SOME FUNCTION THEORETIC PROPERTIES OF THE GAUSS MAP FOR HYPERBOLIC COMPLETE MINIMAL SURFACES

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Introduction. Let M be a simply connected complete minimal surface (immersed) in $\mathbf{R}^3$. It is classical that M can be parameterized by pairs (f, g) where f is analytic, g is meromorphic, and the zeros of f occur precisely at the poles of g, the order of the zero being twice that of the pole. The Weierstrass representation (cf. [9, p. 63]) of M given in terms of f and g is the parameterization

$$\begin{aligned} x_1(z) &= \frac{1}{2}\operatorname{Re}\int^z f(1-g^2)\,dz, \\ x_2(z) &= \frac{1}{2}\operatorname{Re} i\int^z f(1+g^2)\,dz, \\ x_3(z) &= \operatorname{Re}\int^z fg\,dz. \end{aligned} \tag{1}$$

The metric and curvature are given by

$$\begin{aligned} \lambda(z)|dz| &= \frac{1}{2}|f|(1+|g|^2)|dz|, \\ K &= -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2. \end{aligned} \tag{2}$$

An important feature of g is that, after composition with stereographic projection, it represents the Gauss map of the surface. The universal covering surface of a hyperbolic minimal surface is a simply connected surface conformally equivalent to the unit disk $\mathbf{D}$, and can therefore be given as above, where the parameter space is the unit disk. In particular, if the surface itself is simply connected we can and do take f and g as defined in $\mathbf{D}$. The completeness condition then means that $\int_\alpha \lambda|dz| = \infty$ for every path α tending to $\partial\mathbf{D}$.

A fundamental problem in the theory of complete minimal surfaces is the determination of which meromorphic functions g arise as Gauss maps of these surfaces. It is known that if g is holomorphic it cannot be in the Nevanlinna class [6, pp. 394–5] and that g cannot omit seven points [12]. In the present note we shall give some further restrictions.

It is perhaps important to point out that in [12], as well as in Theorems 1 and 2 below, an essential ingredient in the proofs is a general result of Yau [13, p. 661] for complete Riemannian manifolds. It is interesting that, although in the present

Received May 13, 1986. Revision received September 8, 1986.
Research of the authors was supported by the National Science Foundation.
Michigan Math. J. 34 (1987).

setting the additional structure of minimality of the surfaces allows for a complete function-theoretic description via the Weierstrass representation, it seems difficult to frame proofs in strictly function-theoretic terms.

We wish to thank Joe Lipman and Brian Smyth for helpful discussions regarding Theorem 4.

I. Connections with normal functions. Let g be meromorphic in the open unit disk $\mathbf{D}$. The order of normality α of g [10, p. 1] is defined by

$$\alpha = \sup_{z\in\mathbf{D}}(1-|z|^2)\frac{|g'|}{1+|g|^2}.$$

If $\alpha<\infty$, then g is said to be a normal function. In geometric terms, a meromorphic function is normal of order α when it is a Lipschitzian map with Lipschitz constant $\alpha/2$ from the Poincaré disk into the Riemann sphere. Connections between complete minimal surfaces and normal functions were recently exploited in [12].

An analytic function in $\mathbf{D}$ is a Bloch function if it satisfies an estimate of the type $|g'(z)|\le C(1-|z|^2)^{-1}$. For further properties of normal and Bloch functions, see [2].

THEOREM 1. *Let M be a simply connected hyperbolic complete minimal surface in $\mathbf{R}^3$. Then the order of normality α of the Gauss map of M satisfies the inequality $\sqrt{2}/2\le\alpha\le\infty$. If in addition M has bounded curvature, then $1\le\alpha\le\infty$.*

It is interesting to note that the condition $\alpha\ge 1$ does not depend on the actual bound on the curvature. We have been unable to determine if these constants are sharp.

Let f and g correspond to a simply connected hyperbolic minimal surface. Now $|f|(1+|g|^2)|dz|$ is complete if and only if $|f|(1+\beta^2|g|^2)|dz|$ is complete for $\beta>0$. Since β can be taken arbitrarily small it follows from Theorem 1 that g cannot satisfy the Bloch condition. Thus we have the following.

COROLLARY 1. *The Gauss map of a simply connected hyperbolic complete minimal surface in $\mathbf{R}^3$ cannot be a Bloch function.*

Let R be the Riemann surface over the sphere of a meromorphic function g in the unit disk. For $g(z)\in R$, let Δ be the maximal schlicht disk on R centered at $g(z)$ having angular radius $\delta(z)$ measured from the origin. Then the plane projection of Δ is the set

$$\{w: |(w-g(z))/(1+\bar{g}(z)w)|<d(z)\},\quad d(z)=\tan\delta(z)/2.$$

The following criterion for normality is due to Pommerenke [10, p. 4].

THEOREM A. *Let $g(z)$ be meromorphic in the unit disk $\mathbf{D}$ and suppose that $\delta(z)\le\beta\le\pi/3$ $(z\in\mathbf{D})$. Then the order of normality α of g satisfies*

$$\alpha\le\frac{2\sin\beta}{(4\cos^2\beta-1)^{1/2}}<\infty. \tag{2.1}$$

For the Gauss map of a general hyperbolic minimal surface in $\mathbf{R}^3$ we may pass to the universal cover and, as a direct consequence of Theorem 1 and Theorem A, obtain the following geometric information.

COROLLARY 2. *Let g be the Gauss map of a complete hyperbolic minimal surface M in $\mathbf{R}^3$. If (2.1) is satisfied for $\beta=\beta_1<\pi/3$ corresponding to $\alpha=\sqrt{2}/2$, then the Riemann surface of g contains a schlicht disk of angular radius at least β_1. If in addition M has bounded curvature, the same conclusion holds for $\beta=\beta_2<\pi/3$ corresponding to $\alpha=1$. Here $\beta_1\approx .524$ and $\beta_2\approx .659$.*

II. The Nevanlinna characteristic. As pointed out in the introduction, it has been known for some time that if the Gauss map of a simply connected hyperbolic surface misses a point on the sphere, then it is of unbounded characteristic. Our next result shows that this effect persists in general.

THEOREM 2. *Let g be the Gauss map of a simply connected hyperbolic minimal surface M in $\mathbf{R}^3$. Then g has unbounded characteristic.*

From the standpoint of value distribution theory it is important to note that, although g cannot be of bounded characteristic, its Riemann surface need not be regularly exhaustible (cf. [5, p. 145]). In fact, if it were regularly exhaustible, then [5, Theorems 2.4 and 5.4] g could omit at most three values on the sphere. However, examples of complete minimal surfaces do exist whose Gauss maps are nonconstant and omit four values [9, p. 72].

III. Flatness of complete minimal surfaces. The celebrated Efimov theorem [4] states that any complete surface in $\mathbf{R}^3$ with negative curvature must satisfy $\sup K=0$. It is easy to exhibit examples showing that, in general, one cannot expect K to be close to zero on very large regions. Given a complete surface of nonpositive curvature in $\mathbf{R}^3$, the question arises as to what conditions ensure that K be close to zero on a rather large set. Even for minimal surfaces it seems necessary to impose some restrictions on the Gauss map. Consider, for example, a triply periodic minimal surface in $\mathbf{R}^3$. Its Gauss map descends to a meromorphic map on the (compact) quotient surface inside of a 3-torus. In particular, the Gauss map is surjective. On the other hand, for small enough $\epsilon>0$ the set of points in $\mathbf{R}^3$ where the curvature is bigger than $-\epsilon$ is the disjoint union of small neighborhoods around the planar points. In particular, it cannot contain arbitrarily large nearly flat balls.

THEOREM 3. *Let M be a complete minimal surface of bounded curvature in $\mathbf{R}^3$. If the Gauss map omits three points, then for any given $r,\epsilon>0$ there is a geodesic ball $B(r)$ of radius r in M where the curvature satisfies $K\geq-\epsilon$.*

IV. Omitted values of the Gauss map. Let M be a complete minimal surface in R^3. A natural question in the description of M is that of determining limitations on the number of values omitted by the Gauss map g.

As we pointed out in Section II, there are examples where g omits four values, and in [12] it is shown that g can never omit more than six. Whether or not there exists a complete minimal surface whose Gauss map omits six values, or for that matter even five, seems unknown. Even if one restricts attention to surfaces having bounded curvature, this question appears to be unresolved.

It is only in the case where the total curvature $\iint_M K\,dS$ is finite that we have further information. Such a surface can only be of the form $M = M' - \{p_1, \ldots, p_k\}$, where M' is a compact Riemann surface with $f\,dz$ a meromorphic differential on M' and g an n-sheeted meromorphic function [7, p. 356] on M'. Using these facts, Osserman [7, Theorems 3.3 and 3.3A] proved the following.

THEOREM B. *If M is a nonflat complete minimal surface of finite total curvature then g can omit at most three values. If g omits three values then the genus of M', as above, is at least* 1 *and*

$$\iint_M K\,dS \le -12\pi.$$

The catenoid provides an example of total curvature -4π whose Gauss map omits two values. It seems that there are no known examples of complete minimal surfaces of finite total curvature whose Gauss map omits three values. Although we feel it is quite possible that such surfaces exist, even in the case of genus 1, it follows from our next result that there are obstructions beyond those appearing in Theorem B.

THEOREM 4. *Let M be a nonflat complete minimal surface in $\mathbf{R}^3$ whose Gauss map omits three values. Then, the total curvature satisfies*

$$\iint_M K\,dS \le -16\pi. \tag{4.1}$$

V. Proof of Theorem 1. Let the metric λ for M be given by $2\lambda = |f|(1+|g|^2)$, and define $\mu(z) = 2/(1-|z|^2)$ and $v(z) = \mu^p/|f|^2(1+|g|^2)^2$. Then

$$\Delta \log v = p\mu^2 - \frac{8|g'|^2}{(1+|g|^2)^2}.$$

Hence $\Delta \log v \ge 0$, provided that

$$\sup_{z\in\mathbf{D}}(1-|z|^2)\frac{|g'|}{(1+|g|^2)} \le \frac{\sqrt{p}}{\sqrt{2}}. \tag{5.1}$$

Condition (5.1) means that g is normal of order $\alpha \le \sqrt{p}/\sqrt{2}$. On the other hand, since $dS = (1/4)|f|^2(1+|g|^2)^2\,dx\,dy$ we have

$$\int_M v\,dS = 2^{p-2}\int_{\mathbf{D}} \frac{dx\,dy}{(1-|z|^2)^p} < \infty \tag{5.2}$$

if $p < 1$. A theorem of Yau [13, p. 661] asserts that a complete Riemannian manifold of infinite volume supports no integrable function v satisfying $\Delta \log v \ge 0$.

It follows from this, (5.1), and (5.2) that the Gauss map of a complete minimal surface cannot be normal with order of normality $\alpha<\sqrt{2}/2$.

Suppose now that the curvature K of M is bounded: $0\geq K\geq -C$. An application of Yau's form of Schwarz's lemma shows that the metric of M dominates that of the disk with curvature $-C$ [14, p. 201]. Since $\mu/C^{1/2}$ has curvature $-C$ we have

$$\frac{|f|}{2}(1+|g|^2)\geq \frac{\mu}{C^{1/2}}. \tag{5.3}$$

Now let $v=\mu^p/|f|^{2+p}(1+|g|^2)^{2+p}$. As before,

$$\Delta \log v = p\mu^2-(2+p)\frac{4|g'|^2}{(1+|g|^2)^2}.$$

Hence, $\Delta\log v\geq 0$ if

$$\sup(1-|z|^2)\frac{|g'|}{1+|g|^2}\leq\left(\frac{p}{p+2}\right)^{1/2}.$$

Using (5.3), we see that $\int v\,dS<\infty$ for every $p>0$. Letting $p\to\infty$ and arguing as before, we see that the Gauss map of a complete minimal surface of bounded curvature cannot be normal of order $\alpha<1$. □

VI. Proof of Theorem 2. Let $2\lambda=|f|(1+|g|^2)$ and suppose g has bounded characteristic. Then [5, p. 176] $g=g_1/g_2$ where g_1, g_2 are analytic in $\mathbf{D}$, have no common zeros, and $|g_1(z)|<1$, $|g_2(z)|<1$ for $z\in\mathbf{D}$. Define $v(z)=g_2^4(z)/f^2(z)$. Since the zeros of f coincide with the poles of g^2 in position and multiplicity, it follows that $\Delta\log|v|=0$.

As in the proof of Theorem 1,

$$\int_M v\,dS=\frac{1}{4}\int_{\mathbf{D}} v|f|^2(1+|g|^2)^2\,dx\,dy=\frac{1}{4}\int_{\mathbf{D}}(|g_2|^2+|g_1|^2)^2\,dx\,dy<\infty. \tag{6.1}$$

Again, by [13, p. 661], we see that (6.1) is not possible if M is complete, and hence g must have unbounded characteristic. □

VII. Proof of Theorem 3. It suffices to prove the result for the universal cover of M. In fact, suppose the theorem were proved for the universal cover $\tilde{M}$ of M, and let $\tilde{B}(r)$ be a geodesic ball of radius r such that $K\geq-\epsilon$ there. If $\tilde{p}$ denotes the center of $\tilde{B}(r)$, and p its projection in M, then (since M is complete and cannot be compact) M contains a ball $B(r)$ centered at p. Now, $B(r)$ can be taken as the union of all curves starting at p and of length not exceeding r. Since these curves all lift to $\tilde{B}(r)$, and the curvature is preserved under projection, it follows that $B(r)$ satisfies the requirements of the theorem.

Since the Gauss map omits three points, M must be hyperbolic and the Gauss map is thus considered defined on the unit disk $\mathbf{D}$. Also, by a rotation, it can be arranged that the north pole is among the omitted points. It is no restriction to assume that $K\geq -1$. In particular, g is analytic and, as in the proof of Theorem 1, we again have that the metric $\lambda = (|f|/2)(1+|g|^2)$ dominates the hyperbolic metric μ in the disk. Since an analytic function that omits two points is normal [5, pp. 153, 156], it follows from a result of Bagemihl and Seidel [3, p. 16] that g has Fatou points, that is, points on $\partial\mathbf{D}$ where g has (finite or infinite) angular limit. Let $z_0 = e^{i\theta_0}$ be such a point.

We recall an estimate of Osserman [7, p. 340] for geodesic balls $B(p, d)$ where the Gauss map makes an angle at least $\alpha > 0$ with a fixed direction. For such a ball the curvature $K(p)$ satisfies

$$|K(p)| \le \frac{1}{d^2}\frac{32}{\sin^4\alpha}. \tag{7.1}$$

Given $r > 0$, choose $\rho > r$ such that

$$\frac{32}{(\rho - r)^2 \sin^4(\pi/4)} < \epsilon, \tag{7.2}$$

and consider the angular region T centered on z_0 that corresponds to taking the union of all hyperbolic balls of radius ρ centered on the segment $(0, z_0]$. As remarked above, as $z \to z_0$ on T, $\lim g(z)$ exists in $C \cup \{\infty\}$ so if $z \in T$ and $1-|z|$ is small enough, say $|z| > 1-\delta$, $g(z)$ makes an angle at least $\pi/4$ with a fixed direction in the sphere. Since the surface metric $\lambda = (|f|/2)(1+|g|^2)$ dominates the hyperbolic metric $\mu = 2/(1-|z|^2)$ (cf. [14, p. 201]) we must have $B_\lambda(z, s) \subset B_\mu(z, s)$ for $|z| < 1$ and $s > 0$. Here, $B_\lambda(z, s)$ and $B_\mu(z, s)$ are balls centered at z of radius s in the respective metrics λ and μ. Let $\eta < 1$, $1-\eta$ small enough so that $|z| > 1-\delta$ in $B_\mu(\eta, \rho)$ and $p \in B_\lambda(\eta, r)$. Then $B_\lambda(p, \rho - r) \subset B_\lambda(\eta, \rho) \subset B_\mu(\eta, \rho) \subset T \setminus \{z : |z| \le 1-\delta\}$. Applying (7.1) with $d = \rho - r$ and $\alpha = \pi/4$ to any $p \in B_\lambda(\eta, r)$ we then have

$$|K(p)| \le \frac{32}{(\rho - r)^2 \sin^4(\pi/4)} < \epsilon$$

in view of (7.2). □

VIII. Proof of Theorem 4. Let $M = M' - \{p_1, \dots, p_k\}$, where M' has genus γ, and suppose that the total curvature of M is -12π and that g omits three values. Then [7, p. 358] g is a 3-sheeted cover of the sphere and g tends to the exceptional values at the punctures so that $k \ge 3$. On the other hand, equation (11) of [7, p. 360] implies $k \le 3$ so that $k = 3$. Also, from Theorem B we have that $\gamma \ge 1$ and from equation (10) of [7, p. 360] that $\gamma \le 1$. Thus $\gamma = 1$.

We are therefore led to consider elliptic functions which attain three values a_1, a_2, a_3 (the exceptional values) at points z_1, z_2, z_3 (the punctures) with multiplicity three on the period parallelogram. Let T be a period parallelogram associated with M' and assume without loss of generality that $a_1 = \infty$ and $z_1 = 0$.

In general, $f\,dz$ is a differential, but since $\gamma = 1$ it follows that f is a well-defined function on T. Furthermore, since f can only vanish at $z_1 = 0$, and by completeness and equation (2) its multiplicity is at most four, this implies that f is at most a 4-sheeted covering of M'. On the other hand, again by completeness and (2), f must have poles at z_2 and z_3 of order at least two at each point, so it follows that

f is a 4-sheeted cover. Finally, since the poles and zeros of f coincide in position and multiplicity with those of $1/g'$ we have that, for some constant $k \neq 0$,

$$f = k/g'. \tag{8.1}$$

We shall now proceed to express g in terms of the Weierstrass p function for T which is a 2-sheeted cover of the sphere. In fact, since g is 3-sheeted with a pole of order three at 0, there exist constants a, b such that $a \neq 0$ and $g - ap' - bp$ has at most a pole of order one at 0. Since, however, none of the functions g, p', p have poles at other points, this implies that $g - ap' - bp$ is constant. Thus, for appropriate constants a, b, c, we may write $g(z) = ap'(z) + bp(z) + c$ for all $z \in T$. We shall now reduce this expression and show that in fact we may write

$$g(z) = ap'(z) + c \quad (a \neq 0,\ z \in T), \tag{8.2}$$

and that p can be taken as equianharmonic [1, p. 652] satisfying

$$p'^2(z) = 4p^3(z) - 1 \quad (z \in T). \tag{8.3}$$

To see this, we begin with the basic identities [1, p. 640] in T:

$$\begin{aligned} p'^2(z) &= 4p^3(z) - g_2 p(z) - g_3, \\ p''(z) &= 6p^2(z) - g_2/2, \\ p'''(z) &= 12p(z)p'(z), \end{aligned} \tag{8.4}$$

where g_2 and g_3 are constants. Now for $j = 2, 3$ we have

$$\begin{aligned} 0 &= g'(z_j) = ap''(z_j) + bp'(z_j), \\ 0 &= g''(z_j) = ap'''(z_j) + bp''(z_j), \end{aligned} \tag{8.5}$$

which imply

$$b^2 p'(z_j) - a^2 p'''(z_j) = 0 \quad j = 2, 3. \tag{8.6}$$

Now $p'(z_j) \neq 0$. Indeed, if this were not the case then (since $a \neq 0$) it would follow from the first equation in (8.5) that $p(z_j)$ would be covered three times. Thus, from (8.6) and the third equation in (8.4) we have

$$p(z_j) = b^2/12a^2 \quad j = 2, 3, \tag{8.7}$$

and hence with (8.4) and (8.7), (8.5) becomes

$$\begin{aligned} a(6p^2(z_j) - g_2/2) + bp'(z_j) &= 0 \\ 12ap(z_j)p'(z_j) + b(6p^2(z_j) - g_2/2) &= 0 \quad j = 2, 3. \end{aligned} \tag{8.8}$$

We can now determine that $b = 0$. In fact, if $b \neq 0$ then the first equation in (8.8) along with (8.7) would give

$$p'(z_j) = -\frac{a}{b}\left(\frac{b^4}{24a^4} - \frac{g_2}{2}\right) \quad j = 2, 3,$$

which with (8.7) would imply $g(z_1) = g(z_2)$, a contradiction. Thus $b = 0$ and, when substituted into (8.8), this gives $p^2(z_j) = g_2/2$ and $p(z_j)p'(z_j) = 0$. Since $p'(z_j) \neq 0$ we finally obtain $g_2 = 0$. By scaling T we may then arrange to have $g_3 = 1$.

Using relationships (8.1), (8.2), and (8.3) we now rewrite the third parameterization formula in (1):

$$x_3 = \operatorname{Re} \int^z \frac{k(ap'(z)+c)}{6ap^2(z)}\, dz. \tag{8.9}$$

In order for (8.9) to be well defined, the real part of integrals across the period parallelogram must vanish. Clearly the integral of p'/p^2 contributes nothing, since this is the derivative of $-1/p$ which is well defined on the surface. Thus, the problem reduces to the existence of constants a, c, and k such that if γ_1 and γ_2 are curves joining the pairs of opposite sides in T then

$$0 = \operatorname{Re} \frac{ck}{a} \int_{\gamma_j} \frac{dz}{p^2(z)} \quad j=1,2. \tag{8.10}$$

Let $\eta_j = \int_{\gamma_j} p(z)\, dz$ and $\tau_j = \int_{\gamma_j} dz/p^2$ $(j=1,2)$. Then [1, p. 653] $\eta_1 = \alpha e^{i\pi/3}$, $\eta_2 = \alpha e^{-i\pi/3}$ for some constant α, and [11, p. 109] $\tau_1 = 2\eta_1$, $\tau_2 = 2\eta_2$. These values are obviously incompatible with (8.10) unless $c=0$.

Now the conditions that the real parts of the integrals on γ_j in the first and second parameterization formulae (1) can be written in the equivalent form

$$\overline{\int_{\gamma_j} f(z)\, dz} = \int_{\gamma_j} f(z) g^2(z)\, dz \quad j=1,2. \tag{8.11}$$

Again using (8.1), (8.2), and (8.3) in (8.11) along with the fact that $c=0$, we have for $j=1,2$

$$\begin{aligned}\overline{\frac{k}{6a}\int_{\gamma_j} \frac{dz}{p^2(z)}} &= \frac{k}{6a}\int_{\gamma_j} \frac{a^2 p'(z)}{p^2(z)}\, dz = \frac{ak}{6}\int_{\gamma_j} \frac{4p^3(z)-1}{p^2(z)}\, dz \\ &= \frac{2ak}{3}\int_{\gamma_j} p(z)\, dz - \frac{ak}{6}\int_{\gamma_j} \frac{dz}{p^2(z)}.\end{aligned}$$

With η_j and τ_j as before this becomes

$$\overline{k\tau_j/a} + ak\tau_j = 4ak\eta_j = 2ak\tau_j \quad j=1,2,$$

or

$$1/|a|^2 - k\tau_j/\overline{k\tau_j} = 0 \quad j=1,2,$$

which is evidently incompatible with the values of τ_j given above for any choice of a and k.

Since we have now ruled out the possibility of g being 3-sheeted and since the total curvature is -4π times the number of sheets of g [9, p. 77], we have established (4.1). □

REFERENCES

1. M. Abromowitz and A. Stegun, *Handbook of mathematical functions,* Dover, New York, 1965.
2. J. M. Anderson, J. Clunie, and Ch. Pommerenke, *On Bloch functions and normal functions,* J. Reine Angew. Math. 270 (1974), 12–37.

3. F. Bagemihl and W. Seidel, *Koebe arcs and Fatou points of normal functions,* Comment. Math. Helv. 36 (1961), 9–18.
4. N. V. Efimov, *Appearance of singularities on surfaces of negative curvature,* Mat. Sb. (N.S.) 64(106) (1964), 286–320. Am. Math. Soc. Transl., Series 2, 66 (1968), 154–190.
5. W. Hayman, *Meromorphic functions,* Clarendon Press, Oxford, 1964.
6. R. Osserman, *On complete minimal surfaces,* Arch. Rational Mech. Anal. 13 (1963), 392–404.
7. ———, *Global properties of minimal surfaces in E^3 and E^n*, Ann. of Math. (2) 80 (1964), 340–364.
8. ———, *Global properties of classical minimal surfaces.* Duke Math. J. 32 (1965), 565–573.
9. ———, *A survey of minimal surfaces*, Van Nostrand, New York, 1969.
10. Ch. Pommerenke, *Estimates for normal meromorphic functions,* Ann. Acad. Sci. Fenn. Ser. A I No. 476 (1970), 1–10.
11. J. Tannery and J. Molk, *Éleménts de la théorie des fonctions elliptiques,* Vol. 4, Gauthier-Villars, Paris, 1902.
12. F. Xavier, *The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere,* Ann. of Math. (2) 113 (1981), 211–214. *Erratum,* Ann. of Math. (2) 115 (1982), 667.
13. S.-T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry,* Indiana Univ. Math. J. 25 (1976), 659–670.
14. ———, *A general Schwarz lemma for Kähler manifolds,* Amer. J. Math. 100 (1978), 197–203.

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