TRANSVERSAL JACOBI FIELDS FOR HARMONIC FOLIATIONS

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1. Introduction. A foliation $\mathfrak{F}$ on a manifold M is given by the exact sequence of vectorbundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0,$$

where L is the tangent bundle and Q the normal bundle of $\mathfrak{F}$. If $V(\mathfrak{F})$ denotes the Lie algebra of infinitesimal automorphisms of $\mathfrak{F}$, we have an exact sequence of Lie algebras

$$0 \to \Gamma L \to V(\mathfrak{F}) \xrightarrow{\pi} \Gamma Q^L \to 0,$$

where ΓQ^L denotes the invariant sections of Q under the action of ΓL by Lie derivatives [4; 9]. We assume throughout that $\mathfrak{F}$ is Riemannian, with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q \cong L^\perp$ [10]. ∇ denotes the unique metric and torsion-free connection in Q (see, e.g., [3; 9; 10]). Associated to ∇ are transversal curvature data, in particular the (transversal) Ricci operator $\rho_\nabla: Q \to Q$ and the Jacobi operator $J_\nabla = \Delta - \rho_\nabla: \Gamma Q \to \Gamma Q$ [4]. In this paper we study geometric properties of infinitesimal automorphisms $Y \in V(\mathfrak{F})$ such that $\bar{Y} = \pi(Y) \in \Gamma Q^L$ satisfies the Jacobi condition $J_\nabla \bar{Y} = 0$. In view of the variational meaning of J_∇ [4], it is then natural to assume $\mathfrak{F}$ to be harmonic; that is, all leaves of $\mathfrak{F}$ are minimal submanifolds [3].

THEOREM A. *Let $\mathfrak{F}$ be a transversally orientable harmonic Riemannian foliation on a compact orientable Riemannian manifold (M, g_M), and Y an infinitesimal automorphism of $\mathfrak{F}$. Then the following properties are equivalent:*

(i) *$\bar{Y}$ is a transversal Killing field, that is, $\theta(Y)g_Q = 0$;*

(ii) *$\bar{Y}$ is a transversally divergence-free Jacobi field;*

(iii) *$\bar{Y}$ is transversally affine, that is, $\theta(Y)\nabla = 0$.*

REMARKS. (1) If $\mathfrak{F}$ is given by the fibers of a harmonic submersion $f: M \to N$, the equivalence of (i) and (ii) specializes to the statement that a projectable vector field $v = V \circ f$ ($V \in \Gamma TN$) along f is a divergence-free Jacobi field along f if and only if V is a Killing vector field on N (see [12] for the particular case $N = S^n$). Note, however, that the Jacobi condition for v in the harmonic map theory uses the pull-back of the Riemannian connection of N which, in general, differs from the canonical connection ∇ in Q [3].

(2) For the foliation of M by points, the equivalence of (i) and (ii) is the classical characterization of Killing vector fields given by Lichnerowicz [7] and Yano [14]; the implication (iii)$\Rightarrow$(i) is due to Yano [14].

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THEOREM B. *Let $\mathfrak{F}$ and Y be as in Theorem A with* $\operatorname{codim}\mathfrak{F}=2$. *Then the following properties are equivalent:*

(i) *$\bar{Y}$ is a transversal conformal field, that is, $\theta(Y)g_Q=\sigma\cdot g_Q$;*
(ii) *$\bar{Y}$ is a transversal Jacobi field.*

REMARK. For the point foliation this result goes back to Lichnerowicz [7]. A sharpening for Jacobi fields along conformal diffeomorphisms was given by Smith [11] (see also [12]).

The key to obtaining Theorems A and B is the transversal divergence theorem given in Section 2 (Theorem C). In Section 3, we generalize the operators δ, δ^* occurring in the Berger–Ebin decomposition [1] to the foliation context. They play a crucial role in deriving a basic identity relating the trace Laplacian and curvature (Theorem D, (i)). The proofs of Theorem A and B are given in Section 4. Finally, in Section 5, we give a few examples.

The terminology for foliations is based on [3–6; 9; 10]. For the related concepts and results in harmonic map theory we refer to Eells–Lemaire [2].

2. Transversal divergence theorem. Let $\mathfrak{F}$ be as in Theorem A and let $\Omega_B^{\cdot}(\mathfrak{F})\subset\Omega(M)$ be the subcomplex of basic forms (forms killed by $i(X)$, $\theta(X)$ for $X\in\Gamma L$, cf. [5; 6; 9; 10]). The transversal orientation and g_Q give rise to a transversal volume form $\nu\in\Omega_B^q(\mathfrak{F})$, $q=\operatorname{codim}\mathfrak{F}$. Clearly $d\nu=0$. The characteristic form of $\mathfrak{F}$ (a volume form along the leaves) is given by $\chi_{\mathfrak{F}}=*\,\nu\in\Omega^p(M)$, $p+q=n=\dim M$, with respect to the Hodge star operator of g_M on $\Omega(M)$. Then $\mu=\nu\wedge\chi_{\mathfrak{F}}$ is the Riemannian volume form of (M,g_M). Given an infinitesimal automorphism Y of $\mathfrak{F}$, the transversal divergence $\operatorname{div}_B\bar{Y}$ is defined as the unique basic scalar satisfying

$$\theta(Y)\nu=\operatorname{div}_B\bar{Y}\cdot\nu.$$

It depends only on $\bar{Y}=\pi(Y)$.

THEOREM C. *Let $\mathfrak{F}$ and Y be as in Theorem A. Then*

$$\int_M \operatorname{div}_B\bar{Y}\cdot\mu=0.$$

Proof. We have

$$\begin{aligned}\operatorname{div}_B\bar{Y}\cdot\mu&=(\operatorname{div}_B\bar{Y}\cdot\nu)\wedge\chi_{\mathfrak{F}}=\theta(Y)\nu\wedge\chi_{\mathfrak{F}}=(di(Y)\nu)\wedge\chi_{\mathfrak{F}}\\&=d(i(Y)\nu\wedge\chi_{\mathfrak{F}})+(-1)^q i(Y)\nu\wedge d\chi_{\mathfrak{F}}.\end{aligned}$$

By Stokes' theorem it suffices to show that the second term is in fact zero. To prove this, we consider the canonical multiplicative filtration [5; 6]

$$F^r\Omega^m=\{\omega\in\Omega^m\mid i(X_1)\cdots i(X_{m-r+1})\omega=0,\ X_j\in\Gamma L,\ j=1,\dots,m-r+1\},$$

which breaks off above q. We have $\nu\in F^q$ and $i(Y)\nu\in F^{q-1}$. The harmonicity of $\mathfrak{F}$ is expressed by the $\mathfrak{F}$-triviality of $d\chi_{\mathfrak{F}}$ or equivalently by $d\chi_{\mathfrak{F}}\in F^2$ [5; 6]. Thus $i(Y)\nu\wedge d\chi_{\mathfrak{F}}$ has filter degree $(q-1)+2=q+1$ and hence vanishes. □

3. Operators δ, δ^* and fundamental identities. To introduce various differential operators below it is convenient to use the following special (orthonormal) moving frames on M. For $x \in M$, let $\{e_A\}_{A=1}^n \subset T_x M$ be an (oriented) orthonormal basis with $\{e_i\}_{i=1}^p \subset L_x$ and $\{e_\alpha\}_{\alpha=p+1}^n \subset Q_x \cong L_x^\perp$. Let U be a distinguished (flat) neighborhood of x for $\mathfrak{F}$ with local (Riemannian) submersion $f: U \to B$. For $\alpha = p+1, \ldots, n$, let $E_\alpha \in \Gamma(U, Q)$ be the pull-back of the extension of $f_* e_\alpha$ to a vector field on B by parallel transport along geodesic segments emanating from $f(x)$ (use [10, Prop. 4.2]). Then we complete $\{E_\alpha\}_{\alpha=p+1}^n$ by the Gram–Schmidt process to a moving frame $\{E_A\}_{A=1}^n$ by adding $E_i \in \Gamma(U, L)$ with $(E_i)_x = e_i$, $i = 1, \ldots, p$. We have then for $\alpha, \beta = p+1, \ldots, n$:

$$\nabla_{e_\alpha} E_\beta = (\nabla_{E_\alpha} E_\beta)_x = 0;$$

and, as a consequence of torsion-freeness [3, 1.5], $[E_\alpha, E_\beta]_x \in L_x$. Furthermore, as the E_α are infinitesimal automorphisms, we have

$$\nabla_X E_\alpha = \pi[X, E_\alpha] = 0, \quad X \in \Gamma(U, L).$$

Generalizing to the foliation context the operators occurring in the Berger–Ebin decomposition [1], we define $\delta: \Gamma S^2 Q^* \to \Gamma Q^*$, $S^2 =$ symmetric square, by the local formula

$$\delta h = -\sum_{\alpha=p+1}^{n} (\nabla_{E_\alpha} h)(E_\alpha, \cdot), \quad h \in \Gamma S^2 Q^*$$

and $\delta^*: \Gamma Q^* \to \Gamma S^2 Q^*$ by

$$(\delta^* \omega)(V, W) = \tfrac{1}{2}\{(\nabla_V \omega)(W) + (\nabla_W \omega)(V)\}, \quad \omega \in \Gamma Q^*, \ V, W \in \Gamma Q.$$

Note that $\Omega_B^1(\mathfrak{F}) \subset \Gamma Q^*$. Similarly, the basic symmetric 2-forms (killed by $i(X), \theta(X)$ for $X \in \Gamma L$) will be identified with a subspace of $\Gamma S^2 Q^*$.

PROPOSITION 1. *δ and δ^* map basic forms to basic forms.*

Proof. For $X \in \Gamma L$, a direct calculation yields the commutation relations

$$([\theta(X), \delta]h)(e_\alpha) = -\sum_{\beta=p+1}^{n} [X, E_\beta]_x \{h(E_\alpha, E_\beta)\}, \quad h \in \Gamma S^2 Q^*,$$

$$(2[\theta(X), \delta^*]\omega)(E_\alpha, E_\beta) = [X, E_\alpha]\{\omega(E_\beta)\} + [X, E_\beta]\{\omega(E_\alpha)\}, \quad \omega \in \Gamma Q^*.$$

If h is basic, then $\theta(X)h = 0$ and the right-hand side of the first formula vanishes, as $h(E_\alpha, E_\beta) \in \Omega_B^0(\mathfrak{F})$ and $[X, E_\beta]_x \in L_x$. Thus $\theta(X)\delta h = 0$ and δh is basic. The argument for δ^* is similar. □

PROPOSITION 2. *For basic $h \in \Gamma S^2 Q^*$ and $\omega \in \Omega_B^1(\mathfrak{F})$ we have $\langle \delta h, \omega \rangle = \langle h, \delta^* \omega \rangle$ with respect to the global scalar product on basic forms.*

Proof. Indeed, by local computation, we find for the pointwise scalar product that

$$(\delta h, \omega)_x = -(\operatorname{div}_B Z)_x + (h, \delta^* \omega)_x,$$

where $Z \in \Gamma Q^L$ is the g_Q-dual of the basic 1-form λ given locally by

$$\lambda = \sum_{\beta=p+1}^{n} h(\cdot, E_\beta)\,\omega(E_\beta).$$

The proposition follows now by integration, applying Theorem C. □

Let d_B be the restriction of d to basic forms. The adjoint is denoted by $\delta_B : \Omega_B^r(\mathfrak{F}) \to \Omega_B^{r-1}(\mathfrak{F})$. For a harmonic Riemannian $\mathfrak{F}$ it follows from [5; 6] that δ_B on $\omega \in \Omega_B^1(\mathfrak{F})$ is given by the formula

$$\delta_B\omega = -\sum_\alpha (\nabla_{E_\alpha}\omega)(E_\alpha).$$

The range of a summation over a greek index is here, and everywhere below, to extend from $p+1$ to n.

THEOREM D. *Let $\mathfrak{F}$ and Y be as in Theorem A, and $\omega \in \Omega_B^1(\mathfrak{F})$ the g_Q-dual of $\bar{Y} = \pi(Y)$. Then we have the following identities:*

(1) $2\delta\delta^*\omega = -\operatorname{trace}\nabla^2\omega - \rho_\nabla(\omega) + d_B\delta_B\omega$;

(2) $\operatorname{div}_B \bar{Y} = -\delta_B\omega = (\delta^*\omega, g_Q)$;

(3) $|\delta^*\omega - 1/q\cdot\operatorname{div}_B\bar{Y}\cdot g_Q|^2 = |\delta^*\omega|^2 - (1/q)(\operatorname{div}_B\bar{Y})^2$ (*pointwise norms*).

Proof. At $x \in M$, we have

$$\begin{aligned}
2(\delta\delta^*\omega)(e_\beta) &= -2\sum_\alpha (\nabla_{e_\alpha}(\delta^*\omega))(e_\alpha, e_\beta) = -2\sum_\alpha e_\alpha\{(\delta^*\omega)(E_\alpha, E_\beta)\}\\
&= -\sum_\alpha e_\alpha\{(\nabla_{E_\alpha}\omega)(E_\beta) + (\nabla_{E_\beta}\omega)(E_\alpha)\}\\
&= -(\operatorname{trace}\nabla^2\omega)(e_\beta) - \sum_\alpha e_\alpha E_\beta\{\omega(E_\alpha)\} + \sum_\alpha e_\alpha\{\omega(\nabla_{E_\beta}E_\alpha)\}\\
&= -(\operatorname{trace}\nabla^2\omega)(e_\beta) - \sum_\alpha e_\beta E_\alpha\{\omega(E_\alpha)\} - \sum_\alpha \omega(\nabla_{e_\alpha}\nabla_{E_\beta}E_\alpha),
\end{aligned}$$

where in the last equality we used the fact that $[E_\alpha, E_\beta]_x\{\omega(E_\alpha)\} = 0$, as $\omega(E_\alpha)$ is basic and $[E_\alpha, E_\beta]_x \in L_x$. On the other hand,

$$\begin{aligned}
(d_B\delta_B\omega)(e_\beta) &= -e_\beta\left\{\sum_\alpha (\nabla_{E_\alpha}\omega)(E_\alpha)\right\}\\
&= -\sum_\alpha e_\beta E_\alpha\{\omega(E_\alpha)\} + \sum_\alpha \omega(\nabla_{e_\beta}\nabla_{E_\alpha}E_\alpha).
\end{aligned}$$

Noting that $\nabla_{[E_\alpha, E_\beta]_x}E_\alpha$ does not contribute to the curvature $R_\nabla(e_\alpha, e_\beta)$, as $[E_\alpha, E_\beta]_x \in L_x$, we obtain (1). As for (2), we have

$$\begin{aligned}
(\Theta(Y)\nu)(E_{p+1}, \ldots, E_n) &= Y\{\nu(E_{p+1}, \ldots, E_n)\} - \sum_\alpha \nu(E_{p+1}, \ldots, \pi[Y, E_\alpha], \ldots, E_n)\\
&= -\sum_\alpha g_Q(\pi[Y, E_\alpha], E_\alpha)\\
&= \sum_\alpha g_Q(\nabla_{E_\alpha}\bar{Y}, E_\alpha)\\
&= \sum_\alpha (\nabla_{E_\alpha}\omega)(E_\alpha)\\
&= -\delta_B\omega,
\end{aligned}$$

while the second equality is immediate. Using (2), the left-hand side of (3) can be rewritten as

$$|\delta^*\omega|^2-\frac{2}{q}\operatorname{div}_B\bar{Y}\cdot(\delta^*\omega,g_Q)+\frac{1}{q}(\operatorname{div}_B\bar{Y})^2=|\delta^*\omega|^2-\frac{1}{q}(\operatorname{div}_B\bar{Y})^2,$$

and (3) follows. □

4. Proof of Theorems A and B. We first observe that $\bar{Y}$ is transversally Killing if and only if $\delta^*\omega=0$, $\bar{Y}$ is transversally Jacobi if and only if trace $\nabla^2\omega+\rho_\nabla(\omega)=0$ (by duality), and $\bar{Y}$ is transversally divergence-free if and only if $\delta_B\omega=0$. The equivalence of (i) and (ii) in Theorem A follows readily from Theorem D. As shown in [4], (i)⇒(iii). It therefore suffices to prove (iii)⇒(ii). We use the characterization of transversally affine infinitesimal automorphisms by the identity

$$\nabla_V A_\nabla(Y)=R_\nabla(\bar{Y},V),\quad V\in\Gamma Q,$$

where $A_\nabla(Y)\colon Q\to Q$ is given by the difference $\theta(Y)-\nabla_Y$ (and depends only on $\bar{Y}$, see [4]). Evaluating this identity for $V=E_\alpha$ and summing over α, we obtain $-\operatorname{trace}\nabla^2\bar{Y}=\rho_\nabla(\bar{Y})$, which is precisely the Jacobi condition. It remains to show $\operatorname{div}_B\bar{Y}=0$. By Theorem C, it suffices to show that $\operatorname{div}_B\bar{Y}$ is a constant function. Since $\operatorname{div}_B\bar{Y}\in\Omega_B^\circ(\mathfrak{F})$, it remains to verify that $e_\beta\operatorname{div}_B\bar{Y}=0$, $\beta=p+1,\dots,n$. Indeed, we have

$$\begin{aligned}e_\beta\operatorname{div}_B\bar{Y}&=e_\beta\Big\{\sum_\alpha g_Q(\nabla_{E_\alpha}\bar{Y},E_\alpha)\Big\}\\&=\sum_\alpha g_Q(\nabla_{e_\beta}\nabla_{E_\alpha}\bar{Y},E_\alpha)\\&=-\sum_\alpha g_Q((\nabla_{e_\beta}A_\nabla(Y))(E_\alpha),E_\alpha)\\&=-\sum_\alpha g_Q(R_\nabla(\bar{Y}_x,e_\beta)e_\alpha,e_\alpha)\\&=0,\end{aligned}$$

which completes the proof of Theorem A.

To prove Theorem B, we first note that the transversal conformality condition translates to $2\delta^*\omega=\sigma\cdot g_Q$. By (2) and $(g_Q,g_Q)=q$, this identity is further equivalent to $\delta^*\omega=1/q\operatorname{div}_B\bar{Y}\cdot g_Q$. Applying δ to both sides, and observing that the holonomy invariance of g_Q implies $\delta g_Q=0$, we have

$$\delta\delta^*\omega=\frac{1}{q}\delta(\operatorname{div}_B\bar{Y}\cdot g_Q)=-\frac{1}{q}d_B(\operatorname{div}_B\bar{Y})=\frac{1}{q}d_B\delta_B\omega.$$

For $q=2$, this reduces (1) to the Jacobi condition for $\bar{Y}$. Assuming conversely the Jacobi condition for $\bar{Y}$, we have, again by (1), $2\delta\delta^*\omega=d_B\delta_B\omega$. Taking the global scalar product with ω, we obtain

$$2\|\delta^*\omega\|^2-\|\operatorname{div}_B\bar{Y}\|^2=0.$$

For $q=2$, identity (3) implies the transversal conformality of $\bar{Y}$. □

5. Examples. (1) Given a compact oriented Riemannian manifold M with positive semidefinite Ricci tensor, the Albanese (Jacobi) map $J: M \to A(M)$ is the totally geodesic projection of a fibre bundle over the flat Albanese torus $A(M)$, of dimension equal to the first Betti number of M [8; 12]. From the geometric properties of this bundle it follows that the linear space of parallel vector fields on M is isomorphic (via J_*) with the linear space of parallel vector fields on M. By Theorem A, this space is further isomorphic with the linear space of transversal divergence-free Jacobi automorphisms of the corresponding harmonic Riemannian foliation.

(2) By Theorem B, the linear space of transversal Jacobi automorphisms of the harmonic Hopf fibration $f: S^3 \to S^2$ is isomorphic with the linear space of infinitesimally conformal fields on S^2, in particular, it is 6-dimensional. Note further that the nullity of f as a harmonic map equals 8 [13].

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