ON ANALYTIC FUNCTIONS WITH CLUSTER SETS OF FINITE LINEAR MEASURE

Ch. Pommerenke

1. Introduction. Let f be a non-constant complex-valued function defined in the unit disk $\mathbf{D}$. The total cluster set $C(f)$ consists of all limit points of $f(z)$ as $|z| \to 1$, $z \in \mathbf{D}$. The linear Hausdorff measure of $E \subset \mathbf{C}$ is defined by

$$\Lambda(E) = \lim_{\epsilon \to 0} \inf_{(D_n)} \sum_n \operatorname{diam} D_n \tag{1.1}$$

where the infimum is taken over all systems (D_n) of disks with $\operatorname{diam} D_n < \epsilon$ that cover E.

THEOREM. *If f is bounded and analytic in $\mathbf{D}$ and if*

$$\Lambda(C(f)) < \infty, \tag{1.2}$$

then f has a continuous extension to $\bar{\mathbf{D}}$.

This result was proved by Globevnik and Stout [4, Theorem 2] under the additional assumption that

$$\iint_{\mathbf{D}} |f'(z)|^2 \, dx \, dy < \infty, \tag{1.3}$$

and they conjectured that (1.3) is redundant. They applied their result to study proper analytic maps of $\mathbf{D}$ into the unit ball of $\mathbf{C}^N$; see [2] and [3] for related results. I want to thank Professor Globevnik for writing to me about this problem.

Note that (1.2) and (1.3) do not imply ([4], [5]) that f' belongs to the Hardy space H^1. See [5] for further results that follow from (1.2).

2. Auxiliary results. In the following lemma, it is probably possible to replace the factor π by 2.

LEMMA 1. *If B is a continuum with $\Lambda(B) < \infty$ and if V_j are the bounded components of $\mathbf{C} \setminus B$, then*

$$\sum_j \Lambda(\partial V_j) \le \pi \Lambda(B). \tag{2.1}$$

Proof. In each component V_j we fix a point w_j. By (1.1) the compact set B can be covered by finitely many disks $D_{n\mu}$ $(\mu = 1, \dots, m_n)$ such that

$$\sum_{\mu=1}^{m_n} \operatorname{diam} D_{n\mu} < \Lambda(B) + \frac{1}{n} \quad \text{for } n = 1, 2, \dots, \tag{2.2}$$

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and diam $D_{n\mu}<1/n$, $D_{n\mu}\cap B\neq\emptyset$ for $\mu=1,\dots,m_n$. Since B is connected it follows that $B_n=\bar{D}_{n1}\cup\cdots\cup\bar{D}_{nm_n}$ is connected.

Let $N=1,2,\dots$. Since the disks $D_{n\mu}$ lie in a $1/n$-neighborhood of B, we see that, for $j\le N$ and sufficiently large n, there is a component V_{nj} of $\mathbf{C}\backslash B_n$ containing w_j. It follows from (2.2) that

$$\sum_{j\le N}\Lambda(\partial V_{nj})\le\sum_{\mu=1}^{m_n}\text{length}\,\partial D_{n\mu}<\pi\Lambda(B)+\frac{\pi}{n}. \tag{2.3}$$

The domain V_{nj} is simply connected because B_n is connected, and $V_{nj}\subset V_j$ because $B_n\supset B$. Furthermore, V_{nj} tends to V_j as $n\to\infty$ in the sense of Carathéodory kernel convergence [6, p. 28] with respect to the center w_j. Using conformal mapping and then letting $N\to\infty$, we deduce from (2.3) as in [6, p. 321] that (2.1) holds. □

For technical reasons we also consider

$$\Lambda'(E)=\inf_{(D_n)}\sum_n\text{diam}\,D_n \tag{2.4}$$

where now the infimum is taken over all systems (D_n) of disks covering E; here diam D_n is not restricted as it was in (1.1). We prove an analogue of the well-known result of Lavrent'ev, Privalov and Smirnov [6, pp. 320, 322].

LEMMA 2. *Let h be analytic and univalent in $\mathbf{D}$ and suppose that $\Lambda(\partial h(\mathbf{D}))<\infty$. Let $G_n\subset\{\frac{1}{2}<|z|<1\}$ be open sets such that $\partial\mathbf{D}\cup\partial G_n$ is connected. If*

$$\Lambda(\mathbf{D}\cap\partial G_n)\to 0\ \textit{as}\ n\to\infty, \tag{2.5}$$

then (see (2.4))

$$\Lambda'(h(G_n))\to 0\ \textit{as}\ n\to\infty. \tag{2.6}$$

Perhaps the assertion holds already without the assumption that $\partial\mathbf{D}\cap\partial G_n$ is connected.

Proof. Let $K_1,K_2,\dots$ denote suitable absolute constants. Since $\partial\mathbf{D}\cup\partial G_n$ is connected, the components G_{nk} of G_n are simply connected and satisfy $\partial\mathbf{D}\cap\partial G_{nk}\neq\emptyset$. If $G^*_{n\mu}=\bigcup_{k\le\mu}G_{nk}$ then, for all μ,

$$\Lambda(\partial G^*_{n\mu})\le\Lambda(\mathbf{D}\cap\partial G^*_{n\mu})+\Lambda\left(\left\{\frac{z}{|z|}:z\in G^*_{n\mu}\right\}\right)\le K_1\Lambda(\mathbf{D}\cap\partial G_n) \tag{2.7}$$

because $G_n\subset\{\frac{1}{2}<|z|<1\}$. For large n there is a disk D_{nk} containing G_{nk} such that ∂D_{nk} and $\partial\mathbf{D}$ are orthogonal and that

$$\Lambda(\partial\mathbf{D}\cap D_{nk})\le K_2\,\text{diam}\,\partial G_{nk}\le K_2\Lambda(\partial G_{nk}).$$

Applying Lemma 1 to the (finitely many) components of $\partial G^*_{n\mu}$ we conclude that, for all μ,

$$\sum_{k\le\mu}\Lambda(\partial\mathbf{D}\cap D_{nk})\le K_2\sum_{k\le\mu}\Lambda(\partial G_{nk})\le\pi K_2\Lambda(\partial G^*_{n\mu}). \tag{2.8}$$

Consider now the disjoint arcs I_{nj} of which $\bigcup_k(\partial\mathbf{D}\cap D_{nk})$ is composed and let H_{nj} be the domain bounded by I_{nj} and by the circle orthogonal to $\partial\mathbf{D}$ through the endpoints of I_{nj}. Then

$$G_n=\bigcup_k G_{nk}\subset\bigcup_k(\mathbf{D}\cap D_{nk})\subset\bigcup_j H_{nj}. \tag{2.9}$$

It follows from (2.8) and (2.7) that

$$\sum_j \Lambda(I_{nj})\le\sum_k \Lambda(\partial\mathbf{D}\cap D_{nk})\le K_3\,\Lambda(\mathbf{D}\cap G_n). \tag{2.10}$$

Since h is univalent and $\Lambda(\partial h(\mathbf{D}))<\infty$, it follows [6, pp. 320, 322] that $h'\in H^1$. A theorem of Gehring and Hayman [1; 6, Lemma 10.5] shows that

$$\Lambda(h(\partial H_{nj}))\le K_4\int_{I_{nj}}|h'(s)|\,|ds|.$$

Hence we see from (2.4) and (2.9) that

$$\Lambda'(h(G_n))\le K_5\sum_j\int_{I_{nj}}|h'(s)|\,|ds|,$$

and since the arcs I_{nj} are disjoint, it follows from (2.10) and (2.5) that (2.6) holds. □

3. Proof of the theorem. The proof uses the component method of Gnuschke-Hauschild [5]. Let V_j denote the bounded components of $\mathbf{C}\setminus C(f)$. Then, for each j, there are only finitely many components G_{jk} of $f^{-1}(V_j)$. Furthermore [5] the domains $V_j\subset\mathbf{C}$ and $G_{jk}\subset\mathbf{D}$ are simply connected, and if φ_j and ψ_{jk} map $\mathbf{D}$ conformally onto V_j and G_{jk}, then

$$f\circ\varphi_{jk}=\psi_j\circ b_{jk} \tag{3.1}$$

where b_{jk} is a finite Blaschke product.

Let $\zeta\in\partial\mathbf{D}$ and $\epsilon>0$ be given. It follows from Lemma 1 applied to $B=C(f)$ that

$$\sum_{j>N}\Lambda'(V_j)\le 2\sum_{j>N}\Lambda(\partial V_j)<\epsilon \tag{3.2}$$

for some $N=N(\epsilon)$. Let now $j\le N$ and

$$\ell_{jk}(r)=\Lambda(\{s\in\mathbf{D}:|\varphi_{jk}(s)-\zeta|=r\})\quad(0<r<1). \tag{3.3}$$

The standard length-area estimate applied to the bounded univalent function φ_{jk}^{-1} in G_{jk} shows that

$$\int_0^1\left(\sum_{j\le N}\sum_k \ell_{jk}(r)\right)^2\frac{dr}{r}<\text{const}\sum_{j\le N}\sum_k\int_0^1 \ell_{jk}(r)^2\frac{dr}{r}<\infty$$

because there are only finitely many k for each $j\le N$. Hence there exist $r_n\to 0$ such that

(3.4) $$\ell_{jk}(r_n) \to 0 \text{ as } n \to \infty$$

for each $j \le N$ and each k. We define

(3.5) $$U_n = \{z \in \mathbf{D} : |z - \zeta| < r_n\}, \quad H_{jkn} = \varphi_{jk}^{-1}(U_n).$$

Then $\Lambda(\mathbf{D} \cap \partial H_{jkn}) = \ell_{jk}(r_n) \to 0$ as $n \to \infty$, by (3.3) and (3.4), and since b_{jk} is a finite Blaschke product we conclude that

(3.6) $$\Lambda(\mathbf{D} \cap \partial b_{jk}(H_{jkn})) \to 0 \text{ as } n \to \infty$$

for each $j \le N$ and each k.

We can now apply Lemma 2 with $h = \psi_j$ and $G_n = b_{jk}(H_{jkn})$; indeed,

$$\Lambda(\partial \psi_j(\mathbf{D})) = \Lambda(V_j) \le \Lambda(C(f)) < \infty$$

by (2.1), $b_{jk}(H_{jkn}) \subset \{\frac{1}{2} < |z| < 1\}$ for large n by (3.5), and (2.5) holds because of (3.6). We obtain from (2.6) that

$$\Lambda'(\psi_j \circ b_{jk}(H_{jkn})) \to 0 \text{ as } n \to \infty$$

and thus, by (3.5) and (3.1),

$$\Lambda'(f(U_n \cap G_{jk})) = \Lambda'(f \circ \varphi_{jk}(H_{jkn})) \to 0.$$

Hence we obtain that

$$\begin{aligned} \Lambda'(f(U_n) \setminus C(f)) &\le \sum_{j \le N} \sum_{k} \Lambda'(f(U_n \cap G_{jk})) + \sum_{j > N} \Lambda'(V_j) \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned}$$

if n is sufficiently large, by (3.2). Hence we see that

(3.7) $$\Lambda'(f(U_n) \setminus C(f)) \to 0 \text{ as } n \to \infty.$$

We claim now that diam $f(U_n) \to 0$ as $n \to \infty$. Otherwise there would exist $\delta > 0$ such that diam $f(U_n) \ge \delta$. Since $f(U_n)$ is a domain we could therefore find polygonal arcs $P_n \subset f(U_n)$ with diam $P_n > \delta/2$ that intersect the set $C(f)$ of finite linear measure only finitely often. But then (3.7) implies diam $P_n \to 0$ as $n \to \infty$. Hence we have shown that diam $f(U_n) \to 0$, and it follows from (3.5) that f has a continuous extension to $\mathbf{D} \cup \{\zeta\}$ for each $\zeta \in \partial \mathbf{D}$ and therefore to $\bar{\mathbf{D}}$.

Added in proof: H. Alexander (*Polynomial hulls and linear measure,* preprint 1986) has independently proved the same result. It has been recently generalized by J. J. Carmona and J. Cufí (*Analytic functions with locally connected cluster sets,* preprint 1986).

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Fachbereich Mathematik
Technische Universität
D-1000 Berlin 12
Germany