THE SAMELSON SPACE OF A FIBRATION

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Introduction. If G is a compact connected Lie group, then there exists a real graded vector space P_G such that $\Lambda(P_G)\cong H^*(G;\mathbf{R})$, where Λ denotes the exterior algebra. Moreover, if G acts smoothly on a connected manifold M, then there is a graded subspace $P\subset P_G$ and an algebra isomorphism $A\otimes\Lambda(P)\cong H^*(M;\mathbf{R})$ which makes the following diagram commutative:

$$\begin{array}{ccc} A\otimes\Lambda(P) & \longrightarrow & \Lambda(P_G) \\ \cong\downarrow & & \downarrow\cong \\ H^*(M;\mathbb{R}) & \xrightarrow{\ \omega^*\ } & H^*(G;\mathbb{R}) \end{array}$$

The map ω^* is induced by the orbit map $\omega: G\to M$, $\omega(g)=g\cdot x$ (for fixed $x\in M$) and $A\otimes\Lambda(P)\to\Lambda(P_G)$ denotes projection onto $\Lambda(P)$. (See [12, p. 312]).

The action of G on M gives rise to the Borel fibration $M\to MG\to BG$, and it is well known that the orbit map ω corresponds to the "transgression" $\partial:\Omega BG\to M$ via the homotopy equivalence $\Omega BG\simeq G$. The commutative diagram above then provides an isomorphism $\operatorname{Im}\partial^*\cong\Lambda(P)$.

Because these notions are extensions of the classical Lie theoretic approach of Samelson, we say that P is the Samelson subspace of the action.

It is natural to ask if analogous results hold for arbitrary fibrations $F\to E\to B$ and the associated "action" $F\times\Omega B\to F$. This question was answered in [16], where it was shown that F has a rational decomposition $\mathfrak{F}\times K$ with $K\subset\Omega B$ and $H^*(K)\cong\operatorname{Im}(\partial^*: H^*(F)\to H^*(\Omega B))$. The space K is called the *Samelson space* of the fibration because of the obvious analogy to the classical result stated earlier. In fact, the classical theorem is simply a special case of the rational decomposition described above.

The purpose of this paper is to present rational versions of various topological results within the unifying framework of the Samelson space method. In particular, we obtain an elementary proof of the Transgression Theorem [3] and a generalization of the Allday–Halperin inequality [1].

The main result of [16] forms the starting point for this paper, so we recall it in Section 1. Although minimal model theory was the fundamental tool of [16], it shall not be emphasized here. It is hoped that, by stating the results of this paper in customary topological language, a wider audience will be introduced to the efficacy of the Samelson space technique. Furthermore, with the exception of some results on rational holonomy [6] and on elliptic spaces ([13]; [4]), all the ingredients for the results of this paper were present years ago. It seems only right, then, to approach this work in the spirit of classical homotopy theory.

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In the first three sections all spaces are assumed to be rational and simply connected with finite betti numbers (unless stated otherwise). Therefore, it shall be understood that all homology, cohomology, and homotopy is to be taken with rational coefficients.

1. Basic structure theorems. We begin this section by recalling the main result of [16]. The proof given in [16] relies on minimal model theory, so we shall sketch a topological argument here.

RATIONAL FIBRE DECOMPOSITION THEOREM (RFDT). *If $F \to E \to B$ is a fibration, then there is a subproduct $K \subset \Omega B$ and a space $\mathfrak{F}$ such that $F \simeq \mathfrak{F} \times K$ and $H^*(K) \cong \operatorname{Im}(\partial^*: H^*(F) \to H^*(\Omega B))$.*

Sketch of Proof. Recall that, subject to our conventions, ΩB has the homotopy type of a product of Eilenberg-MacLane spaces $K(Q, n)$. Consider the composition

$$\pi_*(\Omega B) \xrightarrow{\partial_\#} \pi_*(F) \xrightarrow{h} H_*(F),$$

where $\partial_\#$ arises from the transgression in the dual Puppe sequence $\partial: \Omega B \to F$ and h is the Hurewicz map. Let the Samelson space K be the maximal subproduct of ΩB such that $h\partial_\#$ restricted to $\pi_*(K)$ is an isomorphism onto $\operatorname{Im}(h\partial_\#)$. By dualizing to cohomology and then representing the basis corresponding to $\operatorname{Im}(h\partial_\#)$, we obtain a map $F \to K$ which is surjective on π_*. If $\mathfrak{F}$ denotes the homotopy fibre then clearly $\pi_*(F) \cong \pi_*(\mathfrak{F}) \oplus \pi_*(K)$. Furthermore, the holonomy of the fibration, $F \times \Omega B \to F$, provides a composition

$$\mathfrak{F} \times K \to F \times \Omega B \to F$$

which induces an isomorphism on π_*. Hence $F \simeq \mathfrak{F} \times K$. The reader is referred to [16] for a proof of the final assertion of the theorem. □

Certain applications of the RFDT present themselves immediately. In particular, various global conditions on the fibre F impose corresponding restrictions on any such decomposition. As a typical example of a fibre condition, we recall that a space X is said to be *quasifinite* if its homology is finite dimensional. We then have the following obvious result.

COROLLARY 1. *If F is quasifinite, then the Samelson space K has the homotopy type of a product of odd spheres.*

Proof. K is a product of $K(Q, n)$'s, and it is well known that $H(K(Q, 2i))$ is a polynomial algebra and is therefore infinite dimensional. The assumption about F then implies that $K = \Pi K(Q, 2i+1) \simeq \Pi S^{2i+1}$. □

The simple observation of Corollary 1 has an interesting refinement due to Felix and Thomas. Because their proof is not readily available, we include it here.

COROLLARY 2 [6]. *If $H_*(F)$ is a finitely generated $H_*(\Omega B)$-module via the holonomy $F \times \Omega B \to F$, then $H^*(F)$ is nilpotent if and only if the Samelson space is a product of odd spheres.*

Proof. If $H^*(F)$ is nilpotent then, by the argument above, no $K(Q,2i)$ is included in K. Hence, $K=\Pi S^{2i+1}$.

Conversely, assume $K=\Pi S^{2i+1}$ and let $I_1=\operatorname{Ker}\partial^*$. Define ideals

$$I_n=I_{n-1}\cdot H^+(F)$$

and note that, because ∂^* is an $H_*(\Omega B)$-module map, each I_j is an $H_*(\Omega B)$-module. Denote the finitely generated $H_*(\Omega B)$-module $H_+(F)/\operatorname{Im}\partial_*$ by R and note that the submodules J_n orthogonal to I_n form a strictly increasing sequence. Because R is finitely generated there exists an N such that $J_N=R$. Hence, $I_N=0$. Now, $\operatorname{Im}\partial^*\cong H^*(K)\cong\Lambda(x_1,\dots,x_r)$ so $(\operatorname{Im}\partial^*)^{r+1}=0$. Therefore, $(H^+(F))^{r+1}\subset I_1$ and $(H^+(F))^{(r+1)N}=0$. □

EXAMPLE. The free loop space $\Lambda S^3=\operatorname{Maps}(S^1,S^3)$ splits as a product $S^3\times\Omega S^3$, so it is clear that $H^*(\Lambda S^3)$ is not nilpotent. In fact, the fibration obtained by pulling back the path fibration over $K(Q,4)\times S^3$ by the inclusion $S^3\to K(Q,4)\times S^3$ has Samelson space $K(Q,2)$.

COROLLARY 3 (cf. [10, Theorem 5-2]). *If all cup products in $H^*(F)$ vanish and F does not have the homotopy type of an odd sphere, then the Samelson space is trivial for every fibration having fibre F.*

EXAMPLES. (i) If $F=\Lambda S^2$, the free loop space on S^2, then the cup structure of $H^*(F)$ is trivial (although the Massey product structure is highly non-trivial). See [20] for details.

(ii) Let $F=(S^n\vee S^n)\cup_\alpha e^{3n-1}$, where $\alpha=[i_1,[i_1,i_2]]$.

REMARK. Corollary 3 has the following refinement: If all spherical cup products are trivial in $H^*(F)$ and F is not an odd sphere, then K is trivial. For instance, if F is the sphere bundle of the vector bundle over $S^3\times S^3$ obtained as a pullback of the tangent bundle of S^6 by a degree 1 map $S^3\times S^3\to S^6$, then F is a manifold of dimension 11 with a cohomology basis in degrees 3, 3, 8, 8, and 11. By Poincaré duality there are nontrivial cup products, but the two degree 3 elements are the only spherical co-cycles and their product is zero. Hence, the Samelson space for F always vanishes. (See [1] or [17, pp. 90, 115] for a minimal model description of F and a calculation of $H^*(F)$.)

Recall that the Gottlieb group (or evaluation subgroup) of a space X is defined by $G_*(X)=\bigcup\operatorname{Im}(\partial_\#:\pi_*(\Omega B)\to\pi_*(X))$, where the union is taken over all fibrations $X\to E\to B$. Clearly, then, the Samelson space is a spatial model for the complement of the kernel of the Hurewicz map restricted to $G_*(X)$. Hence, if K is trivial for any fibration having fibre X, then $G_*(X)\subset\operatorname{Ker}h$. The next result is then an easy consequence of the RFDT and is the rational part of Theorem 4-1 of [10].

COROLLARY 4. *If X is quasifinite and $\chi(X)\neq 0$, then $G_*(X)\subset\operatorname{Ker}h$.*

Proof. If K is nontrivial then $K=\Pi S^{2i+1}$, so $\chi(X)=\chi(Y)\cdot\chi(K)=0$ (where $X\simeq Y\times K$ is a fibre decomposition for some fibration). □

REMARK. In [5], Felix and Halperin showed that if X has finite category (e.g., if X is quasifinite) then $G_{2n}(X)=0$. A simple way to see this is the following: Let $X\to E\to B$ be any fibration with $\partial_\#\neq 0$ and take $K'\subset \Omega B$ such that

$$\partial_\#: \pi_*(K') \xrightarrow{\cong} \operatorname{Im}\partial_\# \subset \pi_*(X).$$

Now by the Mapping Theorem of [5] (which, in fact, also has an elementary proof along these lines), we have $\operatorname{cat} K' \leq \operatorname{cat} X < \infty$. Because $\operatorname{cat}(K(Q,2i))=\infty$, it is clear that $K'=\Pi S^{2i+1}$. Since this holds for arbitrary fibrations, the result follows.

If F is quasifinite and, for some fibration, the Samelson space is nontrivial, then (as shown in the proof of Corollary 4) $\chi(F)=0$. The fact that this result follows immediately from the RFDT indicates that an elementary proof of the various Transgression Theorems of Becker-Casson-Gottlieb may be obtained as well. The Transgression Theorem (see [3], [7], or [2]) was proved originally using the Lefschetz transfer for fibrations, but our proof relies only on the RFDT and the $H_*(\Omega B)$-module structure of $H_*(F)$ (see [6]). We note here, however, that the Samelson space technique yields only the rational portion of the Transgression Theorem. It is hoped that a modification of the method will allow a proof for any coefficients.

TRANSGRESSION THEOREM. *Let $F\to E\to B$ be a fibration with F quasifinite. Suppose that $f: E\to E$ is a fibre preserving map which induces the identity on B and $g: F\to F$. If the Lefschetz number of g is nonzero, then*

$$0=\partial^*: H^*(F)\to H^*(\Omega B).$$

Proof. We shall show that $\Lambda(g)\neq 0$ implies the triviality of the Samelson space. Suppose $F\simeq \mathfrak{F}\times K$ with $K\simeq \Pi S^{2i+1}$.

From the proof of the RFDT we see that the $H_*(K)$-module structure of $H_*(F)$ has the form $a\cdot(1\otimes b)=a\otimes b$, where $a\in H_*(K)$ and $b\in H_*(\mathfrak{F})$. This follows because the decomposition $F\simeq \mathfrak{F}\times K$ is accomplished through the holonomy $F\times\Omega B\to F$. Now, because (g,f,id) is a map of fibrations (which is the identity on the base), the naturality of the holonomy ([14, p. 98]) implies that g_* is a map of $H_*(\Omega B)$-modules. Therefore, for $a\in H_*(K)$ and $b\in H_*(\mathfrak{F})$ we have

$$g_*(a\otimes b)=g_*(a\cdot(1\otimes b))=a\cdot g_*(1\otimes b).$$

Now let $a_i\otimes b_j\in H_p(K)\otimes H_q(\mathfrak{F})\subset H_n(F)$ be a basis element $(p+q=n)$, and note that

$$g_*(a_i\otimes b_j)=a_i\cdot g_*(1\otimes b_j)=a_i\cdot \sum_{k,\ell}\gamma_{0j}^{k\ell}a_k\otimes b_\ell,$$

where $a_k\otimes b_\ell$ are basis elements with $a_k\in H_*(K)$, $b_\ell\in H_*(\mathfrak{F})$. Because we are interested in computing the trace of g_n, we need consider only the term with $k=0$, $\ell=j$ (i.e., $1\otimes b_j$). The coefficient $\gamma_{0j}^{0j}=\gamma_j$ occurs in the trace of

$$\bar{g}_*: H_q(\mathfrak{F})\longrightarrow H_q(F)\xrightarrow{g_*} H_q(F)\xrightarrow{pr} H_q(\mathfrak{F})$$

and, since a_i has no effect on γ_j, is the same for each basis element of $H_p(K)$. Therefore, we have

$$\operatorname{tr}(g_n) = \sum_{p=0}^{\infty} \dim H_p(K) \cdot \operatorname{tr}(\bar{g}_{n-p}).$$

Therefore, the Lefschetz number is computed as

$$\begin{aligned}
\Lambda(g) &= \sum_{n=0}^{\infty} (-1)^n \operatorname{tr} g_n \\
&= \sum_{n=0}^{\infty} (-1)^n \sum_{p=0}^{\infty} \dim H_p(K) \cdot \operatorname{tr} \bar{g}_{n-p} \\
&= \sum_{p=0}^{\infty} \sum_{n=p}^{\infty} (-1)^p \dim H_p(K) \cdot (-1)^{n-p} \operatorname{tr} \bar{g}_{n-p} \\
&= \sum_{p=0}^{\infty} (-1)^p \dim H_p(K) \cdot \sum_{n-p=0}^{\infty} (-1)^{n-p} \operatorname{tr} \bar{g}_{n-p} \\
&= \chi(K) \cdot \Lambda(\bar{g}).
\end{aligned}$$

If K is nontrivial, then $\chi(K) = 0$. Therefore, $\Lambda(g) = 0$ as well. This contradiction completes the proof. □

REMARK. The argument given above applies to the E_2-term of the Serre spectral sequence for a map of fibrations (g, f, id). If the Lefschetz number of E is defined, then the Hopf Trace Theorem yields the formula $\Lambda(f) = \chi(K) \cdot \Lambda(g)$. See [3] for applications of this formula.

The Transgression Theorem may be applied to the action of a compact Lie group on a quasifinite space. If an equivariant self-map of the space is provided, then there is induced a self-map of the associated Borel fibration which restricts to the identity on the base. Hence, according to the Transgression Theorem, if the Lefschetz number of the equivariant self-map is nonzero then the orbit map is homologically trivial. (See [3] for example.) As in the introduction, this result uses the correspondence between the orbit map and the transgression provided by the homotopy commutative diagram,

$$\begin{array}{ccc}
G & \xrightarrow{\ \omega\ } & F \\
\wr\!\wr & \nearrow_{\delta} & \\
\Omega BG & &
\end{array}$$

Recently [11], Gottlieb has defined a new invariant of a group action called the *trace* of the action, denoted by $\operatorname{tr}(G, F)$. The following result generalizes the remarks above.

TRACE THEOREM [11]. *If G is a Lie group which acts on a compact manifold F, then $\operatorname{tr}(G, F) \neq 0$ implies $\omega_* = 0 : H_*(G) \to H_*(F)$.*

From our previous discussion, it should not be surprising that the Samelson space approach applies to this situation as well. In actuality, because the definition of the trace is somewhat complicated, we shall prove a version of Gottlieb's result in which the trace is replaced by an invariant known as the fibre number.

Gottlieb demonstrates that if G is a compact, connected Lie group which acts on a closed manifold F in an orientation preserving manner, then the trace $\operatorname{tr}(G,F)$ is equal to the fibre number $\Phi(G,F)$.

We now give the definition of the fibre number. If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration (of integral spaces) with $H^i(F;\mathbf{Z})=0$ for $i>n$ and $H^n(F;\mathbf{Z})\cong\mathbf{Z}$, then

$$\operatorname{Im}(i^*\colon H^n(E;\mathbf{Z})\to H^n(F;\mathbf{Z}))$$

is a subgroup of $\mathbf{Z}$ with generator Φ (non-negative). We write $\Phi=\Phi(p)$ and say that $\Phi(p)$ is the fibre number of the fibration p. For the Borel fibration $F\to FG\to BG$ associated to an action of G on F, the fibre number is denoted by $\Phi(G,F)$. Finally we note that, although the fibre number is an integral invariant, rational homotopy methods are still well suited to determine whether it is trivial or not. In the following we return to our rational conventions.

FIBRE NUMBER THEOREM. *If $\Phi(G,F)\neq 0$, then $\omega_*=0\colon H_*(G)\to H_*(F)$.*

Proof. First, note that we have the usual equivalences $G\simeq\prod_I K(Q,2i-1)$, $BG\simeq\prod_I K(Q,2i)$. We may apply the RFDT to $F\to FG\to BG$, the Borel fibration associated to the action, to obtain the Samelson space K with $\operatorname{Im}\omega_*\cong H_*(K)$. By construction, $K\simeq\prod_J K(Q,2j-1)$ with $J\subset I$. Define $BK=\prod_J K(Q,2j)$ in imitation of the classifying space BG and note that there is an "inclusion" $BK\xrightarrow{\epsilon}BG$ such that $K\simeq\Omega BK\to\Omega BG\simeq G$ is the Samelson space inclusion. We may use ϵ to construct a pullback diagram of fibrations:

$$\begin{array}{ccc}
K & \longrightarrow & G \\
{\scriptstyle\bar\omega}\downarrow & & \downarrow{\scriptstyle\omega} \\
F & = & F \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\
FK & \xrightarrow{\ r\ } & FG \\
\downarrow & & \downarrow \\
BK & \xrightarrow{\ \varepsilon\ } & BG
\end{array}$$

We know that $\bar\omega^*$ is surjective (by the construction of K), so it is immediate that the fibration

$$K\xrightarrow{\bar\omega}F\xrightarrow{j}FK$$

is totally noncohomologous to zero. Hence, the Leray–Hirsch theorem implies that $H^*(F)\cong H^*(FK)\otimes H^*(K)$ as $H^*(FK)$-modules. This isomorphism is in fact an algebra isomorphism, because the surjection $\bar\omega^*$ has an *algebra* splitting $H^*(K)\to H^*(F)$ induced by the map $F\to K$ constructed in the proof of the RFDT.

Suppose K is nontrivial. Then, if $H^n(FK)$ were nontrivial, by taking a nonzero element and forming the cup product with a nonzero element of $H^*(K)$ we would obtain a nonzero element of $H^*(F)$ in a degree greater than n. Since $H^i(F)=0$ for $i>n$ this is impossible, so $H^n(FK)=0$. Now, the pullback diagram provides the equality $i^*=j^*r^*$, so in degree n we must have $i^*=0$. By definition of the fibre number, $\Phi(G,F)=0$, in contradiction to assumption. Thus K is trivial and $\omega_*=0$. □

The following simple consequence of the RFDT provides an example of the effect of the vanishing of the Samelson space on the structure of a fibration.

PROPOSITION 5. *Let $F \to E \to B$ be a fibration with F ℓ-connected, B k-connected and $k \geq \ell$. Suppose $H_j(F) = 0$ for $j \geq k + \ell + 1$. If the Samelson space is trivial, then the fibration is totally nonhomologous to zero.*

Proof. The Serre sequence and the homology suspension provide a commutative diagram,

$$\begin{array}{ccccccccc} & & & \overset{\cong}{\swarrow} & H_{i-1}(\Omega B) & \searrow^{\delta^*} & & & \\ \to H_i(E) & \to & H_i(B) & & \longrightarrow & & H_{i-1}(F) & \to & H_{i-1}(E) \to \end{array}$$

for $i \leq k + \ell + 1$. Now $\partial_* = 0$ because the Samelson space is trivial, so

$$H_*(F) \to H_*(E)$$

is injective by exactness. □

COROLLARY 6. *If $\chi(F) \neq 0$, then any fibration with fibre F satisfying the conditions above is totally nonhomologous to zero.*

2. The homotopy Euler characteristic. If $\pi_*(X)$ is finite dimensional, then the *homotopy Euler characteristic* is defined to be

$$\chi_\pi(X) = \sum_{i \geq 1} (-1)^i \dim \pi_i(X).$$

If $\pi_*(X)$ and $H_*(X)$ are both finite dimensional, then X is called *elliptic* and there are strong restrictions on its structure (see [13]). In [1], Allday and Halperin applied the Borel fibration to study compact Lie group actions on elliptic spaces and obtained the inequality $\chi_\pi(X) \leq -\mathrm{rk}(G)$. Here, because $G \simeq \Pi S^{2i+1}$, the rank of G may be defined by

$$\mathrm{rk}(G) = -\chi_\pi(G) = (\#\text{ of factors in } \Pi S^{2i+1} \simeq G).$$

Of course, the definition of rank may be applied to any space $K \simeq \Pi S^{2i+1}$ and it is natural to ask if the Allday–Halperin inequality may be generalized, in terms of the Samelson space, to arbitrary fibrations.

THEOREM 7. *If F is elliptic, then for any fibration with fibre F,*

$$\chi_\pi(F) \leq -\mathrm{rk}(K).$$

Proof. First note that $\mathrm{rk}(K)$ is defined for the Samelson space because F is, in particular, quasifinite. The product decomposition $F \simeq \mathfrak{F} \times K$ yields the equality $\chi_\pi(F) = \chi_\pi(\mathfrak{F}) + \chi_\pi(K)$. Because F is elliptic, so is $\mathfrak{F}$ and by [13], $\chi_\pi(\mathfrak{F}) \leq 0$. Hence, $\chi_\pi(F) \leq \chi_\pi(K) = -\mathrm{rk}(K)$. □

COROLLARY 8. *If F is elliptic and $\chi_\pi(F) = 0$, then the Samelson space is trivial for every fibration with fibre F.*

REMARKS. (i) An alternative proof of Corollary 8 consists of using Halperin's observation that $\chi_\pi(F)=0$ if and only if $\chi(F)>0$, and then applying the results of Section 1.

(ii) The assumption of ellipticity is essential for Theorem 7, as is shown by the example of ΛS^3. Clearly, $\chi_\pi(\Lambda S^3)=0$, but the Samelson space may be nontrivial (as we have seen earlier).

By combining Corollary 8 with Proposition 5 we obtain the following result.

PROPOSITION 9. *Let F be ℓ-connected and elliptic with $\chi_\pi(F)=0$, and suppose $H_n(F)$ is the top dimensional homology. Then any fibration with fibre F over an $(n-\ell)$-connected base is totally nonhomologous to zero. In particular, if F is* 1*-connected, then any fibration over S^n is totally nonhomologous to zero.*

Recall that a space is *formal* if its homotopy type is determined by its cohomology algebra. A more precise minimal model definition may be found in [21], for example. Many homogeneous spaces are known to be formal. Since they are elliptic as well, it should not be surprising that arbitrary formal elliptic spaces should behave very much like their homogeneous counterparts. We verify this in the case of the following result.

THEOREM 10. *If F is elliptic, then F is formal if and only if $\chi_\pi(F)=-\mathrm{rk}(K)$ for some fibration with fibre F.*

Proof. Suppose $\chi_\pi(F)=-\mathrm{rk}(K)$ for some fibration. Then, since $F\simeq\mathfrak{F}\times K$, we must have $\chi_\pi(\mathfrak{F})=0$. By the main theorem of [13], this implies that $\mathfrak{F}$ is formal. Now $K\simeq\Pi S^{2i+1}$ is clearly formal and any product of formal spaces is formal, so F must be formal as well.

Conversely, suppose F is formal. From [4] it is known that F is hyperformal; that is, $H^*(F)$ is a polynomial algebra truncated by a Borel ideal. This characterization of F implies that it is amenable to the techniques of [12, Chapter II, §§4, 5]. From these methods we obtain a decomposition $F\simeq\mathfrak{F}\times K$, where K is a certain (maximal) product of odd spheres. Now, by [12, Theorem XI, p. 152], the formality of F implies that $\chi_\pi(F)=-\mathrm{rk}(K)$.

Finally, we need only realize K as a Samelson space for some fibration having F as fibre. Simply take the product fibration

$$\mathfrak{F}\times K\to\mathfrak{F}\times P\bar{K}\to *\times\bar{K},$$

where $\bar{K}=\Pi K(Q,2j)$ such that $K\simeq\Omega\bar{K}$ and $P\bar{K}\to\bar{K}$ is the path fibration. □

COROLLARY 11. *If $\chi_\pi(F)=-1$, then F is formal if and only if the Samelson space is nontrivial for some fibration with fibre F.*

EXAMPLES. (i) The homogeneous spaces of [12, Chapter XI, §4] satisfy $\chi_\pi=-\mathrm{rk}(K)$ for the fibration described in the proof of the theorem. Because $\chi_\pi\neq 0$ in general we obtain many examples of spaces with nontrivial Samelson spaces. In particular, $U(n)/U(k)$ is formal, so it has a nontrivial Samelson space for some fibration.

(ii) On the other hand, the homogeneous spaces of [12, Chapter XI, §5] have $\chi_\pi = -1$, but are not formal. By Corollary 11, their Samelson spaces must always be trivial.

(iii) By minimal model methods, a nonformal 7-manifold with $\chi_\pi = -1$ may be constructed as the orbit space of an almost free circle action on $X = S^2 \times S^3 \times S^3$. For the reader familiar with this approach we outline the construction. Model a Borel fibration

$$X \to X \times_{S^1} ES^1 \to BS^1$$

by the KS extension (with differentials below):

$$\begin{array}{lllll}
\Lambda(e) & \to \Lambda(e,x,y,z,w) & \to & \Lambda(x,y,z,w) & \\
de = 0 & De = 0 = Dx & & dx = 0 & dy = x^2 \\
|e| = 2 & Dy = x^2 & & dz = 0 & dw = 0 \\
 & Dz = e^2 & & |x| = 2 & \\
 & Dw = ex & & |y| = |z| = |w| = 3. &
\end{array}$$

Because $Dz = e^2$, the map $H^*(BS^1) \to H^*(X \times_{S^1} ES^1)$ is not injective. Hence, there are no fixed points for the associated circle action on X, and therefore S^1 acts almost freely (i.e., with finite isotropy). The Vietoris–Begle theorem then provides a (rational) equivalence $X \times_{S^1} ES^1 \simeq X/S^1$. Note that $\chi_\pi = -1$, but the nonzero Massey product $ew - zx$ shows that X/S^1 is not formal. Furthermore, it can be shown that X/S^1 has the (rational) homotopy type of a manifold of dimension 7. This is, of course, the first dimension in which a nonformal simply connected manifold may occur.

3. Fibred suspensions and coformal maps. In [8], Gottlieb studied the nonrational situation of a fibration $F \to E \to B$ in which F and B are compact CW complexes and E is a suspension ΣX. His main theorem (restricted to the rationals) may be restated as follows.

THEOREM. *If the compact fibration $F \to \Sigma X \xrightarrow{\alpha} B$ is nontrivial, then*

$$(\Omega\alpha)_*: H_*(\Omega\Sigma X; Q) \to H_*(\Omega B; Q)$$

is injective.

COROLLARY. *The transgression $\partial^*: H^*(F; Q) \to H^*(\Omega B; Q)$ is injective.*

Proof. If $(\Omega\alpha)_*$ is injective, then the fibration $\Omega\Sigma X \to \Omega B \xrightarrow{\partial} F$ is totally nonhomologous to zero. By the Leray–Hirsch theorem,

$$H^*(\Omega B; Q) \simeq H^*(\Omega\Sigma X; Q) \otimes H^*(F; Q)$$

as modules, so ∂^* is clearly injective. □

Now let us consider the rational implications of these results. If F decomposes nontrivially as $\mathfrak{F} \times K$, then it is plain that Ker $\partial^* \neq 0$. Hence $\mathfrak{F} \simeq *$, and we have the following.

COROLLARY 12. *$F \simeq K$. Hence $\chi(F)=0$.*

THEOREM 13. *If $F \xrightarrow{i} \Sigma X \xrightarrow{\alpha} B$ is a fibration with F and B nontrivial and quasi-finite, then the inclusion of the fibre is inessential.*

Proof. Because $F \simeq K$, there is a map $\sigma: F \to \Omega B$ with $\partial\sigma \simeq \mathrm{id}$. Then $i \simeq i(\partial\sigma) = (i\partial)\sigma \simeq *$, since $i\partial \simeq *$. □

REMARK. The "usual" way to construct fibrations $F \to \Sigma X \to B$ is to apply the Hopf construction. In particular, if G is an H-space then the multiplication $G \times G \xrightarrow{\mu} G$ induces

$$G \xrightarrow{i} G * G \xrightarrow{m} \Sigma G,$$

where $*$ denotes the join and $m(\langle g, t, h\rangle) = \langle \mu(g,h), t\rangle$. This fibration is of the desired type since $G * G \simeq \Sigma(G \wedge G)$. In this case it is clear that, even integrally, the inclusion of the fibre is inessential. Theorem 13 expresses the fact that, rationally, this is the general case. There are examples (see [8]) of integral fibrations $F \to \Sigma X \to B$ with essential fibre inclusions, but they are constructed from Moore spaces $M(G,n)$ with G finite and therefore rationally trivialize.

We are now in a position to give a short proof of an integral result which was originally obtained as a consequence of the transfer for fibrations ([3]).

COROLLARY 14. *If*

$$F \xrightarrow{i} \Sigma X \xrightarrow{\alpha} B$$

is a nontrivial compact fibration with $\tilde{H}_*(B;Q) \neq 0$, then $\alpha \in [\Sigma X, B]$ has infinite order.

Proof. If α were of finite order, then the (nilpotent) group localization

$$[X, \Omega B] \xrightarrow{e_*} [X, \Omega B_0]$$

gives $e_*(\alpha) = e \cdot \alpha \simeq 0$. The bijections $[X, \Omega B_0] \simeq [X_0, \Omega B_0] \simeq [\Sigma X_0, B_0]$ then imply $\alpha_0 \simeq 0$ as well. However (as we have seen), i_0 is inessential, so if α_0 were as well then this would imply $\Omega B_0 \simeq F_0$. As can be readily seen, the assumption on $H_*(B;Q)$ forces ΩB_0 to have infinite dimensional cohomology. This then contradicts the compactness (i.e., quasifiniteness) of F. □

Finally, we mention the following non-existence theorem for compact fibrations $F \to \Sigma X \to B$.

PROPOSITION 15. *There do not exist compact fibrations $F \to \Sigma X \to B$ if either B is a product of odd spheres or B is a wedge of odd spheres.*

Proof. For B a product or wedge of odd spheres, we have $\Omega B \simeq \Pi K(Q, 2j)$. This is obvious for the product and follows for the wedge from Lie algebra model methods [21] or the homology decomposition approach of [18]. Hence, any subproduct $K \subset \Omega B$ has infinite dimensional cohomology. In particular, $F \simeq K$ (the Samelson space) cannot be quasifinite. □

Now we turn to another situation where the Samelson space is nontrivial. Recall that a space is *coformal* if its homotopy type is determined by its homotopy vector spaces together with its Whitehead product (see, e.g., [15] or [21]). Spheres and Eilenberg–MacLane spaces are coformal. Similarly, a map between coformal spaces is coformal if its homotopy class is determined by its effect on the underlying Whitehead algebras. The following theorem was proved in [19].

THEOREM 16. *If $p: E \to B$ is a coformal map, then the homotopy fibre F has a decomposition $F \simeq X \times Y$, where Y is a subproduct of ΩB and*

$$\pi_*(Y) \cong \operatorname{Im}(\partial_\#: \pi_*(\Omega B) \to \pi_*(F)).$$

COROLLARY 17. *The coformal decomposition of F coincides with that of the* RFDT.

Proof. For a coformal space, the kernel of the Hurewicz map consists of "Whitehead products" (see, e.g., [15]). Because Y is a product of $K(Q, n)$'s, the Hurewicz map is injective on $\pi_*(Y)$. Recall that the Samelson space K is characterized as the spatial model of that part of $\operatorname{Im} \partial_\#$ on which h is injective. Since $\pi_*(Y) \cong \operatorname{Im} \partial_\#$ and h is injective on $\pi_*(Y)$, then $Y = K$. □

COROLLARY 18. *If $F \to E \xrightarrow{p} B$ is a fibration with p coformal and $p_\#$ not surjective, then the Samelson space is nontrivial.*

EXAMPLES. The Hopf maps $S^1 \to S^3 \to S^2$ and $S^3 \to S^7 \to S^4$ satisfy the conditions of Corollary 18. The Samelson spaces are S^1 and S^3, respectively.

4. Extensions of the method. In this section we do not assume spaces are rational, but we require that they have the homotopy type of CW complexes. It is our intent to show that the "Samelson space method" has a place in ordinary, as well as rational, homotopy theory.

We recall some notation first. The Hurewicz map is denoted by

$$h: \pi_n(X) \to H_n(X),$$

while $G_n(X)$ denotes the subgroup of $\pi_n(X)$ consisting of elements α such that there exists a fibration $X \to E \to B$ with $\alpha \in \operatorname{Im}(\partial_\#: \pi_{n+1}(B) \to \pi_n(X))$.

THEOREM 19. *Let $X = K(\pi, 1)$ with $H_1(\pi)$ finitely generated. If there exists $\alpha \in G_1(X)$ such that $h(\alpha)$ is of infinite order, then there is a finite cover of X, $\tilde{X} \simeq Y \times S^1$.*

Proof. Here, the Hurewicz map has the form $h: \pi \to \pi/[\pi, \pi] \cong F \oplus T$, where F is a finitely generated free abelian group and T is the (finitely generated) torsion subgroup.

First, assume $h(\alpha)$ is a basis element x_1 of F. We can construct a left inverse to the inclusion $\langle \alpha \rangle = \mathbf{Z} \xrightarrow{i} \pi$ by first defining $\theta: F \oplus T \to \mathbf{Z} = \langle \alpha \rangle$ such that $\theta(T) = 0$, $\theta(x_1) = \alpha$ and $\theta(x_i) = 0$, and then letting $\phi = \theta h: \pi \to \mathbf{Z}$. Clearly, $\phi i =$ identity. Let $H = \operatorname{Ker} \phi$ and note that the splitting i induces a semidirect product structure $\pi \simeq H \ltimes \mathbf{Z}$, where $(h, \alpha) \mapsto h\alpha$. (The action of $\mathbf{Z}$ on H is conjugation in π.) Now, by the definition of $G_1(X)$, there is a fibration $X \to E \to B$ and $\beta \in \pi_2(B)$ with $\partial_\#(\beta) = \alpha$. The holonomy of the fibration $X \times \Omega B \to X$ induces a map $c: \pi \times \pi_2(B) \to \pi$ which has the form $c(a, b) = a\partial_\#(b)$. When restricted to the subgroup $H \times \mathbf{Z} \subset \pi \times \pi_2(B)$ (where $\mathbf{Z} = \langle\beta\rangle$), c has the form $(h, \beta) \mapsto h\alpha$. Hence, $H \times \mathbf{Z} \to \pi \cong H \ltimes \mathbf{Z}$ is a bijective homomorphism, so it is an isomorphism. Let $Y = K(H, 1)$ and note that $Y \times S^1 \simeq X$.

If $h(\alpha)$ is of infinite order, but not necessarily a basis element of F, then we may define θ and ϕ as before, where we require x_1 to have nonzero coefficient k in the basis decomposition of $h(\alpha)$. We then have $\phi i(\alpha) = k\alpha$. Let $p: \mathbf{Z} \to \mathbf{Z}/k\mathbf{Z}$ be projection and denote $\mathrm{Ker}(p\phi)$ by G. Let $\tilde{X} = K(G, 1)$ be the covering of X corresponding to $G \hookrightarrow \pi$. Clearly $\alpha \in G$, and it is well known [10, Theorem 6-1] that $\alpha \in G_1(\tilde{X})$ as well. Also, by definition of G, we see that $h(\alpha)$ is a basis element of $G/[G, G]$. Hence the first part of the proof applies, and we are done. □

REMARK. Theorem 19 is somewhat artificial in the sense that it is essentially an exercise in group theory. The component of the proof which is hidden from view is the fact that $G_1(X)$ is contained in the center of π. Hence, it is clear exactly why the action in the semidirect product $H \ltimes \mathbf{Z}$ is trivial. Furthermore, the holonomy of a fibration induces a homomorphism at the fundamental group level precisely because $\mathrm{Im}\ \partial_\#$ is contained in the center of the fibre's fundamental group for any fibration.

COROLLARY 20. *If $X = K(\pi, 1)$ is compact and there exists $\alpha \in Z(\pi)$ such that $h(\alpha)$ has infinite order, then $\chi(X) = 0$.*

Proof. By [9, Corollary I.13] $G_1(X) = Z(\pi)$ (the center of π), so we may apply Theorem 19. Now $\tilde{X} \simeq Y \times S^1$, so $\chi(\tilde{X}) = 0$. Also, $\chi(\tilde{X}) = k\chi(X)$, so $\chi(X) = 0$ as well. □

REMARK. Corollary 20 is a restricted form of Gottlieb's theorem [9, Corollary IV.3]. It and Theorem 19 apply, for example, to compact manifolds of nonpositive sectional curvature. These manifolds are known to be aspherical and to have homologically injective free abelian centers. Also, by considering the Borel fibration associated to a group action, Theorem 19 may be applied to homologically injective torus actions.

In [10], Gottlieb showed that, if X is a suspension, then the existence of $\alpha \in G_n(X)$ with $h(\alpha)$ of infinite order implies that X has the homotopy type of an odd sphere. (In fact, it is sufficient to assume $h(\alpha) \neq 0$. Also, compare Corollary 3 of this paper.) Furthermore, if $h(\alpha)$ is a generator of $H_n(X)$, then X is homotopy equivalent to one of S^1, S^3, S^7. As a final application of the "Samelson space method" we will derive several results in a similar vein.

THEOREM 21. *Let X be a simply connected n-dimensional complex with $H_n(X)$ finitely generated. If $\alpha \in G_n(X)$ has $h(\alpha) \in H_n(X)$ of infinite order, then there exists $\lambda \in \mathbf{Z}$ such that $X[1/\lambda] \simeq S^n[1/\lambda]$.*

REMARKS. (i) Here, $[1/\lambda]$ denotes the localization obtained by inverting the primes which occur in the prime factorization of λ.

(ii) By the remark following Corollary 4, we see that n must be odd.

Proof of Theorem 21. Because $\alpha \in G_n(X)$, there is a fibration $X \to E \to B$ and $\beta \in \pi_{n+1}(B) \cong \pi_n(\Omega B)$ with $\partial_\#(\beta) = \alpha \in \pi_n(X)$. We denote the respective representative maps by $\partial\beta, \alpha: S^n \to X$.

Let ι denote a generator of $\mathbf{Z} \cong \pi_n(S^n) \cong H_n(S^n)$ and note that

$$h(\alpha) = h(\alpha_\#(\iota)) = \alpha_*(h(\iota)) = \alpha_*(\iota).$$

Also, let $H_n(X) \cong F \oplus T$, where F is the free part of $H_n(X)$ with basis $\{f_1, f_2, \ldots, f_k\}$ and T is the torsion part. Because $h(\alpha)$ has infinite order, it may be written as

$$h(\alpha) = \lambda_1 f_1 + \cdots + \lambda_k f_k + t, \quad \text{some } \lambda_i \neq 0.$$

Without loss of generality, we may assume $\lambda_1 \neq 0$. For notational convenience denote λ_1 by λ.

According to the Samelson space method, we wish to construct a left homotopy inverse for $\alpha: S^n \to X$. In general this cannot be done, but it is possible after $[1/\lambda]$-localization. To see this, we begin by using basic properties of localization and the Hopf–Whitney classification theorem to obtain the following chain of isomorphisms:

$$\begin{aligned}[X[1/\lambda], S^n[1/\lambda]] &\cong [X, S^n[1/\lambda]] \\ &\cong H^n(X; \mathbf{Z}[1/\lambda]) \\ &\cong \operatorname{Hom}(H_n(X), \mathbf{Z}[1/\lambda]).\end{aligned}$$

Define $\theta: H_n(X) \to \mathbf{Z}[1/\lambda]$ by $\theta(f_1) = (1/\lambda)\iota$, $\theta(f_i) = 0$ for $i \geq 2$ and $\theta(t) = 0$ for $t \in T$. Let $\phi: X[1/\lambda] \to S^n[1/\lambda]$ denote the homotopy class corresponding to θ by the isomorphisms above. Also, let $\bar{\alpha}: S^n[1/\lambda] \to X[1/\lambda]$ denote the localization of α. We then compute, using $\phi_* = \theta$,

$$\begin{aligned}\phi_* \bar{\alpha}_*(\iota) &= \phi_*(\alpha_*(\iota) \otimes 1) \\ &= \phi_*(h(\alpha) \otimes 1) \\ &= \phi_*((\lambda_1 f_1 + \cdots + \lambda_k f_k + t) \otimes 1) \\ &= \lambda \cdot (1/\lambda)\iota \\ &= \iota.\end{aligned}$$

Because $[S^n[1/\lambda], S^n[1/\lambda]] \cong \mathbf{Z}[1/\lambda]$, we see that $\phi\bar{\alpha} \simeq$ identity.

Now, let $i: Y \to X[1/\lambda]$ denote the inclusion of the homotopy fibre of ϕ into $X[1/\lambda]$ and note that $\phi\bar{\alpha} \simeq \text{id}$ implies that $\pi_k(X[1/\lambda]) \cong \pi_k(Y) \oplus \pi_k(S^n[1/\lambda]) \cong \operatorname{Im} i_\# \oplus \operatorname{Im} \bar{\alpha}_\#$.

Once again, in accordance with the general method, consider the composition

$$S^n[1/\lambda] \times Y \xrightarrow{\beta \times i} \Omega B[1/\lambda] \times X[1/\lambda] \xrightarrow{c} X[1/\lambda],$$

where c denotes the holonomy of the localized fibration. The effect on homotopy groups is described by

$$c_\#(\bar{\beta}_\# \times i_\#)(a, b) = c_\#(\bar{\beta}_\#(a), i_\#(b)) = \bar{\partial}_\# \bar{\beta}_\#(a) + i_\#(b) = \bar{\alpha}_\#(a) + i_\#(b).$$

From the decomposition of $\pi_k(X[1/\lambda])$ given above, it is then clear that $c_\#(\bar{\beta}_\# \times i_\#)$ is an isomorphism. Consequently, $X[1/\lambda] \simeq S^n[1/\lambda] \times Y$.

Now, because X is n-dimensional, $H_i(X[1/\lambda]) = 0$ for $i > n$, so $H_i(Y) = 0$ for $i \geq 1$. Hence, $X[1/\lambda] \simeq S^n[1/\lambda]$. □

COROLLARY 22. *$X[1/\lambda]$ is an H-space. Hence, if λ is odd then $n = 1, 3,$ or 7.*

Proof. The homotopy equivalence $\phi: X[1/\lambda] \to S^n[1/\lambda]$ is a right homotopy inverse for

$$S^n[1/\lambda] \xrightarrow{\bar{\beta}} \Omega B[1/\lambda] \xrightarrow{\bar{\partial}} X[1/\lambda].$$

Therefore, $X[1/\lambda]$ is a weak retract of the H-space $\Omega B[1/\lambda]$, so is an H-space as well. The last statement follows from standard results on localization of spheres. □

REMARK. The results above hold for any $\lambda_i \neq 0$ in the decomposition $h(\alpha) = \lambda_1 f_1 + \cdots + \lambda_k f_k + t$. Thus, if any of the λ_i are odd, then $n = 1, 3,$ or 7.

COROLLARY 23. *If $h(\alpha)$ is a generator of $H_n(X)$ of infinite order, then X has the homotopy type of S^1, S^3, or S^7.*

Proof. If $h(\alpha)$ is a generator, then some $\lambda_i = 1$. Hence $\mathbf{Z}[1/\lambda_i] = \mathbf{Z}$ and the localized homotopy equivalence is actually integral; $X \simeq S^n$. By Corollary 22, $n = 1, 3,$ or 7. □

REMARK. Corollary 23 does not require localization methods or the simple connectivity of X. A proof modeled on that of Theorem 21 shows $X \simeq S^n$, and an argument similar to that given in the proof of Theorem 13 shows that X is an H-space. Hence, $X \simeq S^1$, S^3, or S^7 since these are the only spheres which are H-spaces.

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