HOMOLOGY SPHERES AS STATIONARY SETS OF CIRCLE ACTIONS

Reinhard Schultz

As the title suggests, this paper deals with the following question: To what extent can one describe the fixed point sets of smooth S^1 actions on spheres or—more generally—homotopy spheres?

A few necessary conditions can be stated immediately. Since a smooth action is locally linear (cf. [2, Ch. VI]), it follows that the fixed point set must be a union of smoothly embedded closed submanifolds. Furthermore, cohomological techniques imply that the fixed point set must be an integral homology sphere; in fact, if m is the dimension of the ambient sphere and n is the dimension of the fixed point set, then $m-n$ must be even. During the 1960s the Hsiangs observed that these conditions are very nearly sufficient (cf. [15, Ch. V, §4]).

THEOREM 0. *Let $n \neq 3$ and let F^n be a closed, smooth, integral homology sphere. Then for any $k>0$ there is a smooth S^1 action on S^{n+2k} such that the fixed point set F' is "almost" diffeomorphic to F; in other words, there is a homeomorphism from $F \to F'$ that is either a diffeomorphism or a diffeomorphism on the complement of a point.*

This result foreshadowed in several respects the later results of L. Jones and many others on converses of the P. A. Smith theorem (see [37] and related papers in the same volume for further information).

COMPLEMENT TO THEOREM 0. The group actions in Theorem 0 may be assumed to be *semifree;* that is, S^1 acts freely on the complement of the fixed point set.

Although Theorem 0 gives very strong information on the central question of this paper, it does not specifically address two points:

(1) What happens if $n=3$?
(2) Under what conditions is there an action whose fixed point set is actually diffeomorphic to F?

For many years it has been known that the conclusion of Theorem 0 applies to certain nonsimply connected **Z**-homology 3-spheres. This result, which is due to Montgomery and Samelson [23], was one of the first indications that surgery theory had far-reaching consequences for transformation groups. In fact, one can use surgery theory to state the following realization theorem for $n=3$.

Received April 8, 1985. Revision received October 7, 1986.

The author was partially supported by National Science Foundation (USA) Grant MCS 83-00669 and by Sonderforschungsbereich 170 "Geometrie und Analysis," Universität Göttingen.

Michigan Math. J. 34 (1987).

THEOREM I. *Let* Σ^3 *be a (smooth) homology* 3*-sphere, and let* q *be a positive integer.*

(i) *For all* $q \geq 1$ *there is a topological semifree locally linear* S^1 *action on* S^{3+2q} *that is semifree and has* Σ^3 *as its fixed point set.*

(ii) *If* $q \geq 2$ *then the actions in* (i) *may be assumed to be piecewise linear in an appropriate sense.*

(iii) *If* $q \geq 2$ *is even then the actions in* (i) *correspond to smooth actions on homotopy spheres.*

(iv) *If* q *is odd and the Eells–Kuiper invariant* $\mu(\Sigma^3) \in \mathbf{Z}/2$ *is zero, then the same conclusion as in* (iii) *holds. Conversely, if* $q = 1$ *then the conclusions of* (ii)–(iv) *hold if and only if* Σ *bounds a contractible manifold.*

This result is a fairly routine consequence of product formulas for surgery obstructions and will be discussed in Section 1. In this paper we shall dispose of the remaining case.

THEOREM II. *Suppose that* $q \geq 5$ *is odd and the Eells–Kuiper invariant* $\mu(\Sigma^3)$ *is nonzero. Then* Σ^3 *is* not *the fixed point set of a smooth semifree* S^1 *action on a homotopy* $(2q+3)$*-sphere.*

Since almost diffeomorphic 3-manifolds are diffeomorphic (cf. [22], [24], [38]), Theorem II represents an essential difference in the behavior of stationary sets of smooth circle actions. However, there are some partial results in higher dimensions. One example is the following.

THEOREM III. *Let* Σ^7 *be a homology sphere representing a generator of the Kervaire–Milnor group* Θ_7, *and suppose* $q \geq 5$ *is odd. Then* Σ^7 *is not the fixed point set of a semifree smooth* S^1 *action on a homotopy* $(2q+7)$*-sphere.*

REMARK. The Kervaire–Milnor groups Θ_m are defined for homotopy spheres in [17] but, as noticed by the Hsiangs [15], if $m \neq 3$ these groups also are the groups of homology n-spheres modulo boundaries of contractible manifolds.

We shall prove Theorem III in Section 4.

One can ask a corresponding question in other dimensions of the form $4k-1$. The answer to this question would require a sharpening of the techniques in [29]. However, the following seems reasonable.

CONJECTURE IV. *If* Σ^{4k-1} *represents the generator of* bP_{4k}, *then* Σ *is not the fixed point set of a semifree smooth* S^1 *action on some homotopy* $(4k+2q-1)$*-sphere with* $q \geq 3$ *odd.*

Theorems II and III suggest the following more general problem on realizing homology 3-spheres as fixed point sets.

QUESTION V. Let V^{2q} be a fixed point free representation of S^1, and let Σ^3 be a homology 3-sphere. Is there a smooth S^1-action on some homotopy $(2q+3)$-sphere with fixed point set Σ^3 and local representation V at fixed points?

In Section 1 we shall show that the answer to this question is yes if either $\mu(\Sigma^3)=0$ or $V=W\oplus W$ for some W. Other cases can be treated directly, but in general the problem seems fairly elusive.

The proof of Theorem II begins with an observation essentially due to Browder [3]: If there is a smooth semifree S^1-action on a homotopy $(2q+3)$-sphere with Σ^3 as the fixed point set, then $\Sigma^3\times \mathbf{C}P^{q-1}$ is smoothly homologically h-cobordant to $S^3\times \mathbf{C}P^{q-1}$. This statement can be combined with various results from surgery theory and homotopy theory to prove a related result; namely, if Σ' generates the group bP_{2q+2} of homotopy $(2q+1)$-spheres bounding parallelizable manifolds, then the standard homeomorphism

$$h\colon S^3\times \mathbf{C}P^{q-1}\#\Sigma'\to S^3\times \mathbf{C}P^{q-1}$$

(i.e., h is a diffeomorphism except at one point) is homotopic to a diffeomorphism. A proof of this implication is given in Section 3. On the other hand, a result of Taylor [34] states that h cannot be homotopic to a diffeomorphism. Since a proof of this result has not appeared in print, a verification is included in Section 2 for the sake of completeness.

ACKNOWLEDGMENTS. The proofs of Theorems II and III use a result of Taylor on homotopy inertia groups that is proved in Section 2. I am grateful to Larry Taylor for allowing me to include a proof of his unpublished result. My interest in the topic of this paper was awakened by remarks of Shmuel Weinberger on PL circle actions in the first draft of [37] (this portion of the paper does not appear in the final version); I am grateful to both Weinberger and Sylvain Cappell for discussing their unpublished work with me and commenting on the results presented here. This paper was written during a stay at the newly established Sonderforschungsbereich 170 "Geometrie und Analysis" connected with the Mathematical Institute in Göttingen. Of course I am also grateful for the SFB and MI for their support and hospitality.

1. Realization of fixed point sets. In this section we shall prove Theorem I and partial results on Conjecture IV. We begin with the results in the topological case.

(1.1) *Let Σ^3 be a closed $\mathbf{Z}$-homology 3-sphere, and let V^{2q} be a positive-dimensional fixed point free representation of S^1. Then there is a locally linear (topological) S^1 action on S^{2q+3} with Σ^3 as fixed point set and local representation V at fixed points.*

Proof. By the results of Freedman on 4-manifolds [9], every homology 3-sphere Σ^3 bounds a compact contractible 4-manifold K^4. Construct a circle action on $K\times D^{2q}$ by letting S^1 act trivially on the first factor and linearly via V^{2q} on the second factor. Then $K\times D^{2q}$ is homeomorphic to D^{2q+4} by a standard argument (cf. [21] or [15, Ch. V, §4]), and the induced S^1 action on the boundary has the required properties. □

(1.2) COMPLEMENT TO (1.1). *The group action on S^{2q+3} may be viewed as a locally linear "piecewise linear" S^1 action if $q\geq 2$ and the action is semifree. If*

$q=1$ *then the corresponding statement is true if and only if* Σ^3 *bounds a compact contractible* smooth *(equivalently, piecewise linear)* 4-*manifold.*

We shall not attempt to formulate the concept of a locally linear piecewise linear S^1 action formally in this paper. A general discussion of piecewise linear actions is due to Gluck [10], and recent work of Cappell and Weinberger has shown that such actions can be studied quite effectively. For our purposes it suffices to assume the existence of a category of piecewise linear S^1 actions with a few simple formal properties such as cutting and pasting constructions (cf. [32, Problem 4.1(B)]).

Proof of Complement. Suppose that $q=1$, and suppose a "locally linear piecewise linear" S^1 action with fixed point set Σ exists. Let K be the orbit space S^5/S^1. Then an argument of Hsiang [16] shows that K is a contractible 4-manifold with boundary Σ. If the action is smooth, then in fact one can make K into a smooth 4-manifold (cf. [31]). This process is formal and has a straightforward piecewise linear analog.

Next assume $q \geq 2$. Consider the product $K \times \mathbf{C}P^{q-1}$ as a smooth manifold. By the product formulas for simply connected surgery, the boundary $\Sigma \times \mathbf{C}P^{q-1}$ is $\mathbf{Z}$-homologically h-cobordant to

$$S^3 \times \mathbf{C}P^{q-1} \# \Sigma',$$

where Σ' lies in the group bP_{2q+2} (cf. [3, pp. 37–39]). In other words, there is a cobordism (in fact, a smooth one) V such that

$$\partial V = (-\Sigma^3 \times \mathbf{C}P^{q-1}) \amalg (S^3 \times \mathbf{C}P^{q-1} \# \Sigma')$$

such that V is simply connected and the inclusion of each boundary component induces an isomorphism in integral homology. If we pass to the PL category, then Σ' is PL isomorphic to S^{2q+1} and can therefore be disregarded.

Form the manifold

$$U = D^4 \times \mathbf{C}P^{q-1} \cup_{S \times CP} V.$$

By construction there is a homotopy equivalence

$$(U, \partial U) \to (D^4, S^3) \times \mathbf{C}P^{q-1}.$$

Let $W \to U$ be the principal S^1 bundle classified by the generator of $H^2(U) \cong H^2(\mathbf{C}P^{q-1})$; the results of [10] allow one to view the free S^1 action on W as a piecewise linear S^1 action whose restriction to the boundary is the standard action on $\Sigma^3 \times S^{2q-1}$. Form the PL S^1-manifold

$$W^* = \Sigma^3 \times D^{2q} \cup_{\Sigma \times S} W.$$

By construction W^* is a PL S^1-manifold, it is homotopy equivalent (hence PL isomorphic) to S^{2q+3}, the action of S^1 is locally linear and semifree, and the fixed point set is Σ^3. □

There is also an elementary analog of (1.1) in the smooth category, but the conclusion is significantly weaker; specifically, one must assume that q is even.

(1.3) *The group actions in* (1.1) *and* (1.2) *may in fact be viewed as differentiable S^1-actions on (possibly exotic!) homotopy spheres provided $q \geq 2$ is* even.

Proof. The argument is essentially the same as in (1.2) with one change; that is, if V is the homological h-cobordism from $\Sigma \times \mathbf{C}P^{q-1}$ to $S^3 \times \mathbf{C}P^{q-1} \# \Sigma'$, then (because q is even) one in fact knows that Σ' will be the standard sphere (cf. Browder [3, p. 38, ¶2]). Using this, one can carry through the balance of the proof of (1.2) in the smooth category; the only difference is that the manifold W^* is only a homotopy sphere and not necessarily the standard sphere. □

If q is odd then one has a somewhat weaker conclusion in the smooth category. Recall that the Eells–Kuiper invariant (often called the Rochlin invariant) $\mu(\Sigma^3)$ of a homology 3-sphere is defined as follows: Since every homology 3-sphere bounds a framed manifold, choose one such coboundary P^4 for Σ^3. General algebraic considerations imply that the signature of P is divisible by 8, and $\mu(\Sigma^3)$ is the residue class of this signature in $8\mathbf{Z}/16\mathbf{Z} \cong \mathbf{Z}/2$. A classical theorem of Rochlin implies that this number does not depend on the choice of 0. For more information see [8].

(1.4) *The group actions in* (1.1) *and* (1.2) *may be viewed as differentiable S^1 actions on homotopy spheres provided $q \geq 2$ and $\mu(\Sigma^3) = 0$.*

Proof. By assumption $\Sigma^3 = \partial P^4$, where P^4 is parallelizable and has index divisible by 16. Let Q^4 be a smooth manifold such that $Q^4 - \{\text{pt.}\}$ is parallelizable and signature $(Q^4) = 16$; one can in fact take Q^4 to be Kummer's quartic surface, but for our purposes this is not important. A connected sum of P with an appropriate number of copies of $\pm Q$ will yield a new parallelizable manifold P' such that $\partial P' = \Sigma^3$ and signature $P' = 0$. Let $P'' = P' - \operatorname{Int} D$, where $D \subseteq \operatorname{Int} P'$ is a smoothly embedded closed 4-disk. Now $\partial P'' = \Sigma \amalg S^3$, and one in fact has a normal map of triads

$$h: (P''; \Sigma, S^3) \to (S^3 \times [0,1]; S^3 \times \{0\}, S^3 \times \{1\}).$$

Form the Cartesian product of h with $\mathbf{C}P^{q-1}$. By the arguments of (1.1)–(1.3), it suffices to show that one can perform surgery on $h \times \mathbf{C}P^{q-1}$, holding the boundary fixed, so that one obtains a homotopy equivalence h^*. The domain of h^* will then serve as a substitute for the manifold V constructed in (1.2).

But the product formulas for simply connected surgery show that the surgery obstruction of $h \times \mathbf{C}P^{q-1}$ depends only upon the signature of P'' (cf. [26]). Since this signature vanishes, the surgery obstruction also vanishes. □

Theorem II states that (1.3) and (1.4) are the best possible results for semifree S^1 actions. In Section 2 we shall develop some auxiliary material needed for the proof of that result, and in Section 3 we shall complete the proof. The remainder of this section deals with the realization of homology spheres as fixed point sets of *nonsemifree* S^1 actions (i.e., Conjecture IV).

The first result is a straightforward extension of (1.4).

PROPOSITION 1.5. *Let Σ^3 be a closed smooth homology* 3*-sphere with $\mu(\Sigma^3) =$* 0, *and let Ω be a positive-dimensional fixed point free representation of S^1. Assume* *that no irreducible representation of S^1 has multiplicity* 1 *in* Ω. *Then there is a smooth semifree S^1 action on some homotopy* $(3+\dim\Omega)$*-sphere with the following properties:*

(i) *The fixed point set is* Σ^3.

(ii) *The local normal representation at fixed points is* Ω.

Proof. Let P'' be the manifold constructed in (1.4), and let $h: P''\to S^3\times[0,1]$ be the corresponding normal map. The Cartesian product of h with the unit sphere $S(\Omega)$ may be viewed as an equivariant normal map, and in fact it is a transverse-linear isovariant normal map in the sense of Browder and Quinn [4]. Consequently there is a well-defined surgery obstruction

$$\sigma(S^1, h\times S(\Omega))\in L(S^1, S^4\times S(\Omega)),$$

where $L(S^1, S^4\times S(\Omega))$ is the appropriate Browder–Quinn surgery obstruction group. The multiplicity assumption implies that $h\times S(\Omega)$ can be transverse-linearly and isovariantly surgered to an equivariant equivalence if and only if this obstruction vanishes.

Wall's surgery theory contains an important periodicity relationship of the form $\sigma(f)=\sigma(f\times \mathbf{C}P^2)$ (cf. [36, Ch. 9]), and the same sort of relationship holds for the Browder–Quinn groups (see [4]). Consequently $\sigma(S^1, h\times S(\Omega))=0$ if and only if $\sigma(S^1, \mathbf{C}P^2\times h\times S(\Omega))=0$. On the other hand, the geometrical construction of the Browder–Quinn theory shows that the latter vanishes if $\sigma(\mathbf{C}P^2\times h)=0$. But in (1.4) we observed that this is the case, and therefore $\sigma(S^1, h\times S(\Omega))=0$.

Therefore we have an S^1-manifold triad $(W; \Sigma^3\times S(\Omega), S^3\times S(\Omega))$ such that the inclusion of $S^3\times S(\Omega)$ is an isovariant homotopy equivalence. Form the smooth S^1-manifold

$$W^*=\Sigma^3\times D(\Omega)\cup W\cup D^4\times S(\Omega)$$

as in (1.4). It follows that W^* is a homotopy sphere and the S^1 action has the prescribed properties. □

The next result may be viewed as a generalization of (1.3).

PROPOSITION 1.6. *Let Σ^3 be a closed smooth homology sphere, and let Ω be a positive dimensional fixed point free representation of S^1 such that $\Omega=\Omega_0\oplus\Omega_0$ for some Ω_0. Then the conclusion of Proposition 1.5 holds for Σ and Ω.*

Proof. Step 1 (verification of the result for a specific choice of Σ). Let $2m$ be the real dimension of Ω_0, and consider the Brieskorn variety

$$V=\{z_0^3+z_1^5+z_2^2+\cdots+z_{2m+2}^2=0\}.$$

Standard results imply that the intersection of V with some small sphere ϵS^{4m+5} is a homotopy $(4m+3)$-sphere if $m>1$ (cf. [14]). Let S^1 act on the last $2m$ complex coordinates of V by the complexification of Ω_0 (i.e., extend the linear action of S^1 on $\Omega_0\cong\mathbf{R}^{2m}$ to $\mathbf{C}^{2m}$). The action of S^1 is strictly speaking on $\mathbf{C}^{2m+3}$, but a routine verification shows that S^1 maps V into itself; in fact, S^1 maps the homotopy sphere $\Sigma^*=V\cap\epsilon S^{4m+5}$ into itself. By construction the S^1 action on Σ^* has fixed point set

$$F^3 = \{z_0^3 + z_1^5 + z_2^2 = 0\} \cap \epsilon S^5$$

and local representation $\Omega = \Omega_0 \otimes \mathbf{C}$ at fixed points. But F^3 is known to be a homology 3-sphere with μ-invariant 1; in fact, F^3 is the Poincaré homology 3-sphere $SO_3/(\text{Icosahedral group})$ (cf. [13]).

Step 2 (realization of all Σ in a homological h-cobordism class). The objective is easy to state: If Σ' is $\mathbf{Z}$-homologically h-cobordant to Σ, then we wish to realize Σ'. Let X^4 be an homological h-cobordism between Σ and Σ'. We can perform 1-dimensional surgery on $\operatorname{Int}(X^4)$ to make X simply connected. Suppose now that W_0 is a smooth homotopy sphere such that

(i) W_0 admits a smooth S^1 action with fixed point set Σ^3,

(ii) a neighborhood of the fixed point set is S^1-diffeomorphic to $\Sigma^3 \times D(\Omega)$.

(Notice that (ii) holds for the actions constructed in Step 1.) We can replace Σ by Σ' using X as follows: Form the S^1-manifold

$$W_1 = \Sigma' \times D(\Omega) \cup X \times S(\Omega) \cup W_0 - \operatorname{Int} \Sigma \times D(\Omega).$$

Then it is immediate that W_1 is again a homotopy sphere, the fixed point set of the action is Σ', and a neighborhood of Σ' is S^1-diffeomorphic to $\Sigma^3 \times D(\Omega)$.

Step 3 (the general case). If $\mu(\Sigma^3)$ is zero then the result is a special case of (1.5); notice that condition (ii) in step 2 follows by construction. Suppose now that $\mu(\Sigma^3) = 1$. Let Σ_0 be the Poincaré 3-sphere, and consider the homology sphere $\Sigma_1 = \Sigma \# -\Sigma_0 \# \Sigma_0$. Since $-\Sigma_0 \# \Sigma_0$ is homologically h-cobordant to S^3, it follows that Σ admits an action satisfying (i) and (ii) if and only if Σ_1 does. On the other hand, Σ_0 admits such an action by step 1, and $\Sigma \# -\Sigma_0$ also admits such an action because its μ-invariant is zero. If we take connected sums of these actions along the fixed point set, we obtain the desired S^1 action on Σ_1. □

It is possible to make further statements on the existence of S^1 actions, but none seem especially enlightening at this time.

2. The homotopy inertia group. The main result of this section is an unpublished theorem of Taylor [34]. I would like to reexpress my gratitude to him for his generous attitude toward my inclusion of his result.

Let M^n be a closed differentiable manifold, and let Σ^n be a smooth manifold that is homeomorphic but not necessarily diffeomorphic to S^n. Then there is a standard class of homeomorphisms

$$h(M, \Sigma): M \# \Sigma \to M, \tag{2.0}$$

all of which are homotopic.

DEFINITION. Let $n \geq 5$. The *homotopy inertia group* of M^n, written $I_h(M^n)$, consists of all (oriented diffeomorphism classes of) homotopy spheres Σ^n such that $h(\Sigma, M)$ is homotopic to a diffeomorphism.

It is fairly straightforward to prove that $I_h(M^n)$ is always a subgroup of the Kervaire–Milnor group of homotopy spheres Θ_n (cf. [7]).

EXAMPLE 1. If M^n is a product of spheres, then $I_h(M^n) = 0$. In fact, much stronger results hold [27].

EXAMPLE 2. If $M^7 = S^3 \times \mathbf{C}P^2$, then $I_h(M^7) = 2\Theta_7$. More generally (cf. [3, pp. 37–38]), $I_h(S^3 \times \mathbf{C}P^{2j})$ always contains $2bP_{4j+3}$.

The following result of Taylor states that Example 2 is in some sense the extreme case.

THEOREM 2.1. *Let N be a closed, oriented, smooth manifold of dimension $4h-1 \geq 7$, and let Σ^{4h-1} be a homotopy sphere generating bP_{4h}*. Then $\Sigma \notin I_h(N)$.

REMARK 1. Given M^n, there is a subgroup of Θ_n called the inertia group $I(M^n)$ such that $I_h(M^n) \subseteq I(M^n) \subseteq \Theta_n$. In contrast to Theorem 2.1, for each n there is an M^n such that $I(M^n) = \Theta_n$ (see Winkelnkemper [40]).

REMARK 2. The proof given below is based upon results of Brumfiel [7]. Taylor's original proof is quite different, and it is motivated by results of Frank on the power of two that divides the signature of a manifold and its dependence on the number of cross sections. Taylor obtained analogous results involving fiber homotopy cross sections. These results yield a remarkably simple formula for the mod 2 reduction of the index obstruction of a $4n$-dimensional oriented surgery problem; namely $\sigma(f) = \{\eta(f)^* k_2\} w_{4n-2}[M]$ mod 2, where $\eta(f)$ denotes the normal invariant and k_2 is the first nontrivial cohomology class in $\tilde{H}(F/0; \mathbf{Z}/2)$. Theorem 2.1 follows from this formula.

Proof. The group bP_{4k} is cyclic of finite order, say θ_k; this number can be computed explicitly (cf. [17, §7]), but we do not need such precise information. At this time it suffices to know that θ_k is always even. Let $f_R : \Theta_{4k-1} \to \mathbf{Z}/\theta_k$ be the splitting homomorphism defined by Brumfiel [5] with $f_R \mid bP_{4k}$ an isomorphism.

Suppose that Σ does lie in the homotopy inertia group of N. Then a result of Brumfiel [7, Prop. II.3, p. 403] yields the following formula for $f_R(\Sigma)$:

$$f_R(\Sigma) = -\sum_m L_{k-m}(N) \frac{\theta_m}{a_m(2m-1)!\, j_m} p_m(\xi)[N \times S^1], \tag{2.2}$$

where L_q is the $4q$-dimensional Hirzebruch polynomial; θ_m is the order of bP_{4m}; j_m is the order of the image of J in dimension $4m-1$; $a_m = 2$ if m is odd, 1 if m is even; and $p_m(\xi) = m$th Pontrjagin class, of some fiber homotopically trivial vector bundle over $S^1 \wedge N$, modulo Brumfiel's convention for $p_k(\xi)$. By the formula it is obvious that $f_R(\Sigma)$ is an element in $\mathbf{Q}/\theta_k\mathbf{Z}$, but one actually obtains an element in $\mathbf{Z}/\theta_k\mathbf{Z}$ by the results of [5].

CLAIM. *Each summand in* (2.2) *is a fraction with even numerator and odd denominator*. For if this is true, then $f_R(\Sigma)$ will correspond to an element of $2\mathbf{Z}_{(2)}/\theta_k\mathbf{Z}$ and hence lie in

$$(2\mathbf{Z}_{(2)}/\theta_k\mathbf{Z}) \cap (\mathbf{Z}/\theta_k\mathbf{Z}) = 2\mathbf{Z}/\theta_k\mathbf{Z}.$$

Consequently, if Σ lies in $I_h(N) \cap bP_{4k}$, then Σ must be an even multiple of the generator of the even order group bP_{4k}.

The first step toward proving the claim is to notice that coefficients in the Hirzebruch polynomials always have odd denominators (see [12, §1]). The second step is somewhat messy. It can be verified in many ways; for example, one can use the solutions of the Adams conjecture at all primes.

SUBLEMMA 2.3. *Let ξ be a fiber homotopically trivial vector bundle over some finite suspension complex SK. Then the* mth *integral Pontrjagin class $p_m(\xi)$ is divisible by j_{4m} modulo torsion.*

Sketch of proof. Fix a prime q, and let $r=3$ $(q=2)$ or a generator of the units mod q^2 (q odd). For some integers l and l' prime to q we have $l\xi = l'(\psi^r\omega-\omega)$ for some bundle ω. Use the additivity of Pontrjagin classes over suspensions and the identity $p_m(\psi^r\omega)=r^{2m}p_m(\omega)$ to conclude that $lp_m(\xi)$ is divisible by $(r^{2m}-1)$. Since j_{4m} and $(r^{2m}-1)$ are divisible by the same powers of q, it follows that $p_m(\xi)$ is divisible by the q-primary factor of j_{4m}. □

We next consider the terms $\theta_m p_m(\xi)/a_m(2m-1)!\,j_m$ that arise in (2.2). There are three cases.

Case 1: $2\le m\le k-1$. Since $p_m(\xi)/j_m$ is integral modulo torsion, we must check that

$$\frac{\theta_m}{a_m(2m-1)!}=\frac{a_m2^{2m-2}\text{ odd}}{a_m(2m-1)!}$$

has an even numerator (see [17, §7] for the implicit assertion on θ_m). But the right-hand side may be rewritten as $2^{\alpha(2m-1)-1}u/v$, where u and v are odd and $\alpha(q)$ is the number of ones in the dyadic expansion of q. Since $m\ge 2$, the exponent of 2 is positive.

Case 2: $m=k$. By construction, we have

$$\frac{\theta_m p_m(\xi)}{a_m(2m-1)!\,j_m}=2^{2m-2}\text{ odd }\frac{s_m(\bar\xi)}{(2m)!}=ph(\bar\xi),$$

where $\bar\xi$ is a bundle that *also* may be assumed to be fiber homotopically trivial (look at the construction on the bottom of page 387 in [7]). The same sort of calculation as in case 1 shows that the numerator is even.

Case 3: $m=1$. The calculation of case 1 only shows that the class under consideration has an odd denominator. Thus we need a slight improvement of the previous Sublemma; namely, the first Pontrjagin class $p_1(\xi)$ is in fact divisible by 48 modulo torsion.

Using solution of the Adams conjecture as in (2.3), one can reduce the question to showing that for every vector bundle α over SX, one has that $p_1(\alpha)$ is divisible by 2 modulo torsion. If α has a complex structure, this is immediate from the equation $p_1(\alpha)=2c_2(\alpha)$. Thus it suffices to prove that any α may be (stably) decomposed as $\beta+\gamma$, where β is a complex vector bundle and γ has finite order in $\tilde{KO}(SX)$. This in turn reduces to proving that

$$[X,U]/\text{torsion}\to[X,SO]/\text{torsion}$$

is onto if X is connected and has dimension ≤ 4.

To prove the latter, observe that U and SO have the 5-types of 2-stage Postnikov systems with homotopy in dimensions 1 and 3 and trivial k-invariants (notice that $U \simeq SU \times S^1$ as spaces while $\mathbf{R}P^\infty$ is a retract of SO). Since $\pi_3(U) \to \pi_3(SO)$ is bijective and $\pi_1(SO)$ is torsion the assertion follows. This completes the proof of (2.2). □

REMARK. Strictly speaking, Brumfiel's Proposition II.3 [7] is stated only for 1-connected manifolds. However, the result is true in general because the inclusion $I_h(N) \subseteq d([N_0^k \wedge S^1, F/0])$ (see [7, Prop. II.1, p. 403]) holds without any restrictions on the fundamental group (cf. [7, Prop. II.1]—the point is that in general the normal maps in $[\ldots, F/0]$ could have nontrivial surgery obstructions besides the signature if $\pi_1(N)$ is nontrivial).

3. **Proof of Theorem II.** We shall assume that S^1 acts smoothly and semifreely on some homotopy $(2q+3)$-sphere M with fixed point set Σ^3, where $\mu(\Sigma^3)=1$, and obtain a contradiction. Here is the first step.

(3.1) *If an action of the desired type exists, then $S^3 \times \mathbf{C}P^{q-1}$ is orientation-preservingly diffeomorphic to $S^3 \times \mathbf{C}P^{q-1} \# \Sigma'$, where Σ' represents a generator of bP_{2q+2}.* (Notice that q odd implies $2q+2 \equiv 0 \bmod 4$).

Proof. A direct analog of the argument in [3, §5.6, p. 33] shows that the equivariant normal bundle of the fixed point set Σ in M is equivariantly stably trivial; since $q \geq 3$ implies $2q$ is greater than 3, stable triviality implies triviality. Let S_0 be an invariant sphere in M that links Σ once; such a manifold may be found by taking a sphere of fixed radius in a tubular neighborhood of M. The argument in [3, p. 31] shows that S_0 has a trivial equivariant normal bundle and that

$$M_0 = M - (\Sigma^3 \times \operatorname{Int} D^{2q} \cup D^4 \times S_0)/S^1$$

is a smooth, simply connected $\mathbf{Z}$-homology h-bordism. On the other hand, by the surgery obstruction product formulas we have a similar homology h-cobordism M_1 from $\Sigma^3 \times \mathbf{C}P^{q-1}$ to $S^3 \times \mathbf{C}P^{q-1} \# \Sigma'$, where Σ' is as described (cf. [3, p. 38, first full paragraph]). Join M_0 and M_1 together along $\Sigma^3 \times \mathbf{C}P^{q-1}$. The result is a smooth h-cobordism between $S^3 \times \mathbf{C}P^{q-1}$ and $S^3 \times \mathbf{C}P^{q-1} \# \Sigma'$, and the ends of this manifold are diffeomorphic by the h-cobordism theorem [21]. □

(3.2) COMPLEMENT. *The diffeomorphism may be chosen to send*

$$D_+^3 \times \mathbf{C}P^{q-1} \subseteq S^3 \times \mathbf{C}P^{q-1} \ \textit{to}\ D_+^3 \times \mathbf{C}P^{q-1} \subseteq S^3 \times \mathbf{C}P^{q-1} \# \Sigma'$$

by the "identity."

Proof. The homology h-cobordisms may be constructed as unions

$$M_0 = D_+^3 \times \mathbf{C}P^{q-1} \times [0,1] \cup M_0' \quad \text{and} \quad M_1 = D_+^3 \times \mathbf{C}P^{q-1} \times [0,1] \cup M_1'.$$

For M_1 this is immediate from the product formula, while for M_0 this follows from a standard refinement of Browder's construction. □

The following is immediate from (3.1) and (3.2).

(3.3) *Let $\varphi: S^3 \times \mathbf{C}P^{q-1} \# \Sigma' \to S^3 \times \mathbf{C}P^{q-1}$ be the diffeomorphism in* (3.1) *and* (3.2), *and let $h(S^3 \times \mathbf{C}P^{q-1}, \Sigma')$ be the standard homeomorphism from*

$$S^3 \times \mathbf{C}P^{q-1} \# \Sigma' \to S^3 \times \mathbf{C}P^{q-1}.$$

Then φh^{-1} is a self-homeomorphism which corresponds to a homotopy class in $\pi_3(F_{S^1}(\mathbf{C}^q))$, where the $F\langle$etc.$\rangle$ represents the space of all S^1-equivariant maps of S^{2q-1} with the free linear action (see [1]).

Proof. The homotopy class of a self-map of $S^3 \times \mathbf{C}P^{q-1}$ is determined by its projections onto the factors. Since φh^{-1} sends $D^3_\pm \times \mathbf{C}P^{q-1}$ to itself and is the "identity" on $D^3_+ \times \mathbf{C}P^{q-1}$, the projection of φh^{-1} onto $S^3 = D^3_+ \cup D^3_-$ is homotopic to the usual coordinate projection by straight-line homotopies on each hemisphere. It follows that φh^{-1} comes from an element of $\pi_3(E(\mathbf{C}P^{q-1}))$, where $E(X)$ denotes the monoid of continuous self-maps of X. However, there is a fibration

$$\mathrm{Maps}(\mathbf{C}P^{q-1}, S^1) \longrightarrow F_{S^1}(\mathbf{C}^q) \xrightarrow{p} E(\mathbf{C}P^{q-1})$$

(cf. [28]), and by the homotopy exact sequence, the projection p induces homotopy isomorphisms in dimensions greater than 1. □

Since $q \geq 3$, the results of [1] imply that

$$\begin{aligned} \pi_3(F_{S^1}(\mathbf{C}^q)) &\cong \pi_3^S(S\mathbf{C}P_+^\infty) \\ &\cong \mathbf{Z} \oplus \pi_3^S(S^1) \\ &\cong \mathbf{Z} \oplus \mathbf{Z}_2. \end{aligned} \tag{3.4}$$

The first summand is the image of $\pi_3(U)$ (cf. [1]).

We shall need a computational result.

PROPOSITION 3.5. *Let ψ be a homotopy self-equivalence of $S^3 \times \mathbf{C}P^{q-1}$ ($q \geq 3$) that is induced by an element of $\pi_3(F_{S^1}(\mathbf{C}P^q))$ not in the image of $\pi_3(U_q)$. Then the normal cobordism class of ψ is nontrivial if $q \geq 4$. On the other hand, if $q = 3$ then ψ is homotopic to a self-diffeomorphism of $S^3 \times \mathbf{C}P^2$.*

Proof. Because elements in the image of $\pi_3(U_q)$ induce diffeomorphisms of $S^3 \times \mathbf{C}P^{q-1}$ (up to homotopy), the splitting of (3.4) and composition formulas for normal invariants (cf. [27, §2.2(i), p. 143]) imply that we need only consider the special case where ψ arises from the nonzero element of $\pi_3^S(S^1) \cong \mathbf{Z}_2$. The results of [29, §3, pp. 157–158] then yield the following formula.

(3.6) *Let $\alpha \in \pi_n^S(S^1)$ correspond to the element $\alpha' \in \pi_n(F_{S^1}(\mathbf{C}^r))$, r large. Then the normal invariant of α' is the image of the class*

$$S^n\mathbf{C}P_+^\infty \xrightarrow{\alpha \wedge 1} S^1\mathbf{C}P_+^\infty \xrightarrow{t} S^0 \quad (t = \text{transfer [1]})$$

under the composite $[Y, S^0] \cong [Y, F] \to [Y, F/0]$. □

In the case under consideration α is η^2, the square of the Hopf map. We claim the following.

(3.7a) *The restriction of $t(\eta^2 \wedge 1)$ to $S^3\mathbf{C}P^2$ is trivial.*

(3.7b) *The restriction to $S^3\mathbf{C}P^3$ is nontrivial, and in fact its image in $[S^3\mathbf{C}P^3_+, F/0]$ is nontrivial.*

(3.7c) *The restriction of* $t(\eta^2\wedge 1)$ *to* S^3 *is* $\eta^3=4\nu$.

The second statement immediately implies the proposition if $q\geq 5$. However, more work will be needed if $q=3$; details appear after the verification of (3.7a)–(3.7b).

We begin by displaying the stable homotopy type of $\mathbf{C}P^3$ explicitly as the complex $S^2\cup_{(\eta,2\nu)}e^4\cup e^6$ (cf. Brumfiel [6]). The map $t\colon S\mathbf{C}P_+^\infty\to S^0$ restricts to the class $(\eta,\nu)\in\{S\mathbf{C}P_+^1,S^0\}=\pi_1^S\oplus\pi_3^S$.

Let $y=t(\eta^2\wedge 1)=\eta^2S^2t$. It follows that $y\mid S^3=\eta^3$, proving (3.7c). Furthermore, $y\mid S^5=0$ since $\pi_5^S=0$. Therefore standard Toda bracket considerations say that

$$y\mid 7\text{-cell}=\langle\eta,\nu,\eta^2\rangle=0,\qquad y\mid 9\text{-cell}=\langle 2\nu,\nu,\eta^2\rangle=\epsilon\eta$$

(see Toda [35] for the bracket identities). But $\epsilon\eta\in\pi_q^S$ projects nontrivially into $\pi_q(F/0)$, and this proves (3.7b). Finally, the relation $\langle\eta,\nu,\eta^2\rangle=0$ implies (3.7a).

The case $q=3$. This can be done using the long exact sequence for stably tangential surgery (cf. Madsen, Taylor, and Williams [20]):

$$\begin{tikzcd}
\{S^4\mathbf{C}P_+^2,S^0\} \arrow[r] \arrow[dr, "\sigma"'] & L(1) \arrow[r] \arrow[d, "\cong"] & hS_3^t(\mathbf{C}P^2) \arrow[r, "N"] & \{S^3\mathbf{C}P_+^2,S^0\} \\
 & \mathbf{Z}. \arrow[ur, "\theta"'] & &
\end{tikzcd}$$

There is a canonical homomorphism

$$\gamma\colon\pi_3(F_{S^1}(\mathbf{C}^3))\to hS_3^t(\mathbf{C}P_+^2)$$

defined as in [30] (see Section 4). It suffices to show that γ sends the torsion generator of the codomain to a framed homotopy smoothing of the form

$$((S^3\times\mathbf{C}P^2,\ \textit{exotic framing}),\ \text{identity}).$$

We begin with a digression.

DEFINITION. Let M^n be a smooth manifold with tangent bundle τ_M. A (*stably*) *tangential PL smoothing* of M is a triple (V,t,φ), where V is a smooth manifold, $t\colon V\to M$ is a piecewise differentiable homeomorphism (e.g., in the sense of Munkres [24]), and $\varphi\colon E(\tau_V\oplus k)\to E(\tau_M\oplus k)$ is a vector bundle isomorphism covering t. One defines a concordance relation on such triples as in [11] or [18]; the conditions on the bundle morphism include identification via stabilization.

There is an obvious map from tangential PL smoothings to ordinary PL smoothings in the sense of Hirsch and Mazur [11] or Lashof and Rothenberg [18]. Namely, one forgets the bundle isomorphism φ. One can apply the Cairns–Hirsch theorem as in [11; 18] to extend the basic result.

Classes of PL smoothings $(M)=[M,PD/0]$.

THEOREM 3.8. *The equivalence classes of tangential PL smoothings of M are in* 1–1 *correspondence with the group of homotopy classes* $[M,PL]$. *Furthermore, under this correspondence the forgetful map from tangential smoothings corresponds (up to sign) to the homomorphism induced by the composite*

$$PL\subseteq PD\to PD/0.$$

REMARK. The objects PL and PD are defined in [11] or [18]. A basic result states that the inclusion of PL in PD is a weak homotopy equivalence.

The following result establishes a crucial relationship between PL and the tangential surgery sequence.

THEOREM 3.9. *Let M^n $(n \geq 6)$ be a compact smooth manifold. Then there is a commutative diagram as follows:*

$$\begin{array}{ccccccc}
\longrightarrow [SM, F] & \xrightarrow{\beta} & [SM, F/PL] & \xrightarrow{\Delta} & [M, PL] & \longrightarrow & [M, F] \\
\downarrow\cong & & \downarrow\sigma & & \downarrow H & & \downarrow\cong \\
\longrightarrow \{SM, S^0\} & \xrightarrow{\beta'} & L_{n+1}(\pi_1 M) & \xrightarrow{\Delta'} & hS^t(M^n) & \longrightarrow & \{M, S^0\}.
\end{array}$$

The top row is an exact sequence of abelian groups, the bottom row is the surgery sequence, the map σ is a surgery obstruction homomorphism, and H is a forgetful map that takes a tangential PL smoothing to a tangential homotopy smoothing.

There is a corresponding relative version, say if one considers PL and homotopy smoothings equivalent to the standard smoothing on the boundary; one must replace M by $M/\partial M$ in the top row, in the middle groups of the bottom row one must replace M by $(M, \partial M)$, and $\{S^k M, S^0\}$ must be replaced by $\{S^k(M/\partial M), S^0\}$.

The proof of this result is standard (cf. Sullivan [33]; see [27] for a related reference in print).

We now return to the proof of (3.5) for $q=3$. Consider the relative version of Theorem 3.9 with $M = D^3 \times \mathbf{C}P^2$. The quotient $M/\partial M$ is given by $S^3\mathbf{C}P^2_+$ (= Thom space of the trivial 3-space bundle over $\mathbf{C}P^2$). It follows that the diagram in Theorem 3.9 may be rewritten in more quantitative terms:

$$\begin{array}{ccccccc}
0 \longrightarrow & \pi_4(F/PL) \oplus \pi_8(F/PL) & \longrightarrow & \pi_3(PL) \oplus \pi_7(PL) & \longrightarrow & \pi_3^S \oplus \pi_7^S \longrightarrow 0 \\
& \downarrow\sigma & & \downarrow H & & \downarrow\cong \\
0 \longrightarrow & L_8(1) \cong \mathbf{Z} & \longrightarrow & hS_3^t(\mathbf{C}P^2) & \longrightarrow & \pi_3^S \oplus \pi_7^S \longrightarrow 0.
\end{array}$$

The map β' must be trivial because the domain is finite and the codomain is torsion-free. The group $[S^4\mathbf{C}P^2_+, F/PL]$ is isomorphic to $\pi_4 \oplus \pi_6 \oplus \pi_8(F/PL)$ by Sullivan's results on the homotopy type of F/PL. Clearly σ is trivial on the π_6 summand, and the image of β is also contained in this summand because $\pi_{4s}(F/PL) \cong \mathbf{Z}$. Furthermore, one can use the surjectivity of $\pi_6(F) \to \pi_6(F/PL)$ and obstruction theory to show that β maps onto the torsion subgroup.

It is immediate from the diagram that $hS_3^t(\mathbf{C}P^2)$ is the quotient of $\pi_3(PL) \oplus \pi_7(PL)$ by the image of σ.

Standard calculations (cf. Williamson [39]) show that $\pi_3(0) \cong \pi_3(PL) \cong \mathbf{Z}$ and $\pi_7(PL) \cong \mathbf{Z} \oplus \mathbf{Z}_4$, where the image of $\pi_7(0)$ is the subgroup $7\mathbf{Z} \oplus \{0\}$ (see [39, pp. 28–29]). Furthermore, the following hold:

(i) σ restricted to $\pi_4(F/PL) \cong \mathbf{Z}$ is multiplication by 2,

(ii) σ restricted to $\pi_8(F/PL) \cong \mathbf{Z}$ is the identity,

(iii) the map $\pi_4(F/PL) \to \pi_3(PL)$ corresponds to multiplication by 24 (recall the domain and codomain are infinite cyclic), and

(iv) the map $\pi_8(F/PL) \to \pi_7(PL)$ sends the generator to $(60, 1) \in \mathbf{Z} \oplus \mathbf{Z}/4 \cong \pi_7(PL)$.

All of these except (iv) are well known, and (iv) follows by an elementary diagram chase involving the diagram below:

$$\begin{array}{ccccc} \Omega F/0 & \longrightarrow & 0 & \longrightarrow & F \\ \downarrow & & \downarrow \bar{\Psi} & & \downarrow = \\ \Omega F/PL & \longrightarrow & PL & \longrightarrow & F. \end{array}$$

Therefore H is surjective, and the kernel of H corresponds to all elements of the form

$$(24y; 120y, 2y) \in \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/4 \cong \pi_3(PL) \oplus \pi_7(PL).$$

In other words, $hS_3^t(\mathbf{C}P^2)$ is generated by generators A, B, C subject to relations $4C = 0$, $24A + 120B + 2C = 0$. It follows that $hS_3^t(\mathbf{C}P^2)$ is isomorphic to $\mathbf{Z} \oplus (\mathbf{Z}/48)$, and the torsion is generated by $4A + 5B$.

We are interested in the image of the 2-torsion $\psi \in \pi_3(F_{S^1}(\mathbf{C}^3))$ in $hS_3^t(\mathbf{C}P^2)$. By construction this class has order at most 2, and therefore by the preceding paragraph the image of ψ corresponds to some multiple of $96A + 120B$. If we can show that the latter lies in the image of the composite

$$[S^3\mathbf{C}P_+^2, 0] \to [S^3\mathbf{C}P_+^2, PL] \to hS_3^t(\mathbf{C}P^2)$$

(i.e., classes represented by reframing the identity); then it will follow that the homotopy equivalence corresponding to ψ is homotopic to a diffeomorphism. However, earlier in the proof we observed that the image of this map is generated by A and $7B$. Since $24A + 120B$ has order 2, we clearly have

$$24A + 120B = 7 \cdot 24A + 7 \cdot 120B,$$

and from this description it is clear that the element of $hS_3^t(\mathbf{C}P^2)$ with order 2 lies in the image of $[S^3\mathbf{C}P_+^2, 0]$. $\square$

We are finally ready to prove Theorem II. Suppose that one does have a smooth semifree S^1 action on some homotopy $(2q+3)$-sphere with Σ^3 as fixed point set, where $\mu(\Sigma^3) = 1$ and $q \geq 3$ is odd. By (3.1)–(3.3) there is a diffeomorphism

$$\varphi: S^3 \times \mathbf{C}P^{q-1} \# \Sigma' \to S^3 \times \mathbf{C}P^{q-1}$$

such that $\varphi' h^{-1}$ is homotopic to the identity. This will still suffice to show that Σ' lies in the homotopy inertia group of $S^3 \times \mathbf{C}P^2$, and therefore by Taylor's result we shall have reached a contradiction.

In any case it follows that the homotopy smoothings $(S^3 \times \mathbf{C}P^{q-1} \# \Sigma', h)$ and $(S^3 \times \mathbf{C}P^{q-1}, \varphi h^{-1})$ define the same element of $hS_3(\mathbf{C}P^{q-1})$. Therefore φh^{-1} must be normally cobordant to the identity.

Suppose now that $q > 3$. Then Proposition 3.5 implies that the class in

$$\pi_3(F_{S^1}(\mathbf{C}^q)) \cong \pi_3(U_q) \oplus (\mathbf{Z}/2)$$

corresponding to φh^{-1} must lie in the image of $\pi_3(U_q)$. Because all classes of the latter type have representatives that correspond to self-diffeomorphisms of

$S^3 \times \mathbf{C}P^{q-1}$, it follows that we can find a self-diffeomorphism φ'' of $S^3 \times \mathbf{C}P^{q-1}$ so that $\varphi''\varphi h^{-1}$ is homotopic to the identity. Therefore we may set $\varphi' = \varphi''\varphi$.

If $q=3$ two changes are necessary. First, Proposition 3.5 yields no restriction on the homotopy type of φh^{-1}. However, this result also states that every homotopy class is represented by a diffeomorphism. Therefore it is still possible to find φ'' as in the previous paragraph, and of course one sets $\varphi' = \varphi''\varphi$ once again. □

REMARK. The normal invariant formula in (3.6) can also be derived from results of Novikov [25] on the normal invariants of self-equivalences of manifolds $M^n \to M^n$ obtained by twisting the identity map via an element of $\pi_n(M^n)$ with degree zero; that is, if $\alpha \in \pi_n(M^n)$ is such an element, one constructs the composite

$$M \xrightarrow{d} M \vee S^n \xrightarrow{\alpha \vee 1} M \vee M \xrightarrow{\text{fold}} M,$$

where d is the map which collapses a nicely embedded $(n-1)$-sphere inside some coordinate neighborhood. This requires the description of the $\pi_*^S(S^1)$ summand of $\pi_*(F_{S^1})$ in [28].

4. Proof of Theorem III. The argument is entirely parallel to that of Section 3, so we shall merely explain the necessary changes. First, in (3.1) one obtains a diffeomorphism

$$\varphi: S^7 \times \mathbf{C}P^{q-1} \# \Sigma' \to S^7 \times \mathbf{C}P^{q-1},$$

where Σ' represents a generator of bP_{2q+6}. Wherever one sees S^3 or D^4 in (3.1)–(3.3), one should substitute S^7 or D^8, and similarly for other corresponding changes (such as π_7 instead of π_3). The groups $\pi_7(F_{S^1}(\mathbf{C}^q))$ are stable because $q \geq 5$, and hence they may be computed as

$$\begin{aligned} \pi_7(S\mathbf{C}P_+^\infty) &\cong \pi_7(U) \oplus \pi_7^S(S^1) \\ &\cong \mathbf{Z} \oplus \mathbf{Z}_2. \end{aligned} \tag{4.1}$$

One then has the following analog of Proposition 3.5.

PROPOSITION 4.2. *Let ψ be a homotopy self-equivalence of $S^7 \times \mathbf{C}P^{q-1}$ ($q \geq 5$) that is induced by an element of $\pi_7(F_{S^1}(\mathbf{C}^q))$ not in the image of $\pi_7(U)$. Then ψ is not homotopic (in fact, not normally cobordant) to a diffeomorphism.*

Sketch of proof. Once again it suffices to consider the single element of order 2. Recall that π_6^S is generated by ν^2; i.e., the square of the Hopf map. One can use (3.6) once again to deduce that the normal invariant is equal to the image of $\nu^2 t = t(\nu^2 \wedge 1)$. Since $t \mid S^1 = \eta$ and $t \mid \{S\mathbf{C}P^1 = S^3\}$ is ν, it follows that the normal invariant's restriction in $[S^7\mathbf{C}P^1, F/0] \cong \pi_9(F/0)$ is the image of ν^3; but the latter is not zero. □

The proof of Theorem III may be completed as follows. Suppose there is a smooth semifree S^1 action on some homotopy $(2q+7)$-sphere with fixed point set Σ^7, where $q \geq 5$ is odd and Σ generates Θ_7. By the generalization of (3.1) we know that $S^7 \times \mathbf{C}P^{q-1} \# \Sigma'$ is diffeomorphic to $S^7 \times \mathbf{C}P^{q-1}$, and Proposition 4.2 implies that the diffeomorphism $\varphi: S^7 \times \mathbf{C}P^{q-1} \# \Sigma' \to S^7 \times \mathbf{C}P^{q-1}$ is such that the composite homeomorphism $\varphi h^{-1}: S^7 \times \mathbf{C}P^{q-1} \to S^7 \times \mathbf{C}P^{q-1}$ corresponds to some homotopy class in $\pi_7(F_{S^1}(\mathbf{C}^q))$. As in Section 3, it follows that there is a diffeomorphism $\varphi'\colon S^7 \times \mathbf{C}P^{q-1} \# \Sigma' \to S^7 \times \mathbf{C}P^{q-1}$ such that $\varphi' h^{-1}$ is homotopic to the identity. Once again this contradicts Taylor's theorem on the homotopy inertia group, and therefore there cannot be an S^1 action as described above. □

REFERENCES

1. J. C. Becker and R. E. Schultz, *Equivariant function spaces and stable homotopy theory* I, Comment. Math. Helv. 49 (1974), 1-34.
2. G. Bredon, *Introduction to compact transformation groups,* Academic Press, New York, 1972.
3. W. Browder, *Surgery and the theory of differentiable transformation groups.* Proceedings of the conference on transformation groups (New Orleans, La., 1967), 1-46, Springer, New York, 1968.
4. W. Browder and F. Quinn, *A surgery theory for G-manifolds and stratified sets,* Manifolds—Tokyo 1973, 27-36, Univ. Tokyo Press, Tokyo, 1975.
5. G. Brumfiel, *On the homotopy groups of* BPL *and* PL/0—I, Ann. of Math. (2) 88 (1968), 291-311.
6. ———, *Differentiable* S^1 *actions on homotopy spheres,* mimeo, Univ. of California, Berkeley, 1968.
7. ———, *Homotopy equivalences of almost smooth manifolds,* Comment. Math. Helv. 46 (1971), 381-407.
8. J. Eells and N. Kuiper, *An invariant for certain smooth manifolds,* Ann. Mat. Pura Appl. (4) 60 (1962), 93-110.
9. M. Freedman, *The topology of four-dimensional manifolds,* J. Differential Geom. 17 (1982), 357-453.
10. H. Gluck, *Piecewise linear groups and transformation groups,* Trans. Amer. Math. Soc. 149 (1970), 585-593.
11. M. Hirsch and B. Mazur, *Smoothings of piecewise linear manifolds,* Ann. of Math. Stud., 80, Princeton Univ. Press, Princeton, N.J., 1974.
12. F. Hirzebruch, *Topological methods in algebraic geometry,* 3rd ed., Springer, New York, 1966.
13. ———, *The topology of normal singularities of algebraic surface* (d'après Mumford), Séminaire Bourbaki 15ᵉ année 1962/1963, Exposé 250, 9 pp. Secrétariat mathématique, Paris, 1964. (Reprinted by W. A. Benjamin, New York, 1966).
14. F. Hirzebruch and K. H. Mayer, *O(n)-Mannigfaltigkeiten, exotische Sphären und Singularitäten,* Lecture Notes in Math., 57, Springer, Berlin, 1968.
15. W.-C. Hsiang and W.-Y. Hsiang, *Differentiable actions of compact connected classical groups* I, Amer. J. Math. 89 (1967), 705-786.
16. W.-Y. Hsiang, *On the unknottedness of the fixed point sets of differentiable circle group actions on spheres—P. A. Smith conjecture,* Bull. Amer. Math. Soc. 70 (1964), 678-680.
17. M. Kervaire and J. Milnor, *Groups of homotopy spheres,* Ann. of Math. (2) 77 (1963), 504-537.
18. R. Lashof and M. Rothenberg, *Microbundles and smoothings,* Topology 3 (1965), 357-388.

19. I. H. Madsen and R. J. Milgram, *The classifying spaces for surgery and cobordism of manifolds,* Ann. of Math. Stud., 92, Princeton Univ. Press, Princeton, N.J., 1979.
20. I. Madsen, L. Taylor, and B. Williams, *Tangential homotopy equivalences,* Comment. Math. Helv. 55 (1980), 445–484.
21. J. Milnor, *Lectures on the h-cobordism theorem,* Princeton Univ. Press, Princeton, N.J., 1965.
22. E. E. Moise, *Geometric topology in dimensions 2 and 3,* Springer, New York, 1977.
23. D. Montgomery and H. Samelson, *Examples for differentiable group actions on spheres,* Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1202–1205.
24. J. R. Munkres, *Obstructions to the smoothing of piecewise differentiable homeomorphisms,* Ann. of Math. (2) 72 (1960), 521–554.
25. S. P. Novikov, *Homotopically equivalent smooth manifolds* (Russian), Izv. Akad. Nauk. SSSR Ser. Mat. 28 (1964), 365–474; English translation, Amer. Math. Soc. Transl. (2) 48 (1965), 271–396.
26. C. P. Rourke and D. P. Sullivan, *On the Kervaire obstruction,* Ann. of Math. (2) 94 (1971), 397–413.
27. R. Schultz, *On the inertia group of a product of spheres,* Trans. Amer. Math. Soc. 156 (1971), 137–153.
28. ———, *Homotopy decompositions of equivariant function spaces* I, Math. Z. 131 (1973), 49–75.
29. ———, *Smooth actions of small groups on exotic spheres.* Algebraic and geometric topology (Stanford, Calif., 1976), 155–160, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.
30. ———, *Differentiability and the P. A. Smith theorems for spheres:* I. Current trends in algebraic topology, Part 2 (London, Ont., 1981), 235–273, CMS Conf. Proc., 2, Amer. Math. Soc., Providence, R.I., 1982.
31. ———, *Exotic spheres admitting smooth circle actions with codimension four stationary sets.* Proceedings of the Northwestern homotopy theory conference (Evanston, Ill., 1982), 339–368, Contemp. Math., 19, Amer. Math. Soc., Providence, R.I., 1983.
32. ———, *Problems submitted to the A.M.S. Summer Research Conference on group actions.* Group actions on manifolds (Boulder, Colo., 1983), 513–568, Contemp. Math., 36, Amer. Math. Soc., Providence, R.I., 1985.
33. D. Sullivan, *Smoothing homotopy equivalences,* mimeo, Univ. of Warwick, 1966.
34. L. Taylor, *Divisibility of the index with applications to inertia groups and $H^*(G/0)$,* Notices Amer. Math. Soc. 21 (1974), A-464.
35. H. Toda, *Composition methods in homotopy groups of spheres,* Ann. of Math. Stud., 49, Princeton Univ. Press, Princeton, N.J., 1962.
36. C. T. C. Wall, *Surgery on compact manifolds,* Academic Press, London, 1970.
37. S. Weinberger, *Constructions of group actions: a survey of recent developments.* Group actions on manifolds (Boulder, Colo., 1983), 269–298, Contemp. Math., 36, Amer. Math. Soc., Providence, R.I., 1985.
38. J. H. C. Whitehead, *Manifolds with transverse fields in Euclidean space,* Ann. of Math. (2) 73 (1961), 154–212.
39. R. E. Williamson, *Cobordism of combinatorial manifolds,* Ann. of Math. (2) 83 (1966), 1–33.
40. H. E. Winkelnkemper, *Manifolds as open books,* Bull. Amer. Math. Soc. 79 (1973), 45–51.

41. ———, *On the action of* Θ_n, Trans. Amer. Math. Soc. 206 (1975), 339–346.

Department of Mathematics
Purdue University
West Lafayette, IN 47907