| &= \tfrac{1}{2}\left(\sqrt{|Q'_\Gamma(z)|} + \sqrt{|Q''_\Gamma(z)|}\right)|dz| \\ &= \tfrac{1}{2}[\rho_{\prime}(z) + \rho_{\prime\prime}(z)]\,|dz|\end{aligned}$$

is Γ-admissible for both S' and S''. Therefore

$$\iint_{S'} \rho_{\prime}^2(z)\,d\mu(z) \le \iint_{S'} \rho^2(z)\,d\mu(z) \le \iint_{S'\cup S''} \rho^2(z)\,d\mu(z),$$

$$\iint_{S''} \rho_{\prime\prime}^2(z)\,d\mu(z) \le \iint_{S''} \rho^2(z)\,d\mu(z) \le \iint_{S'\cup S''} \rho^2(z)\,d\mu(z).$$

The arguments similar to those we have used for formula (49) in Section 3 can be applied to show that

$$\iint_{S'\cup S''} |Q'_\Gamma(z)|\,d\mu(z) = \iint_{S'} |Q'_\Gamma(z)|\,d\mu(z) + o(1),$$

$$\iint_{S'\cup S''} |Q''_\Gamma(z)|\, d\mu(z) = \iint_{S''} |Q''_\Gamma(z)|\, d\mu(z) + o(1),$$

as $\tau \to 0$ and $R \to \infty$. Hence we obtain

$$\begin{aligned}\iint_{S'\cup S''} &(\rho_\prime^2 + \rho_{\prime\prime}^2)\, d\mu(z) \\ &\le 2 \iint_{S'\cup S''} \rho^2(z)\, d\mu(z) + o(1) \\ &= \iint_{S'\cup S''} (\rho_\prime^2 + \rho_{\prime\prime}^2)\, d\mu(z) - \frac{1}{2} \iint_{S'\cup S''} (\rho_\prime - \rho_{\prime\prime})^2\, d\mu(z) + o(1).\end{aligned}$$

Letting τ approach zero and R approach ∞ we conclude that

$$\iint_{\mathbf{C}} (\rho_\prime - \rho_{\prime\prime})^2\, d\mu(z) = 0,$$

$$|Q'_\Gamma(z)| = |Q''_\Gamma(z)|,$$

$$Q'_\Gamma(z) = Q''_\Gamma(z).$$

This completes the proof of Theorem 3. □

7. Conclusion and comments. Inequality (11) follows from Proposition 2 and (64).

The first-order coefficients a_j $(j=1,\dots,n)$ appearing in (61)–(62) form a complex vector $a(\Gamma) = (a_1,\dots,a_n)$ in an $(n-2)$-dimensional subspace of $\mathbf{C}^n$ which is orthogonal to the vectors $(1,1,\dots,1)$ and $(\bar{z}_1,\bar{z}_2,\dots,\bar{z}_n)$. Denote by $\mathfrak{J}$ the set of all admissible families $\Gamma = \{\Gamma_1,\dots,\Gamma_N\}$ of homotopy classes of Jordan curves in $\dot{\mathbf{C}} = \mathbf{C} - \{z_1,\dots,z_n\}$. The number of elements in $\mathfrak{J}$ exceeds $2(n-2)$, which shows that the vectors $a(\Gamma)$ are linearly dependent over real numbers as Γ vary $\mathfrak{J}$. It is of particular interest to know whether the convex hull of these vectors contains the zero vector in $\mathbf{C}^n$. Suppose for a moment that there exist positive numbers $t_1, t_2, \dots, t_m$, $t_1 + \cdots + t_m = 1$ and families $\Gamma^1, \Gamma^2, \dots, \Gamma^m$ in $\mathfrak{J}$, so that

$$\sum_{i=1}^{m} t_i a(\Gamma^i) = 0.$$

Then

$$T_E(z) = -\sum_{j=1}^{n} \frac{r_j^2}{(z-z_j)^2} = \sum_{i=1}^{m} t_i Q_{\Gamma^i}(z).$$

The reduced norm of $T_E(z)$ is therefore negative,

$$\langle\langle T_E \rangle\rangle = \left\langle\left\langle \sum_{i=1}^{m} t_i Q_{\Gamma^i} \right\rangle\right\rangle \le \sum_{i=1}^{m} t_i \langle\langle Q_{\Gamma^i} \rangle\rangle < 0,$$

and by Proposition 2,

$$\iint_{\mathbf{B}-E} |T_E(z)|\, d\mu(z) \le |E| \log \frac{\pi}{|E|}. \tag{72}$$

Let us end this paper with the following estimate for $T_E(z)$.

THEOREM 4. *Let E be the union of nonoverlapping disks in* $\mathbf{B}$ *and let* $T_E(z)$ *denote the Hilbert transform of the characteristic function of* E. *We then have*

$$\iint_{\mathbf{B}-E} |T_E(z)|\, d\mu(z) \leq |E| \log \frac{\pi}{|E|} + \inf \|\phi\|_{L^1(\mathbf{C})},$$

where the infimum is taken over $\phi \in \operatorname{conv}\{\phi_\Gamma\}_{\Gamma \in \mathfrak{Z}}$.

The proof is much the same as that given for inequality (72). □

REFERENCES

1. T. Carleman, *Uber ein Minimalproblem der mathematischen Physik,* Math. Z. 1 (1918), 208–212.
2. F. W. Gehring and E. Reich, *Area distortion under quasiconformal mappings,* Ann. Acad. Sci. Fenn. Ser. A I Math. No. 388 (1966), 1–14.
3. T. Iwaniec, *Extremal inequalities in Sobolev spaces and quasiconformal mappings,* Z. Anal. Anwendungen 1 (1982), 1–16.
4. ———, *The best constant in a* BMO-*inequality for the Beurling–Ahlfors transform,* Michigan Math. J. 33 (1986), 387–394.
5. J. A. Jenkins, *Positive quadratic differentials in triply-connected domains,* Ann. of Math. (2) 53 (1951), 1–3.
6. ———, *On the existence of certain general extremal metrics,* Ann. of Math. (2) 66 (1957), 440–453.
7. ———, *On the global structure of the trajectories of a positive quadratic differential,* Illinois J. Math. 4 (1960), 405–412.
8. ———, *Univalent functions and conformal mapping,* Springer, Berlin, 1958.
9. J. A. Jenkins and D. C. Spencer. *Hyperelliptic trajectories,* Ann. of Math. (2) 53 (1951), 4–35.
10. W. Kaplan, *On the three pole theorem,* Math. Nachr. 75 (1976), 299–309.
11. O. Lehto, *Remarks on the integrability of the derivatives of quasiconformal mappings,* Ann. Acad. Sci. Fenn. Ser. A I Math. No. 371 (1965), 1–8.
12. E. Reich, *Some estimates for the two-dimensional Hilbert transform,* J. Analyse Math. 18 (1967), 279–293.
13. ———, *Some estimates for the two-dimensional Hilbert transform* II, Unpublished notes, 1972.
14. K. Strebel, *On quadratic differentials with closed trajectories and second order poles,* J. Analyse Math. 19 (1967), 373–382.
15. ———, *Quadratic differentials: a survey.* On the work of Leonhard Euler (Berlin, 1983), 219–238, Birkhäuser, Basel-Boston, Mass., 1984.

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