REPRESENTING HOMOLOGY CLASSES OF ALMOST DEFINITE 4-MANIFOLDS

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1. Introduction. In this note we wish to apply results of gauge theory (cf. [1], [2], [3]) to study which 2-dimensional homology classes are representable by embedded 2-spheres for almost definite 4-manifolds with odd intersection form. We work throughout in the differentiable category. Most of our results will concern simply connected 4-manifolds. Here results of Wall [15] and Freedman [5] imply that a simply connected 4-manifold with odd intersection form is homeomorphic to the connected sum of p copies of $\mathbf{C}P^2$ and q copies of $\overline{\mathbf{C}P}^2$, where $\overline{\mathbf{C}P}^2$ denotes $\mathbf{C}P^2$ with the other orientation. Let $M(p,q)$ denote a differentiable 4-manifold which is homeomorphic to $p\mathbf{C}P^2\#q\overline{\mathbf{C}P}^2$. Note that Donaldson has shown that there exist examples of manifolds $M(1,9)$ which are not diffeomorphic to $\mathbf{C}P^2\#9\overline{\mathbf{C}P}^2$. Our results will concern almost definite manifolds where $p=1$ or 2.

Let us recall what Donaldson's Theorems A, B, and C say (cf. [1], [2]). Theorem A says that a definite simply connected 4-manifold must have standard intersection form. It was applied by Kuga [8] and Suciu [13] to study the problem of representing homology classes in $S^2\times S^2$ and $\mathbf{C}P^2$ by embedded spheres and to give estimates on the number of double points of immersed spheres that represent the homology class. In the course of proving their results, they also gave results for when some rather special homology classes in manifolds $M(p,q)$ are represented by embedded spheres. An alternate proof of Kuga's theorem was given by Fintushel and Stern [3], and their techniques will be the basis of our results on $M(1,1)$ and $M(1,2)$. Theorem B says that a simply connected spin 4-manifold with $b_2^+=1$ must have intersection form a standard hyperbolic form of rank 2. Theorem C says that a simply connected spin 4-manifold with $b_2^+=2$ must have intersection form the direct sum of two copies of the standard hyperbolic form.

Our main results are the following.

THEOREM 1. *If x,y represent generators of $H_2(M(1,1))$ with $xy=0$, $x^2=-y^2=1$, then $ax+by$ is not represented by an embedded sphere if $||a|-|b||\geq 2$, except for $\pm(0,2)$, $\pm(2,0)$, $\pm(1,-1)$. If $M(1,1)$ is diffeomorphic to $\mathbf{C}P^2\#\overline{\mathbf{C}P}^2$, then $ax+by$ is represented by an embedded sphere if and only if $||a|-|b||\leq 1$ or $(a,b)=\pm(0,2)$ or $\pm(2,0)$ or $\pm(1,-1)$.*

THEOREM 2. *Let x be a characteristic homology class in $H_2(M(1,2))$.*

(i) *If x is represented by an embedded sphere, then $x^2=-1$.*

(ii) *If $M(1,2)$ is diffeomorphic to $\mathbf{C}P^2\#2\overline{\mathbf{C}P}^2$ and $x^2=-1$, then x is represented by an embedded sphere.*

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Note that Rohlin's signature theorem says that in the situation of part (i) we must have $x^2 = -1 \bmod 16$ (cf. [7]). Note also that Theorem 2 implies that any realizable characteristic homology class y in $2\overline{CP}^2$ must have $y^2 = -2$, that is, it must be the sum of the standard generators.

THEOREM 3. *Let x be a characteristic homology class in $H_2(M(p,q))$.*

(i) *If $p=1$, $q>2$, and $x^2 \geq -1$, or if $p=1$, $q>3$, and $x^2 = -2$, then x is not represented by an embedded sphere.*

(ii) *If $p=2$, $q>3$, and $x^2 \geq -1$, then x is not represented by an embedded sphere.*

(iii) *If $p=q=2$ and x is represented by an embedded sphere, then $x^2=0$, that is, the sphere has trivial normal bundle.*

(iv) *If $p=2$, $q=3$, and $x^2 \geq 0$, then x is not represented by an embedded sphere.*

Note that in case (ii) Wall [14, Lemma 6] shows that one can represent any primitive characteristic class x of $H_2(M(2,q))$, $q \geq 3$, with $x^2 = 2-q$ by an embedded sphere. He also shows [14, Theorem 3] that every primitive ordinary class in $H_2(M(2,q))$, $q \geq 2$, can be represented by an embedded sphere. One can use Rochlin's genus theorem [11] to show that for $M(2,2)$ the only divisible classes which could be represented by an embedded sphere would have to be of the form $x = ky$, where $0 \leq k \leq 3$ and $y^2 = \pm 1$. If $k=1$, the class $x=y$ is primitive ordinary and so can be represented. In any event, y can be represented. When $k=2$, one can use [14, Lemma 5] to show that x can be represented. When $k=3$, the same techniques will show only that the class is representable by an embedded torus. For certain special classes, such as $(3,0,3,0)$, geometric techniques will allow one to find spheres which represent them, but the general question—when $k=3$ and the class is not characteristic—remains open.

Theorem 2 has characterized which characteristic elements in $M(1,2)$ are represented by embedded spheres. We now wish to give some partial results on which primitive ordinary elements in $M(1,2)$ are so represented using techniques similar to those used in the proof of Theorem 1. The difficulty in applying this technique more generally is in finding an appropriate pseudofree Euler class to use. We will concentrate on the classes of the form $u = ax+by+cz$, where $a,b,c \geq 1$ and $u^2 \geq 1$. Here x,y,z are the standard generators with $x^2 = -y^2 = -z^2 = 1$. We refer to this class as (a,b,c). Recall that in $M(1,1)$ the class $(k+2,k)$ is not represented by an embedded sphere for $k \geq 1$. We show the following.

THEOREM 4. *With the notation above, the class $u = (2k+3, 2k+1, 2m)$, with $k, m \geq 1$, is not represented by an embedded 2-sphere if $u^2 \geq 5$.*

Recall that in [10] we obtained some restrictions on the normal Euler number of an embedded real projective plane in a positive definite 4-manifold. Here we will apply our current techniques to improve those results in a couple of cases. Our main improvement comes in the case of a characteristic embedding of $\mathbf{R}P^2$ in $\mathbf{C}P^2$. In [10] we showed that the normal Euler number was greater than or equal to -1 and was congruent mod 16 to either -1 or 3. We conjectured that only the values -1 and 3 could occur. Here we will verify that conjecture.

THEOREM 5. *The normal Euler number of a characteristic embedding of* $\mathbf{R}P^2$ *in* $\mathbf{C}P^2$ *is either* -1 *or* 3.

2. Proof of Theorem 1. Our argument follows the same outline as that given by Fintushel and Stern [3] for the case of $S^2\times S^2$. The reader should be familiar with that argument as we will concentrate mainly on the points of the argument which differ in our case. Let $u=ax+by$; by replacing x, y by their negatives if necessary we may assume that $a\geq b+2\geq 3$. Note that Rochlin's genus theorem (cf. [11]) can be used to rule out representing all nonzero classes with a common factor of a and b except $\pm(2,0)$, $\pm(0,2)$, $\pm(1,-1)$ or when $|a|=|b|$. Thus we will assume that a and b are relatively prime. Using the fact that a, b are relatively prime we can choose a', b' so $ab'-a'b=1$, $1\leq a'\leq a-1$, $1\leq b'\leq b-1$ (or $b=b'=1$). Set $v=a'x+b'y$. Then u, v generate $H_2(M(1,1))$. Assume that u is represented by an embedded sphere and let N denote a tubular neighborhood. Then $\partial N\approx L(a^2-b^2,1)$. Let $Y=M(1,1)\setminus \operatorname{int} N$. One computes that $H_1(Y)\approx H_3(Y)\approx 0$ and $H_2(Y)\approx \mathbf{Z}$. Now consider the Poincaré duals of u, v which we will also denote by the same symbols. Then $H^2(Y)$ is generated by the image of v, which we will denote by $\underline{v}$. Now consider the commutative diagram

$$\begin{array}{ccccc} H^2(Y,\partial Y) & \longrightarrow & H^2(Y) & \xrightarrow{i^*} & H^2(\partial Y) \\ \uparrow & & \uparrow & & \uparrow \\ H^2(M,N) & \longrightarrow & H^2(M) & \xrightarrow{j^*} & H^2(N). \end{array}$$

The map j^* is just the cup product with u evaluated on the orientation class times the generator of $H^2(N)$ given by the Poincaré dual to the disk fiber. Note that the generator of $H^2(N)\approx\mathbf{Z}$ maps to the generator of $H^2(\partial N)\approx \mathbf{Z}/(u^2)$. Thus to compute the cup product in the orbifold $X=Y\cup c\partial Y$, we note that this cup product structure is determined by that of the manifold with boundary Y and so $\underline{v}^2=(u^2v^2-(uv)^2)/u^2$. Here we are using the identification of $H^4(M)$ with $\mathbf{Z}$. Also $i^*(\underline{v})=uv \bmod u^2$. Note

$$uv=aa'-bb' \quad \text{and} \quad (aa'-bb')^2=(a^2-b^2)(a'^2-b'^2)+1$$

(using $1=(ab'-a'b)^2$). Thus $i^*(\underline{v})$ is a generator and so we can use the class $\underline{v}$ as the pseudofree Euler class e when forming the bundle over the pseudofree orbifold. Also $e^2=-1/u^2$, and so this will imply $\mu(e)=1$. The argument so far has been a straightforward adaptation of the one in [3] for the case of $S^2\times S^2$. What remains is to show that the index is positive when $a\geq b+2\geq 3$. Using the same techniques as in [3] (or the basic formula in [9]) one computes that the index is $-3+2(uv-v^2)$, so we are reduced to showing that $(uv-v^2)\geq 2$. For the case of $S^2\times S^2$ this follows easily, since $uv-v^2=a'(a-a')+b'(b-b')$ and it is easy to show this is ≥ 2 if $a, b\geq 2$. In our case, $(uv-v^2)=a'(a-a')-b'(b-b')$. We use $1=ab'-ba'$, so $a'(a-a')-b'(b-b')-1=((a+b)-(a'+b'))(a'-b')\geq 1$ since each term in the product is ≥ 1 with our assumption that $a\geq b+2\geq 3$.

We now need to see why those classes that have not been excluded are represented in $\mathbf{C}P^2\#\overline{\mathbf{C}P}^2$. The three exceptional cases are easily represented using embedded spheres in $\mathbf{C}P^2$ representing the generator or twice the generator. For $||a|-|b||\leq 1$, the easiest way to see that these classes are represented by an embedded sphere is to regard the total space as the twisted S^2 bundle over S^2. I am indebted to A. Suciu for pointing out the usefulness of this viewpoint to me. Under the standard diffeomorphism the induced automorphism on H_2 sends x to u, where u represents a cross-section of the bundle and sends y to $v-u$ (where v represents the fiber). Thus $ax+by$ is sent to $(a-b)u+bv$. Thus, if $a=b$, this class is sent to a multiple of the fiber which can be represented by tubing together a number of copies of the fiber. If $a-b=1$, then this is sent to $u+bv$. Since a cross-section will intersect b disjoint fibers in one point each, one can represent the desired class by forming connected sums at those b intersection points. The other cases are reduced to this one by choosing different orientations on the spheres representing x and y. □

3. Proofs of Theorems 2 and 3. The proofs of Theorems 2 and 3 are based on the following lemma, which is a straightforward consequence of Donaldson's Theorems B and C.

BASIC LEMMA. *Let $x \in H_2(M(p,q))$ be characteristic.*

(i) *If $p=1$, $q>2$, and $x^2=-1$, then x is not represented by an embedded sphere.*

(ii) *If $p=2$, $q>3$, and $x^2=-1$, then x is not represented by an embedded sphere.*

(iii) *If $p=1$, $q>3$, and $x^2=-2$, then x is not represented by an embedded sphere.*

Proof. For (i), excising a neighborhood of an embedded sphere which would represent x and replacing it by a disk would yield a simply connected 4-manifold W with rank q and $b_2^+=1$. Since x is characteristic, W is spin and thus gives a contradiction to Donaldson's Theorem B. Part (ii) follows similarly using Theorem C. For (iii), first take a connected sum with (CP^2, CP^1) and then use (ii). □

Proof of Theorem 3. Parts (i) and (ii) follow from the basic lemma after taking connected sum with x^2+1 copies of $(\overline{CP}^2, CP^1)$. For part (iii), first note that classes with $x^2 \geq 1$ can be ruled out by taking two connected sums with $(\overline{CP}^2, CP^1)$ and then applying part (ii). By reversing orientations we see that we also get a contradiction if $x^2 \leq -1$. For part (iv), we first add a copy of $(\overline{CP}^2, CP^1)$ and then apply part (ii).

Proof of Theorem 2. (i) This follows from 3(iii) by taking connected sum with $(\overline{CP}^2, CP^1)$.

(ii) Suppose x is characteristic and $x^2=-1$. Expressing $CP^2\#2\overline{CP}^2$ as $\overline{CP}^2\# S^2\times S^2$ allows us to decompose the quadratic form as $\langle -1\rangle \oplus H$, with H the standard hyperbolic form and $\langle -1\rangle$ represented by $CP^1 \subset \overline{CP}^2$. Since $x^2=-1$, the form also decomposes as $\langle x\rangle \oplus \langle x\rangle^\perp$ is even. The classification of even indefinite forms of rank 2 implies there is an isomorphism from H to $\langle x\rangle^\perp$. This determines an automorphism of $\langle -1\rangle \oplus H$, where $\langle -1\rangle$ is sent to $\langle x\rangle$. By Wall [14, Theorem 2], this automorphism is induced by a diffeomorphism and so x is represented by an embedded sphere, the image of $CP^1 \subset \overline{CP}^2$ under this diffeomorphism.

4. Proof of Theorem 4. This follows the same pattern as the proof of Theorem 2. The main difficulty is in finding the appropriate class to use as the pseudofree Euler class e.

If it were so represented then we could take a regular neighborhood N and form the orbifold $X = Y \cup c\partial Y$ where $Y \cup N = M(1, 2)$. We again identify u with its Poincaré dual, which we also denote by u, and extend u to a basis u, v, w of $H^2(M)$ where $v = (k+1, k, m)$, $w = (2m(k+1), 2mk, 2m^2+1)$. Note that $u^2 = 8k+8-4m^2 = 0 \bmod 4$. Thus $u^2 \geq 5$ implies that $u^2 \geq 8$. The map $H^2(M) \to H^2(Y)$ is surjective with kernel generated by u, and so v, w map to a basis $\underline{v}$, $\underline{w}$ of $H^2(Y) \approx 2\mathbf{Z}$. If $X = Y \cup c\partial Y$ as before then the intersection form on X (with appropriate orientation and basis $\underline{v}$, $\underline{w}$) is given by the matrix

$$1\Big/u^2 \begin{pmatrix} 1 & 0 \\ 0 & u^2 \end{pmatrix}.$$

The class $\underline{v}$ maps to $uv = (u^2/2) - 1$ times the generator of $H^2(\partial Y)$ given by the Poincare dual of the circle fiber. Since $(uv)^2 - u^2v^2 = 1$, $\underline{v}$ maps to a generator of $H^2(\partial Y)$ and so may be used as a pseudofree Euler class e for the orbifold X. From the intersection form we see $\mu(e) = 1$. Using the technique given in [3, Section 10] (or [9]) one computes $R(x, e) = (u^2/2) - 3$. Thus $u^2 \geq 8$ implies that $R \geq 1$, which is a contradiction. □

REMARK. Theoretically, the same technique could be used to exclude representing other classes with positive square. The main difficulty comes from selecting an appropriate pseudofree Euler class. There are examples where it is impossible to select such a class which will work in applying the results of [3] as above. The situation would improve dramatically, however, if one could prove the analogue of [4, Conjecture 5.7] for minimal classes.

5. Normal bundles of an embedded $\mathbf{R}P^2$. In this section we wish to apply Theorems 2 and 3 in order to study the normal bundle of an embedded $\mathbf{R}P^2$ in certain 4-manifolds and to give a proof of Theorem 5. Our starting point will be a lemma which is essentially contained in [12].

CONNECTING LEMMA (cf. [12]). *Let M be a simply connected oriented 4-manifold. If $\mathbf{R}P^2 \subset M$ is a characteristic embedding with normal Euler number n, then there is a characteristic embedding of S^2 in $M \# S^2 \times S^2$ representing a homology class c with c^2 equal to either $n+2$ or $n-2$.*

Proof. We regard M as the connected sum of S^4 with M, and use the decomposition of S^4 as $S^1 \times D^3 \cup D^2 \times S^2$. Let B denote the Möbius band and fix a standard embedding of B in $S^1 \times D^2 \subset S^1 \times D^3$. Since M is simply connected we can adjust the embedding of $\mathbf{R}P^2$ so that it agrees with the embedding of B after an isotopy. Thus all of the information from the embedding is contained in a relative embedding of a disk $(D^2, \partial D^2)$ in $(D^2 \times S^2 \# M, S^2 \times S^2)$ which is standardized on the boundary. In particular, the normal Euler number n can be regarded as coming from comparing a standard section of the normal bundle over the Möbius band restricted to the boundary circle and the restriction of a section from the remaining disk. Now in the case where M is itself S^4 we know that the normal Euler number must be either 2 or -2, and both values occur. Thus there are two embedded disks D_1, D_2 in $D^2 \times S^2$ which are standard on the boundary giving normal Euler numbers -2 and $+2$.

Consider now our assumed characteristic embedding of $\mathbf{R}P^2$ in M which we have made standard on the Möbius band B. Let us now call the remaining embedded disk D_3 in $D^2 \times S^2 \# M$. Using each of the disks D_1, D_2 we can form two different embeddings of the sphere in the connected sum $S^2 \times S^2 \# M$ via $D_i \cup D_3 \subset D^2 \times S^2 \cup D^2 \times S^2 \# M$. These will represent elements of the form $2x + c_i y + z$ in $H_2(S^2 \times S^2 \# M)$. The two possible Euler numbers (= self intersection numbers) are $n \pm 2$. Thus $4c_i + z^2 = n \pm 2$. Hence $|c_2 - c_1| = 1$ and one of the c_i is even. Since the original $\mathbf{R}P^2$ was characteristic, the class z will be characteristic. Thus with the choice of D_i with c_i even, we obtain a characteristic homology class in $H_2(S^2 \times S^2 \# M)$ which is represented by an embedded sphere and whose square is either $n+2$ or $n-2$. □

We have called the above lemma a connecting lemma since it serves as a mechanism to connect up the two problems of representing a given homology class by an embedded sphere and determining the normal bundle of an embedded $\mathbf{R}P^2$ which is characteristic. Although it might have been stated more generally without the emphasis on the characteristic elements, we have chosen this statement since it sets up the application of Theorems 2 and 3. We now prove Theorem 5.

Proof of Theorem 5. From our characteristic embedding of $\mathbf{R}P^2$ in $\mathbf{C}P^2$ we obtain (by the connecting lemma) a characteristic embedding of S^2 in $M(2,1)$ with square $n \pm 2$. But Theorem 2(i), with the orientation reversed, says that this square must be 1. Hence $n = -1$ or 3. Both values are easily realized (cf. [10]). □

REMARKS. The connecting lemma can be used to get other results about embedded characteristic $\mathbf{R}P^2$'s in $M(p,0)$ or $M(p,1)$. A characteristic $\mathbf{R}P^2$ in $M(p,0)$ leads by the connecting lemma to a characteristic sphere in $M(p+1,1)$ with square $n \pm 2$. Now if $p > 1$, Theorem 3(i) implies $n \pm 2 \geq 2$ and hence $n \geq 0$. If $p > 2$ then we can conclude that $n \geq 1$, using the second part of Theorem 3(i). Recall that in [10] we showed that when $p < 8$ we actually have $n \geq -2+p$, but the technique used there broke down when $p \geq 8$. Thus we will obtain some new results when $p \geq 8$ but unfortunately not as strong as the expected result that $n \geq -2+p$ (which we get only for $p = 1, 2, 3$). Also note that these new results apply only to characteristic $\mathbf{R}P^2$'s where the results of [10] applied more generally. A new result can be obtained concerning characteristic embeddings of $\mathbf{R}P^2$ in $M(p,1)$. If we apply the connecting lemma and Theorem 3(ii), we can conclude that $n \pm 2 \geq 2$, hence $n \geq 0$, when $p > 2$. Applying Theorem 3(iii) will imply that for a characteristic embedding of $\mathbf{R}P^2$ in $M(1,1)$ we must have $n = \pm 2$. Note that this implies Theorem 5. Finally, Theorem 3(iv) implies that a characteristic $\mathbf{R}P^2$ in $M(2,1)$ will have $n \geq -1$.

Rochlin also gave a technique in [12] for connecting problems about normal Euler numbers of embedded Klein bottles in M to embedded spheres in $2(S^2 \times S^2) \# M$. The square of the homology class of the corresponding sphere will be n, $n+4$, or $n-4$, where n is the original normal Euler number of the Klein bottle. If the Klein bottle is characteristic, the sphere may be chosen to be characteristic. Using this technique and Theorem 3(iv) one can show that a characteristic Klein bottle in $\mathbf{C}P^2$ must have normal Euler number $n \geq -3$. It was already known that $n-1 \equiv 0, \pm 4 \bmod 16$ (cf. [6]).

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