HOLOMORPHIC FUNCTIONS ON THE POLYDISC HAVING POSITIVE REAL PART

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Let D^N denote the open unit polydisc in C^N and let

$$\mathcal{P}_N = \{f \mid f \text{ is holomorphic on } D^N, \ \mathrm{Re} f > 0, \text{ and } f(\theta) = f(0,0,\ldots,0) = 1\}.$$

Of course $\mathcal{P}_N$ is compact in the topology of uniform convergence on compacta. Thus, it follows from the Krein-Milman theorem that $\mathcal{P}_N$ is the closed convex hull of its extreme elements. In the case $N=1$ the extreme elements of $\mathcal{P}_N$ are easily found via Herglotz's theorem. For $N>1$, however, a complete description of the extreme elements of $\mathcal{P}_N$ is not known, although Forelli has found a necessary condition for a member of $\mathcal{P}_N$ to be extreme. (See [1].) Forelli [1] and McDonald [3; 4] have also constructed several examples of extreme elements of $\mathcal{P}_2$.

In this paper, we study certain faces of the convex set $\mathcal{P}_N$. We recall that a face F of a convex set S is a convex subset which satisfies: $(c, x, y) \in (0,1) \times S \times S$ and $cx + (1-c)y \in F$ together imply $x, y \in F$. For our purposes, it is important to note that an extreme point of the face F is also an extreme point of S. For each $x \in S$ there is a smallest face $\mathfrak{F}(x)$ containing x. $\mathfrak{F}(x)$ is simply the union of all line segments from S which contain x as a relative interior point. If S is a compact convex subset of some locally convex vector space, then the closed faces will always contain extreme elements. Faces of the form $\mathfrak{F}(x)$ are, however, not closed in general, but, if it can be shown that $\mathfrak{F}(x)$ is finite-dimensional, then $\mathfrak{F}(x)$ will necessarily be closed. Furthermore, if it is known that $\mathfrak{F}(x)$ is finite-dimensional, then it follows from a theorem of Carathéodory that x can be written as a finite convex combination of extreme elements of S. (See, e.g., [5].)

Our main result is that $\mathfrak{F}(G)$ is a finite-dimensional face of $\mathcal{P}_N$ when G is of the form $G = (1+g)/(1-g)$, where g is a rational inner function satisfying $g(\theta) = 0$. We also show that each member of $\mathfrak{F}(G)$ is the Cayley transform of a rational inner function and that the set of extreme elements of sets of the form $\mathfrak{F}(G)$ is dense in the set of extreme elements of $\mathcal{P}_N$. Finally, we study some particular examples of faces of the form $\mathfrak{F}(G)$.

1. The main result. In this section g will denote a rational inner function on D^N which satisfies $g(\theta) = 0$. It is known that g must have the form

$$g = MQ^*/Q, \tag{1}$$

where Q is a polynomial having no zero on D^N, where

$$Q^*(z) = Q^*(z_1, \ldots, z_N) = \overline{Q(1/\bar{z}_1, \ldots, 1/\bar{z}_N)},$$

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and where M is a monomial such that MQ^* is a polynomial. (See [6, Th. 5.2.5].) The notation $\delta(g)$ will be used to denote $\prod_{j=1}^{N}(d(j)+1)$, where $d(j)$ denotes the degree of the monomial M in z_j.

When F is a function on D^N and ζ belongs to $T^N=\{(\xi_1,\xi_2,\ldots,\xi_N)\mid|\xi_j|=1$ for $j=1,2,\ldots,N\}$, the expression F_ζ will indicate the function defined on D^1 by $F_\zeta(z)=F(z\zeta)$. If F_ζ happens to have radial limits at almost every point of the unit circle T^1, the function given by $F_\zeta(z)$ for $z\in D^1$ and by $\lim_{r\to 1^-}F_\zeta(rz)$ for $z\in T^1$ will also be denoted by F_ζ. Finally, in the case of the inner function g above it is important to observe that g_ζ is a finite Blaschke product.

We are now ready to state our main result.

THEOREM. *Let $G=(1+g)/(1-g)$. Then the (real) dimension of the face $\mathfrak{F}(G)$ is $\leq\delta(g)-2$.*

Proof. It will be shown that every function in $\mathfrak{F}(G)$ is of the form

$$F=\frac{1+g+v/Q}{1-g}, \tag{2}$$

where v is a polynomial which vanishes at θ and satisfies

$$M\bar{v}=v \quad \text{in } T^N. \tag{3}$$

The theorem will then follow from the fact that the set of polynomials satisfying (3) and vanishing at θ is a real vector space having dimension $\delta(g)-2$.

To verify (2) it should first be observed that a function F in $\mathcal{P}_N$ belongs to $\mathfrak{F}(G)$ if and only if F is of the form $F=G+U$, where U is holomorphic on D^N and, for some $a>0$, $G-aU\in\mathcal{P}_N$. Let $\zeta\in T^N$. By the classical theorem of Herglotz

$$G_\zeta(z)=\int\frac{\xi+z}{\xi-z}\,d\mu(\xi)=\mu^{\#}(z),$$

where μ is a positive measure on the circle. (μ of course depends on ζ.) Likewise $F_\zeta=G_\zeta+U_\zeta=\mu_1^{\#}$. Then

$$\mu_1\ll\mu \tag{4}$$

because

$$G=\frac{a}{1+a}(G+U)+\frac{1}{1+a}(G-aU).$$

Put $\lambda=\mu_1-\mu$; then $U_\zeta=\lambda^{\#}$. By (4) $g_\zeta=1$ almost everywhere with respect to λ. Thus, letting $u=(1-g)U$, it can be asserted that

$$u_\zeta(z)=\int(g_\zeta(\xi)-g_\zeta(z))\frac{\xi+z}{\xi-z}\,d\lambda(\xi).$$

Hence, u_ζ is bounded in the disc D^1 because g_ζ is holomorphic on $\bar{D}^1$. Consequently, u_ζ has radial limits a.e. on T^1.

Because $\operatorname{Re}U_\zeta$ vanishes almost everywhere on T^1, so does $\operatorname{Re}((1-\bar{g}_\zeta)u_\zeta)=0$, that is, $\operatorname{Re}u_\zeta=\operatorname{Re}(\bar{g}_\zeta u_\zeta)$. In other words $u_\zeta+\bar{u}_\zeta=\bar{g}_\zeta u_\zeta+g_\zeta\bar{u}_\zeta$, hence

$$(1-\bar{g}_\zeta)u_\zeta=(g_\zeta-1)\bar{u}_\zeta=(1-\bar{g}_\zeta)g_\zeta\bar{u}_\zeta$$

because $g_\zeta\bar{g}_\zeta=1$ in T^1, which means that

$$g_\zeta\bar{u}_\zeta=u_\zeta \quad \text{a.e. in } T^1. \tag{5}$$

By (1), $g_\zeta=M_\zeta\bar{Q}_\zeta/Q_\zeta$ in T^1. Thus, by (5),

$$M_\zeta\overline{Q_\zeta u_\zeta}=Q_\zeta u_\zeta \quad \text{a.e. in } T^1. \tag{6}$$

Put $v=Qu$, and let k be the degree of the monomial M. Because v is holomorphic in D^N, $v=\sum_{\ell=1}^{\infty}v_\ell$ there, where the ℓth term of the series is a homogeneous polynomial of degree ℓ. Then $\sum_{\ell=1}^{\infty}v_\ell(\zeta)e^{i\ell t}$ is the Fourier series of the bounded function v_ζ, and $\sum_{\ell=-\infty}^{k-1}M(\zeta)\bar{v}_{k-\ell}(\zeta)e^{i\ell t}$ is the Fourier series of $M_\zeta\bar{v}_\zeta$. Hence, by (6),

$$v_\ell(\zeta)=0 \quad \text{if } \ell\geq k, \tag{7}$$

while

$$v_\ell(\zeta)=M(\zeta)v_{k-\ell}(\zeta) \quad \text{if } 1\leq\ell\leq k-1. \tag{8}$$

The constraint (7) means that v is a polynomial. Then (3) follows by (6), or by (7) and (8). Finally, (2) follows from

$$\frac{v/Q}{1-g}=U. \qquad \square$$

2. **Corollaries and examples.** In this section G and g will continue to be as in Section 1.

COROLLARY 1. *Let $F\in\mathfrak{F}(G)$. Then F is of the form $F=(1+f)/(1-f)$, where f is a rational inner function.*

Proof. By (2),

$$F=\frac{1+g+v/Q}{1-g}, \tag{9}$$

where v is a polynomial satisfying (3), or (equivalently) satisfying

$$Mv^*=v \tag{10}$$

and vanishing at θ. Let f be the Cayley transform of F, that is,

$$f=(F-1)/(F+1). \tag{11}$$

Then $|f|<1$ in D^N because $\operatorname{Re}F>0$ there. By (9), (10), and (11),

$$\begin{aligned} f&=\frac{2g+v/Q}{2+v/Q}\\ &=\frac{2MQ^*+v}{2Q+v}\\ &=\frac{M(2Q^*+v^*)}{2Q+v}, \end{aligned}$$

which implies that the radial limit of f is unimodular. Thus f, like g, is a rational inner function. Furthermore $F=(1+f)/(1-f)$. □

If g is a rational inner function such that $G=(1+g)/(1-g)$ is an extreme element of $\mathcal{P}_N$, then of course the dimension of $\mathfrak{F}(G)$ is 0. But the following shows that the bound $\delta(g)-2$ for the dimension of $\mathfrak{F}(G)$ can be attained.

COROLLARY 2. *If g is continuous on $\bar{D}^N$, then the dimension of $\mathfrak{F}(G)$ is $\delta(g)-2$.*

Proof. If g is continuous on $\bar{D}^N$, then without loss of generality we may assume that $|Q|\geq\frac{1}{2}$ there. (See [6, Th. 5.2.5.].) This means that if v is a polynomial and if Mv^* is too, then $M(2Q^*+v^*)/(2Q+v)$ is bounded by 1 in D^N, provided $|v|<1$ there. In other words, if v vanishes at θ and satisfies (10), and $|v|<1$ in D^N, then the right side of (9) belongs to $\mathfrak{F}(G)$. It follows immediately that the dimension of $\mathfrak{F}(G)$ is $\delta(g)-2$. □

COROLLARY 3. *Let G and g be as in Section 1. Then every element of $\mathfrak{F}(G)$ can be written as a convex combination of at most k extreme elements of $\mathfrak{F}(G)$, where $k\leq\delta(g)-1$.*

Proof. By our main result, there is a real vector space $\mathcal{W}$ of holomorphic functions on D^N of dimension $\leq\delta(g)-2$ such that $\mathfrak{F}(G)=(G+\mathcal{W})\cap\mathcal{P}_N$. It follows that $\mathfrak{F}(G)$ is a compact convex subset of $\mathcal{P}_N$. Also, a result due to Carathéodory implies that each $G_1\in\mathfrak{F}(G)$ can be written as a convex combination of at most $\delta(g)-1$ extreme elements of $\mathfrak{F}(G)$. □

Since $\mathfrak{F}(G)$ is a face of $\mathcal{P}_N$, it follows that the set $\operatorname{ex}\mathfrak{F}(G)$ of extreme elements of $\mathfrak{F}(G)$ is contained in the set $\operatorname{ex}\mathcal{P}_N$ of extreme elements of $\mathcal{P}_N$. Thus,

$$\operatorname{ex}\mathcal{P}_N\supseteq\bigcup_{G\in\mathcal{G}}\operatorname{ex}\mathfrak{F}(G),$$

where $\mathcal{G}$ consists of all members of $\mathcal{P}_N$ of the form $G=(1+g)/(1-g)$, where g is a rational inner function.

COROLLARY 4. $\bigcup_{G\in\mathcal{G}}\operatorname{ex}\mathfrak{F}(G)$ *is dense in* $\operatorname{ex}\mathcal{P}_N$.

Proof. Let $H\in\mathcal{P}_N$. We can write $H=(1+h)/(1-h)$, where h is holomorphic on D^N, vanishes at θ, and satisfies $|h|\leq 1$. By [6, Theorem 5.1] we can find a sequence $\{g_n\}$ of rational inner functions which vanish at θ and converge uniformly on compact subsets of D^N to h. Let $G_n=(1+g_n)/(1-g_n)$. By Corollary 3 there are extreme elements $F_{1n},F_{2n},\ldots,F_{\ell(n)n}$ of $\mathfrak{F}(G_n)$ such that

$$G_n=\alpha_{1n}F_{1n}+\alpha_{2n}F_{2n}+\cdots+\alpha_{\ell(n)n}F_{\ell(n)n},$$

where $\alpha_j^n\geq 0$ and $\sum\alpha_j^n=1$. It follows that H belongs to the closed convex hull of $\bigcup_{G\in\mathcal{G}}\operatorname{ex}\mathfrak{F}(G)$.

We can now apply Milman's converse to the Krein–Milman theorem to assert that $\bigcup_{G\in\mathcal{G}}\operatorname{ex}\mathfrak{F}(G)$ is dense in $\operatorname{ex}\mathcal{P}_N$. (See, e.g., [5].) □

REMARK. Corollaries 1 and 4 combined with the main result of Forrelli's paper [1] yield the existence of a class S of irreducible rational inner functions such that $\{(1+g)/(1-g)\mid g\in S\}$ is a dense subset of $\mathcal{P}_N$.

EXAMPLE 1. We consider the function

$$G_0(z,w)=\frac{1+zw}{1-zw}.$$

By preceding discussions, $\mathcal{F}(G_0)$ consists of all functions of the form

$$G_1(z,w)=\frac{1+zw+u(z,w)}{1-zw}, \tag{12}$$

where $G_1\in\mathcal{P}_2$, where

$$\frac{1+zw-au(z,w)}{1-zw}\in\mathcal{P}_2 \tag{13}$$

for some positive constant $a>0$, and where u satisfies

$$zw\overline{u(z,w)}=u(z,w) \tag{14}$$

for all $(z,w)\in T^2$. Condition (14) implies that u is of the form

$$u(z,w)=cz+\bar{c}w,$$

where c is a constant. Conditions (12) and (13) can therefore be reformulated as

$$0<1-|z|^2|w|^2+(1-|w|^2)\operatorname{Re}(cz)+(1-|z|^2)\operatorname{Re}(\bar{c}w) \tag{15}$$

and

$$0<1-|z|^2|w|^2-a(1-|w|^2)\operatorname{Re}(cz)-a(1-|z|^2)\operatorname{Re}(\bar{c}w) \tag{16}$$

respectively, where $(z,w)\in D^2$. Now (15) and (16) are equivalent to

$$0<1-|z|^2|w|^2-(|z|(1-|w|^2)+|w|(1-|z|^2))|c| \tag{17}$$

and

$$0<1-|z|^2|w|^2-(|z|(1-|w|^2)+|w|(1-|z|^2))a|c|. \tag{18}$$

But (17) and (18) hold for some $a>0$ if and only if

$$|c|\le\frac{1+|z||w|}{|z|+|w|} \tag{19}$$

for all $(z,w)\in D^2$. From (19) it becomes clear that $|c|\le 1$ and that the extreme elements of $\mathcal{F}(G_0)$ are exactly the functions of the form

$$G_\alpha(z,w)=\frac{1+zw+e^{i\alpha}z+e^{-i\alpha}w}{1-zw}=\frac{(1+e^{i\alpha}z)(1+e^{-i\alpha}w)}{1-zw},$$

where $\alpha\in[0,2\pi]$.

EXAMPLE 2. Let g be an inner function on D^1. For simplicity's sake, we will assume that $g(0)=0$. We consider the element of $\mathcal{P}_2$ defined by

$$G(z,w)=\frac{1+zg(w)}{1-zg(w)}.$$

It follows from the proof of Theorem 1 of [4] that $\mathfrak{F}(G)$ consists of all functions of the form

$$G_1(z,w)=\frac{2zF(w)}{1-zg(w)}+F_1(w), \tag{20}$$

where F is a function in the Hardy space $H_1(D^1)$ satisfying

$$F(e^{i\theta})\overline{g(e^{i\theta})}\geq 0 \text{ a.e.} \tag{21}$$

and

$$(2\pi)^{-1}\int_0^{2\pi} F(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta=1, \tag{22}$$

and where

$$F_1(w)=(2\pi)^{-1}\int_0^{2\pi}\left(\frac{e^{i\theta}+w}{e^{i\theta}-w}\right)F(e^{i\theta})\overline{g(e^{i\theta})\,d\theta}.$$

Another consequence of the proof of Theorem 1 of [4] is that $G_1\in \operatorname{ex}\mathfrak{F}(G)$ if and only if F is an outer function.

EXAMPLE 2(i). We now consider the case where the inner function g of example 2 is an infinite Blaschke product

$$g(w)=w\prod_{k=1}^{\infty}\frac{\bar{\alpha}_k}{|\alpha_k|}\frac{\alpha_k-w}{1-\bar{\alpha}_k w}.$$

For $N=1,2,\ldots$ we let

$$h_N(w)=w\prod_{k=1}^{N}\frac{\bar{\alpha}_k}{|\alpha_k|}\frac{\alpha_k-w}{1-\bar{\alpha}_k w}$$

and

$$g_N(w)=g(w)/h_N(w).$$

Next, we define

$$F_N(w)=g(w)+(g_N(w)+g(w)h_N(w))/2.$$

It is not hard to show that F_N satisfies conditions (21) and (22). Replacing F by F_N in (20), we obtain the expression

$$G_{1N}(z,w)=G_1(z,w)+\frac{h_N(w)+zg_N(w)}{1-zg(w)}.$$

Since the infinite set of functions $\{h_N(w)+zg_N(w)\mid N=1,2,\ldots\}$ is linearly independent, it follows that the face $\mathfrak{F}(G)$ is not finite dimensional.

EXAMPLE 2(ii). Next, we consider the case where $g(w)=w^2$. It is not hard to show that functions which satisfy (21) and (22) must have the form

(23) $$F_{ab}(w)=\bar{a}+\bar{b}w+w^2+bw^3+bw^4.$$

Using (23) in (20), we obtain the expression

$$G_{ab}(z,w)=\frac{1+2bw+2\bar{a}z+2aw^2+2\bar{b}zw+azw^2}{1-zw^2}.$$

It is clear that G_{ab} is an extreme element of $\mathfrak{F}(G)$ if and only if $e^{-2i\theta}F_{ab}(e^{i\theta})$ is an extreme element of the class Q_2 of non-negative trigonometric polynomials having constant coefficient 1. The extreme elements of Q_2 have been determined in [3]. They are exactly the members of Q_2 of the form

$$q(e^{i\theta})=\Lambda|e^{i\theta}+\lambda_1|^2|e^{i\theta}+\lambda_2|^2,$$

where $|\lambda_1|=|\lambda_2|=1$ and

$$\Lambda^{-1}=(2\pi)^{-1}\int_0^{2\pi}|e^{i\theta}+\lambda_1|^2|e^{i\theta}+\lambda_2|^2\,d\theta.$$

Two special examples of interest are as follows:

$$F_{1/6,2/3}(w)=(1+w)^4/6$$

and

$$F_{1/2,0}(w)=(w^2+1)^2/2.$$

The corresponding members of (G) are as follows

$$G_{1/6,2/3}(z,w)=\frac{1+(1/3)z+(4/3)w+(1/3)w^2+(4/3)zw+zw^2}{1-zw^2}$$

and

$$G_{1/2,0}(z,w)=\frac{1+z+w^2+zw^2}{1-zw^2}=\frac{(1+z)(1+w^2)}{1-zw^2}.$$

(Of course, $G_{1/6,2/3}$ and $G_{1/2,0}$ are extreme elements of $\mathfrak{F}(G)$.) We observe that

$$\frac{(1+zw)(1+w)}{1-zw^2}=\frac{3}{4}G_{1/6,2/3}(z,w)+\frac{1}{4}G_{1/2,0}(-z,iw).$$

Since $G_{1/2,0}(-z,iw)\in\mathfrak{F}(G)$, we deduce the failure of Theorem 1 of [4] in the case where q is allowed to depend on both z and w.

OPEN QUESTION. Is there some simple way of characterizing the extreme elements of $\mathfrak{F}(G)$?

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