A COVERING LEMMA FOR MAXIMAL OPERATORS WITH UNBOUNDED KERNELS

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I. Introduction. Calderon and Zygmund [1] proved that certain maximal operators are bounded on $L^p(\mathbf{R}^n)$ for $p>1$, using the rotation method. It is unknown whether they take $L^1(\mathbf{R}^n)$ into Weak $L^1(\mathbf{R}^n)$. We prove a positive result for a certain subclass of these operators. The method is to prove an analog of the usual covering lemma [4], even though the kernels are unbounded.

More specifically, let $g(\theta)$ be a positive, integrable, decreasing function on the interval $(0,1)$ such that $\theta g(\theta)$ is increasing. For $(x_1,x_2)=x\in\mathbf{R}^2$, set

$$\Omega(x)=\begin{cases} g(x_2/x_1) & \text{if } 0<x_2<x_1 \text{ and } |x|\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

For $r>0$, let $\Omega_r(x)=r^{-2}\Omega(x/r)$. Define, for $f\in L^1(\mathbf{R}^2)$,

$$M_\Omega f(x)=\sup_{r>0}(\Omega_r*|f|)(x)=\sup_{r>0}\int_{\mathbf{R}^2}\Omega_r(x-y)\,|f(y)|\,dy.$$

THEOREM. *M_Ω is weak-type $(1,1)$. That is, there is a constant C such that, for every $f\in L^1(\mathbf{R}^2)$ and every $\alpha>0$,*

$$|\{x\in\mathbf{R}^2=M_\Omega f(x)>\alpha\}|\le\frac{C}{\alpha}\|f\|_{L^1}\|g\|_{L^1}.$$

There is a similar result on $\mathbf{R}^n$, $n>2$, if $\theta g(\theta)$ is replaced by $\theta^{n-1}g(\theta)$ and $g(x_2/x_1)$ is replaced by $g(|x-(x_1,0,0,\dots,0)|/x_1)$, for $|x|\le 1$. Soria has proved such a result without restriction on $\theta g(\theta)$, but with a stronger size condition than $g\in L^1$ [3]. The idea of the proof is to use a covering lemma. However, the usual type of covering lemma does not apply because Ω may be an unbounded function. We will use the following substitute.

DEFINITION. $\Omega\in L^1(\mathbf{R}^2)$ has the selection property with constant C if, given any positive continuous function $r(x)$ defined on a measurable set $D\subseteq B_1(0)$, the unit ball of $\mathbf{R}^2$, there is a measurable subset $E\subseteq D$ such that

$$|E|\ge\frac{1}{2}|D|, \tag{1}$$

$$S(E,\Omega,r)(y)\equiv\int_E\Omega_{r(x)}(x-y)\,dx\le C \quad\text{for almost every } y\in\mathbf{R}^2. \tag{2}$$

Here, $|E|$ denotes the Lebesgue measure of E.

LEMMA. *If Ω has the selection property with constant C, then M_Ω is weak-type $(1,1)$.*

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Proof of Lemma. f and α are given. We may assume there is a continuous function $r(x)$ such that $M_\Omega f(x) = (\Omega_{r(x)} * |f|)(x)$ for all $x \in \mathbf{R}^2$. Set

$$D = \{x \in \mathbf{R}^2 : M_\Omega f(x) > \alpha\}.$$

By dilating, we may assume $D \subseteq B_1(0)$. By (1) and (2),

$$|D| \le 2|E| \le \frac{2}{\alpha} \int_E M_\Omega f(x)\, dx$$
$$= \frac{2}{\alpha} \int_{\mathbf{R}^2} |f(y)| \int_E \Omega_{r(x)}(x-y)\, dx\, dy \le \frac{2C\|f\|_1}{\alpha}. \qquad \square$$

It seems unlikely, but possible, that the lemma has a converse.

II. Proof of the Theorem. We will show that Ω has the selection property with constant $C \cdot \|g\|_{L^1}$. We may assume $\|g\|_{L^1} = 1$, so that $\|\Omega\|_{L^1} \le 1$. Also, we may assume that $r(x) > \epsilon > 0$ on D, for some ϵ.

Now, if the function $g(\theta)$ is replaced by a Dirac mass supported at 0, M_Ω is essentially the one-dimensional Hardy–Littlewood maximal operator, which has the selection property. More specifically, we claim that there is a measurable set $\tilde{D} \subseteq D$ such that $|\tilde{D}| \ge |D|/2$, and

$$\int \frac{dx_1}{r(x)} \le C' < \infty \quad y \in \mathbf{R}^2,$$

where the integral is over $\{x \in \tilde{D} : x_2 = y_2 \text{ and } y_1 < x_1 < r(x) + y_1\}$. The proof of this claim is nontrivial, but is contained in the following argument for the more general M_Ω and so we omit it. Now $\tilde{D}$ plays the role of D; we look for $E \subseteq \tilde{D}$ with $|E| \ge c|\tilde{D}| \ge c/2|D|$. It is irrelevant that $c/2 < 1/2$ (see condition (1)), because the argument can be repeated on $D_2 = D \setminus E$ to build a larger "E".

Let $\ell(q)$ be the side length of the dyadic square $q \subseteq \mathbf{R}^2$. We want to cover $\tilde{D}$ with disjoint squares q_k and let $E = \bigcup_k \{x \in \tilde{D} \cap q_k : r(x) > \ell(q_k)\}$. The q_k are chosen in stages. At stage $i \ge 0$, we have chosen all desired q_k such that $\ell(q_k) > 2^{-i}$ (so q won't be chosen if $\ell(q) > 1$).

Stage i. (Choosing q_k with $\ell(q_k) = 2^{-i}$); set

$$E_i = \text{points certain to belong to } E, \text{ at stage } i$$
$$\equiv \{x \in \tilde{D} : \exists q_k, \text{ chosen before stage } i,\ x \in q_k \text{ and } r(x) > \ell(q_k)\}$$
$$\bigcup \{x \in \tilde{D} : x \notin q_k,\ \forall \text{ chosen } q_k, \text{ and } r(x) > 2^{-i}\}.$$

We can't define E_{i+1} yet, but we will have $E_i \subseteq E_{i+1}$.

The square q is chosen into $\{q_k\}$ at stage i if its interior is disjoint from the chosen squares, if $\ell(q) = 2^{-i}$, and if one of the following holds:

(a) $S(E_i)(q) \equiv |q|^{-1} \int_q S(E_i, \Omega, r)(y)\, dy > \frac{1}{2}$;

(b) $|E_i \cap q|/|q| > \frac{1}{2}$;

(c) q touches some q_k (their boundaries intersect), where q_k has been chosen prior to stage i, or during stage i for reason (a) or (b).

(Condition (a) is the crucial one; condition (c) merely insures that adjacent q_k will have comparable sizes.) This completes stage i.

We repeat this for $i=0,1,2,\dots$ and define $E=\bigcup_{i=1}^{\infty} E_i$.

Claim 1. The $\{q_k\}$ cover $\tilde{D}$, a.e.

It is trivial that they cover $x\in\tilde{D}\setminus E$; $r(x)>2^{-i}$ for some i, so $x\notin E_i$ implies $x\in$ some q_k. Also, they cover each E_i, by Lebesgue's differentiation theorem and condition (b).

Claim 2. $|E|\geq c\sum_k |q_k|\geq c|\tilde{D}|$.

The first claim gives the second inequality. Let $k\in K_\alpha$ (resp. K_b, K_c) if q_k was chosen by condition (a) (resp. conditions (b), (c)). By simple geometry,

$$\sum_{k\in K_c} |q_k|\leq 25 \sum_{k\in K_a\cup K_b} |q_k|,$$

and

$$\sum_{k\in K_a} |q_k|\leq 2\int_{\mathbf{R}^2}\int_E \Omega_{r(x)}(x-y)\,dx\,dy=2|E|\cdot\|\Omega\|_1$$

by (a). And by (b),

$$\sum_{k\in K_b} |q_k|\leq 2\sum_{K_b} |q_k\cap E|\leq 2|E|.$$

These prove Claim 2. We must now prove condition (2) of the definition, that $S(E,\Omega,r)(y)\leq C$ on $\mathbf{R}^2$.

Choose y. We may assume $S(E,\Omega,r)(y)>1/2$. So $S(E_j,\Omega,r)(y)>1/2$ for some j (monotone convergence), and by the differentiation theorem, $S(E_j)(q)>1/2$ for some dyadic $q, y\in q$, $\ell(q)\leq 2^{-j}$. By condition (a), q is contained in a chosen square. Thus $y\in q_k$, a chosen square.

Let

$$H=\{x\in E\colon |x-y|\leq 10\cdot\ell(q_k)\}$$
$$A=\{x\in E\colon y_2\leq x_2\leq y_2+2\ell(q_k) \text{ and } y_1+2\ell(q_k)\leq x_1\}$$
$$B=E\setminus(H\cup A).$$

Therefore, $S(E,\Omega,r)(y)\leq S(H,\Omega,r)(y)+S(A,\Omega,r)(y)+S(B,\Omega,r)(y)$. H is the region near q_k, A is the region where $g((x_2-y_2)/(x_1-y_1))$ is large, and B is the large remaining region.

The estimate for H. We claim that if $x\in H$ and $\Omega_{r(x)}(x-y)\neq 0$, then $r(x)\geq \ell(q_k)/2$. This is obvious if $|x-y|\geq \ell(q_k)/2$. If $|x-y|<\ell(q_k)/2$, then condition (c) insures that $x\in$ some q_j with $\ell(q_j)\geq \ell(q_k)/2$. But since $x\in E$, $r(x)\geq \ell(q_j)$, and the claim is proved. So,

$$S(H,\Omega,r)(y)=\int_H \Omega_{r(x)}(x-y)\,dx$$
$$\leq C\int_H \Omega_{10\cdot\ell(q_k)}(x-y)\,dx\leq C\|\Omega\|_{L^1}.$$

The estimate for B. Let q_k^* be the (non-dyadic) square with the same center as q_k, and $\ell(q_k^*)=3\ell(q_k)$. Let $4\ell(q_k)=2^{-i}$ and set $B_i=B\cap E_i$. Since g decreases,

$$S(B_i)(q_k^*) > c \cdot S(B_i, \Omega, r)(y).$$

Also, there is an absolute constant c' and a dyadic square $\tilde{q}_k$ such that $\tilde{q}_k$ touches q_k (as in condition (c)), $\ell(\tilde{q}_k) = 2^{-i}$, and

$$S(B_i)(\tilde{q}_k) \geq c' \cdot S(B_i)(q_k^*).$$

By condition (c), $\tilde{q}_k$ is not, and is not contained in, a chosen square. Thus $S(B_i)(\tilde{q}_k) \leq 1/2$, by (a). We have shown that $S(B_i, \Omega, r)(y) \leq C$. The case of $x \in B \backslash B_i$ proceeds as for H; if $\Omega_{r(x)}(x-y) \neq 0$, then

$$2 \cdot \ell(q_k) \leq |x-y| \leq r(x) < 2^{-i} = 4\ell(q_k).$$

Therefore,

$$S(B \backslash B_i, \Omega, r)(y) \leq c \cdot S(B \backslash B_i, \Omega, 4\ell(q_k))(y) \leq C.$$

The estimate for A. For $x = (x_1, x_2) \in A$, let $\pi(x) = (y_1 + 2\ell(q_k), x_2)$ be its projection onto the left edge of A. The hypothesis that $\theta \cdot g(\theta)$ is increasing implies

$$\Omega_{r(x)}(y) \leq \frac{c\Omega_1(\pi(x) - y)}{2 \cdot \ell(q_k) r(x)}.$$

And, since $A \subseteq \tilde{D}$,

$$\int_{\{x \in A = x_2 = x' \text{ and } r(x) > x_1 - y_1\}} \frac{dx_1}{r(x)} < C$$

for each x', $y_2 < x' < y_2 + 2\ell(q_k)$. Thus,

$$\begin{aligned} S(A, \Omega, r)(y) &\leq \frac{C}{2\ell(q_k)} \int_{y_2}^{y_2 + 2\ell(q_k)} \Omega_1((y_1 + 2\ell(q_k), x') - y)\, dx' \\ &\leq C\|g\|_1 = C. \end{aligned}$$

This completes the proof of (2), and of the theorem. □

III. Remarks. 1. This theorem is a slight improvement on the observation of R. Fefferman and F. Soria, that M_Ω is weak-type (1.1) when g is decreasing and in $L \log L$ (see [3]). The latter result follows from the lemma of Stein and N. Weiss about summing weak-type operators [5]. Any decreasing h in $L \log L$ can be majorized by a g, as in the theorem. Any decreasing h in L^1 can be majorized by a g such that $\theta^{1+\epsilon} g(\theta)$ increases, for any preassigned $\epsilon > 0$. We have no reason to believe the theorem is false without these restrictions on g.

2. We can hope that the selection property will aid the study of other maximal operators, for example, those with lower-dimensional kernels. M. Christ has shown that the theorem of this paper can be proved without it, however.

IV. Sketch of M. Christ's proof. The motivation and notation may be found in Stein [4]. Given $\alpha > 0$, Calderon–Zygmund decompose $f = g' + \sum_k b_k$, where $\|g'\|_{L^\infty} \leq \alpha$ (so that g' may be discarded), and where $\int b_k = 0$ and $\operatorname{supp} b_k \subseteq q_k$. Define A, B, and H as before, using the lower left corner of q_k for y. Let q_k^* be a dilation of q_k about its center so that $\bigcup_k H_k \subseteq Q = \bigcup_k q_k^*$. Since $|Q| \leq c\|f\|_{L^1}/\alpha$, we may discard the H_k. Since $\int b_k = 0$, one can show that

$$\|\sup_{r>0}|\Omega_r(x-y)b_k(y)\,dy|\|_{L^1(B_k)}\le C\|b_k\|_{L^1}.$$

Summing over k, $|\{M_\Omega(\Sigma_{x\in B_k} b_k)(x)>\alpha\}|\le C\|f\|_1/\alpha$. A similar result for $x\in A_k$ proves the theorem; since $\theta g(\theta)$ increases,

$$\int\Omega_{r(x)}(x-y)\cdot|b_k(y)|\,dy\le\frac{c}{r(x)}\int_0^{100r(x)}a_k(x_1-t,x_2)\,dt$$

where $\|a_k\|_{L^1}\le c\|b_k\|_{L^1}$. Now, sum over k and note that the right-hand side is in Weak L^1 by the Hardy–Littlewood theorem on $\mathbf{R}^1$.

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