SEMILINEAR BOUNDARY VALUE PROBLEMS FOR UNBOUNDED DOMAINS

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1. Introduction. Let A be a non-negative self adjoint elliptic partial differential operator of order m on a (bounded or unbounded) domain $\Omega \subset \mathbf{R}^n$. We consider the Dirichlet problem for equations of the form

$$Au = f(x, u), \tag{1.1}$$

where $f(x,u)$ is a function defined on $\Omega \times \mathbf{R}$. Examples of the functions we consider include

$$f(x,u) = V(x)e^u(W(x)\cos e^u - 1), \tag{1.2}$$

where $V(x) \geq 0$, $V \in L^1$, $W \in L^\infty$. We show that for this choice of $f(x,u)$ the Dirichlet problem for (1.1) always has a solution (no matter what A, m, Ω are). The same is true for

$$f(x,u) = W(x) - V(x)ue^{u^2}, \tag{1.3}$$

where $W \in L^t$ for some t satisfying $1/2 \leq 1/t \leq 1/2 + m/2$ and V satisfies the assumptions above. Another example is

$$f(x,u) = V(x)[W(x)u^k \sin u^{k+1} - \sinh u + 1], \tag{1.4}$$

with V, W satisfying the same hypotheses as for (1.2) and $V \in L^t$ with t as above. We can also consider expressions such as

$$f(x,u) = W(x) - V(x)u^{2k-1}, \tag{1.5}$$

where V, W satisfy the same assumption as for (1.3).

In some instances we find a constant $\lambda_0 > 0$ such that

$$Au = \lambda f(x,u) \tag{1.6}$$

has a solution for each λ such that $0 < \lambda < \lambda_0$. This is done for the case

$$f(x,u) = V(x)|u|^q u + W(x), \tag{1.7}$$

where $q \geq -1$, $V \in L^\infty$, and $W \in L^t$ with $1/(q+2) + m/2 = 1/2 \leq 1/t \leq 1/2 + m/n$.

Another example is

$$f(x,u) = V_1(x)|u|^{q_1}u + V_2(x)|u|^{q_2}u, \tag{1.8}$$

with $-2 < q_1 < 0 < q_2$. In this case we give sufficient conditions for (1.6) to have a non-trivial solution.

We present two methods of attack. The first is to find a stationary point of a functional corresponding to (1.1). One of the major stumbling blocks in this

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approach is the fact that our assumptions on $f(x, u)$ are so weak that the functional is unbounded from above and below. We restrict the functional to a bounded region in hope of obtaining a minimum. If we are successful in obtaining such a minimum we must then show that the minimum is not located on the boundary of the region. Details are given in Section 3.

Another approach is to replace u in (1.1) with a truncated function $\Psi_k(u)$ which satisfies $|\Psi_k(u)| \le \min(|u|, k)$. For fixed k, the functional corresponding to

$$Au = f(x, \Psi_k(u)) \tag{1.9}$$

is bounded from below. We can then try to obtain a solution to (1.9) by minimizing the functional. Even when we are successful in this, we are left with the task of showing either that the solution obtained satisfies $\Psi_k(u) = u$ (highly unlikely), or that we can obtain a sequence $\{u_k\}$ of solutions of (1.9) which converge in some way to a solution of (1.1). We give the details in Section 4. Our main results are stated in Section 2.

2. The main results. In stating our hypotheses we shall use a family of norms depending on four parameters. Put

$$\begin{aligned} \omega_\alpha(x) &= |x|^{\alpha - n}, && 0 < \alpha < n, \\ &= 1 - \log|x|, && \alpha = n, \\ &= 1, && \alpha > n. \end{aligned}$$

For a function $V(x)$ defined on $\mathbf{R}^n$ we define

$$\begin{aligned} M_{\alpha,r,t,\delta}(V) &= \left(\int \left(\int_{|x-y|<\delta} |V(x)|^r \omega_\alpha(x-y)\, dx \right)^{t/r} dy \right)^{1/t} \quad (1 \le t < \infty) \\ &= \sup_y \left(\int_{|x-y|<\delta} |V(x)|^r \omega_\alpha(x-y)\, dx \right)^{1/r} \quad (t = \infty), \end{aligned} \tag{2.1}$$

$$M_{0,r,t,\delta}(V) = \|V\|_t = \text{the } L^t(\mathbf{R}^n) \text{ norm of } V$$

$$M_{0,r,t}(V) = M_{0,r,t,1}(V).$$

We let $M_{\alpha,r,t}$ be the set of those $V(x)$ such that $M_{\alpha,r,t}(V) < \infty$. The space $H^{s,p}$ is the completion of the set of test functions (C^∞ with compact supports) with respect to the norm

$$\|u\|_{s,p} = \|\bar{F}(1 + |\xi|^2)^{s/2} Fu\|_p, \tag{2.2}$$

where F denotes the Fourier transform, ξ its argument and $\bar{F}$ its inverse. When s is a positive integer and $1 < p < \infty$, the norm (2.2) is equivalent to the sum of L^p norms of u and all its derivatives up to order s. We shall need the following result, proved in [5; 6].

LEMMA 2.1. *If* $1 \le t \le \infty$, $1 < q$, $r < \infty$,

$$1 \le q/2 + 1/t, \quad 0 \le \alpha/nr \le mq/n + 1 - q/2 - 1/t, \tag{2.3}$$

then there is a constant $C(m, n, q, \alpha, r, t) < \infty$ *such that*

$$\int V(x)\,|u(x)|^q\,dx \le C(m,n,q,\alpha,r,t)M_{\alpha,r,t}(V)\,\|u\|_{m,2}^q. \tag{2.4}$$

If $t<\infty$, then

$$M_{\alpha,r,t,\delta}(V)\to 0 \quad \text{as } \delta\to 0, \tag{2.5}$$

and multiplication by $|V(x)|^{1/q}$ is a compact operator from $H^{s,p}$ to L^q. If $t=\infty$, the same will be true if we assume (2.5) *and*

$$\int_{|x-y|<1} |V(x)|^r\,|x-y|^{\alpha-n}\,dx\to 0 \quad \text{as } |y|\to\infty. \tag{2.6}$$

We are now ready to describe the problems considered. For m a positive integer and Ω an arbitrary (bounded or unbounded) domain in $\mathbf{R}^n$, let $W=H_0^{m,2}(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ (the set of test functions with supports in Ω) in the space $H^{m,2}$. Let $a(u,v)$ be a symmetric bilinear form on W satisfying

$$K_1^{-2}\|u\|_{m,2}^2\le a(u)\le C^2\|u\|_{m,2}^2 \quad (u\in W) \tag{2.7}$$

where $a(u)=a(u,u)$. The linear operator A associated with $a(u,v)$ is defined as follows. We shall say that $u\in D(A)$ and $Au=f$ if $u\in W$, $f\in L_{\text{loc}}(\Omega)$, and

$$a(u,\varphi)=(f,\varphi) \tag{2.8}$$

for all $\varphi\in W\cap L^\infty$ with compact supports in Ω. Let $g(x,y)$ be a function defined on $\Omega\times\mathbf{R}$ which is measurable in x for each $u\in\mathbf{R}$. We assume that

$$f(x,y)=\partial g(x,u)/\partial u \tag{2.9}$$

exists and is continuous in u for almost every $x\in\Omega$ and all $u\in\mathbf{R}$. For each $G\subset\subset\Omega$ (i.e., G is bounded and $\bar{G}\subset\Omega$) and $M<\infty$ there is a $V(x)\in L^1(G)$ such that

$$|f(x,u)|\le V(x), \quad x\in G, \ |u|\le M. \tag{2.10}$$

Furthermore, we assume that $g(x,0)\in L^1(\Omega)$ and that

$$g(x,y)-g(x,0)\le B(x,y)\equiv\sum_{k=1}^{\infty}V_k(x)\,|u|^{q_k}, \quad x\in\Omega,\ u\in\mathbf{R}, \tag{2.11}$$

where for each k

$$M_{\alpha_k,r_k,t_k,\delta}(V_k)\to 0 \quad \text{as } \delta\to 0 \tag{2.12}$$

holds for some set α_k, r_k, t_k, q_k satisfying

$$1\le q_k/2+1/t_k, \quad 0\le\alpha_k/nr_k\le mq_k/n+1-q_k/2-1/t_k. \tag{2.13}$$

if $t_k=\infty$, then we assume in addition that (2.6) holds for $V(x)=V_k(x)$, $r=r_k$, and $\alpha=\alpha_k$. We also assume that there is an $R>0$ such that

$$M(R)\equiv\sum_{k=1}^{\infty}C_kM_{\alpha_k,r_k,t_k}(V_k)R^{q_k}<\infty, \tag{2.14}$$

where

$$C_k=C(m,n,q_k,\alpha_k,r_k,t_k). \tag{2.15}$$

If $2m\le n$, we make one final assumption.

We assume that for each $G \subset\subset \Omega$ there are functions $c_k(v)$, $W_k(x)$ and constants $\sigma_k \geq 1$ such that $c_k(v) \in L^{\infty}_{\text{loc}}$,

$$c_k(tv)/t \to e_k(v) \quad \text{as } 0 < t \to 0 \quad (v \in \mathbf{R}), \tag{2.16}$$

and

$$W_k(x) \in M_{\beta_k, \rho_k, \tau_k}, \quad 1 \leq k \leq N, \tag{2.17}$$

for some set β_k, ρ_k, τ_k such that

$$1 \leq \sigma_k/2 + 1/\tau_k, \quad 0 \leq \beta_k/n\rho_k \leq m\sigma_k/n + 1 - \sigma_k/2 - 1/\tau_k, \tag{2.18}$$

and

$$\begin{aligned} g(x, y) \leq g(x, u+v) &+ \sum_1^N c_k(v) W_k(x) |u|^{\sigma_k} \\ &+ c_0(v) |g(x, u)|, \quad x \in G, \ u, v \in \mathbf{R}. \end{aligned} \tag{2.19}$$

(The number N may depend on G.)

Our first result is the following.

THEOREM 2.2. *Let*

$$\lambda_0^{-1} = 2 \inf_{R>0} M(K_1 R)/R^2. \tag{2.20}$$

If $0 < \lambda < \lambda_0$, *then there is a* $u \in W$ *such that*

$$Au = \lambda f(x, u). \tag{2.21}$$

If $R > 0$ *is such that* $\lambda M(K_1 R) < R^2$, *then* (2.21) *has a solution* $u \in W$ *satisfying* $a(u) \leq R^2$.

THEOREM 2.3. *Theorem* 2.2 *holds if we replace* (2.19) *with either*

$$[uf(x, u)]_+ \leq b(x, u) \equiv \sum_1^N W_k(x) |u|^{\sigma_k} \quad (x \in G, \ u \in \mathbf{R}) \tag{2.22}$$

or

$$[uf(x, u)]_- \leq b(x, u) \quad (x \in G, \ u \in \mathbf{R}), \tag{2.23}$$

where $h_{\pm} = \max(0, \pm h)$ *and the* $W_k(x)$ *and* σ_k *are as above.*

In Theorems 2.2 and 2.3, the solution of (2.21) may be $u = 0$. The following theorem gives a criterion which guarantees the existence of a non-zero solution.

THEOREM 2.4. *In addition to the hypotheses of either Theorem* 2.2 *or Theorem* 2.3, *assume that: there is an open set* $\Omega_0 \subset \Omega$, *functions* $w(x)$, $w_k(x)$ *in* $L^1(\Omega_0)$ *and positive functions* $\alpha(u)$, $\beta_k(u)$ *such that* $w(x)$ *does not change sign in* Ω_0 $(w(x) \not\equiv 0)$;

$$\alpha(u) \to \infty, \quad \beta_k(u) = 0(u^2), \quad 1 \leq k \leq N \tag{2.24}$$

as $u \to 0$; *and*

(2.25) $$w(x)\alpha(u)|u|u-\sum_{1}^{N} w_k(x)\beta_k(u)\le g(x,u)-g(x,0)$$

holds for $x\in\Omega_0$ *and* $|u|\le 1$. *Then* (2.21) *has a solution* $u\neq 0$ *having the properties described in Theorem* 2.2.

REMARK 2.5. If $\alpha(u)=|u|^{-\theta}$ in Theorem 2.4 with $\theta>0$, then we can allow $w(x)$ to change sign in Ω_0.

THEOREM 2.6. *Let*

$$\lambda_1^{-1}=2\liminf_{R\to\infty} M(K_1R)/R^2.$$

Assume that $0<\lambda<\lambda_1$ *and that the hypotheses of either Theorem* 2.2 *or Therem* 2.3 *hold. Assume that there are functions* $w(x), w_k(x), \alpha(u), \beta_k(u)$ *as described in Theorem* 2.4, *but* (2.24) *holds as* $u\to\infty$ *and* (2.25) *holds for* $x\in\Omega_0$, $u\in\mathbf{R}$. *Then* (2.21) *has a solution* $u\neq 0$.

REMARK 2.7. If $\alpha(u)=|u|^{\theta}$ in Theorem 2.6, where $\theta>0$, then we can allow $w(x)$ to change sign in Ω_0.

Theorem 2.2 is proved in Section 3, while Theorems 2.3–2.6 are proved in Section 4.

Now we present some examples of equations that can be solved by our methods.

1. $$f(x,u)=V(x)e^u(W(x)\cos e^u-1),$$

where $V(x)\ge 0$, $V\in L^1$, $W\in L^\infty$. In this case

$$M(R)\le \|V\|_1(\|W\|_\infty+1),\quad \lambda_0=\infty.$$

Thus (2.21) has a solution for every $\lambda>0$.

2. $$f(x,u)=qV(x)|u|^{q-2}u+W(x),$$

where $V\in L^\infty$, $1/2=1/q+m/n$, and

(2.26) $$W\in M_{\alpha,r,t},\quad 0\le \alpha/nr\le m/n+1/2-1/t,\quad t\le 2.$$

In this case we first approximate the problem with one for the exponent $p<q$. Put

$$V_p(x)=\begin{cases} V(x), & |x|<(q-p)^{-1}\\ 0, & |x|>(q-p)^{-1}\end{cases}$$

$$f_p(x,u)=pV_p(x)|u|^{p-2}u+W(x).$$

By Lemma 2.1,

(2.27) $$\int |W(x)u(x)|\,dx\le M_2\|u\|_{m,2}$$

for some constant M_2. Also,

$$\int |V_p(x)|\,|u(x)|^p\,dx \le \left(\int |V(x)|\,|u(x)|^q\,dx\right)^{p/q} \times \left(\int |V_p(x)|\,dx\right)^{(q-p)/q}$$
$$\le \|V\|_\infty^{p}(C(q-p)^{-n})^{(q-p)/q}\|u\|_{m,2}^{p} = M_1\|u\|_{m,2}^{p}.$$

Thus $M(R) = M_1R^p + M_2R$ and the minimum of $M(K_1R)R^{-2}$ in $R>0$ is at

$$K_1R_{0,p} = (M_2/(p-2)M_1)^{1/p-1},$$

with

$$\lambda_{0,p}^{-1} = \frac{p-1}{(p-2)^{(p-2)/(p-1)}}K_1^2\|V\|_\infty^{1/(p-1)}M_2^{(p-2)/(p-1)}(C(q-p)^{-n})^{(q-p)/q(p-1)}.$$

Note that

$$M_1 \to \|V\|_\infty^q$$
$$R_{0,p} \to (M_2/(q-2)\|V\|_\infty^q)^{1/q-1} = R_0$$
$$\lambda_{0,p}^{-1} \to \frac{q-1}{(q-2)^{(q-2)/(q-1)}}K_1^2\|V\|_\infty^{1/(q-1)}M_2^{(q-2)/(q-1)} = \lambda_0^{-1}$$

as $p \to q$. Thus if $0<\lambda<\lambda_0$, then for p sufficiently close to q there is a solution u_p of

$$a(u_p, v) = \lambda(f_p(u_p), v) \tag{2.28}$$

satisfying $\|u_p\|_{m,2} \le K_1(R_0+1)$. Thus there is a subsequence of $\{u_p\}$ which converges weakly in $H^{m,2}$ and a.e. to some limit u. Also

$$\int |V_p(x)\,|u_p(x)|^{p-1}|^{q'}\,dx$$
$$\le \|V\|_\infty^{q'}\left(\int |u_p(x)|^q\,dx\right)^{(p-1)/(q-1)}(C(q-p)^{-n})^{(q-p)/(q-1)} \to \|V\|_\infty\|u\|_{m,2}^q.$$

Thus there is a subsequence of $\{V_p|u_p|^{p-2}u_p\}$ which converges weakly in $L^{q'}$. Hence

$$(f_p(u_p), v) = (pV_p(x)\,|u_p|^{p-2}u_p + W, v) \to (qV(x)\,|u|^{q-2}u + W, v) = (f(u), v),$$

provided $v \in L^q$. Taking the limit on both sides of (2.28), we obtain

$$a(u,v) = \lambda(f(u), v), \quad v \in H^{m,2} \cap L^q$$

3. $$f(x,u) = W(x) - V(x)ue^{u^2},$$

where W satisfies (2.26) and $V \in L^1$, $V(x) \ge 0$. Here we apply Theorem 2.3. In this case $\lambda_0 = \infty$.

4. $$f(x,u) = V(x)(W(x)u^k \sin u^{k+1} - \sinh u + 1),$$

where $V \in L^1$, $W \in L^\infty$, $V(x) \ge 0$, and V satisfies (2.26). Here $\lambda_0 = \infty$.

5. $$f(x,u)=W(x)-V(x)u^{2k-1},$$

where W satisfies (2.26) and $0\le V(x)\in L^1_{\text{loc}}$. Here we use Theorem 2.3; $\lambda_0=\infty$.

6. $$f(x,u)=V_1(x)|u|^{q_1-2}u+V_2(x)|u|^{q_2-2}u,$$

with $0<q_1<2<q_2$. Assume that

$$\int V_i(x)|u(x)|^{q_i}\,dx\le M_i\|u\|^{q_i}_{m,2}.$$

Then

$$M(K_1R)R^{-2}\le M_1K_1^{q_1}R^{q_1-2}+M_2K_1^{q_2}R^{q_2-2}.$$

A calculation gives

$$\lambda_0^{-1}=M_1^{(q_2-2)/(q_2-q_1)}\times M_2^{(2-q_1)/(q_2-q_1)}(q_2-q_1)(q_2-2)^{(2-q_2)/(q_2-q_1)}(2-q_1)^{(q_1-2)/(q_2-q_1)}K_1^{2-q_1}.$$

By Theorem 2.4 the solution will not be trivial if there is an open set in which $V_1(x)\ge 0$, $V_1(x)\not\equiv 0$.

3. A variational problem. In this section we give the proof of Theorem 2.2. Let $V=\{v\in W\mid g(x,v(x))\in L^1(\Omega)\}$ and put $G(v)=a(v)-2\lambda I(v)$, $v\in V$, where $I(v)=\int_\Omega g(x,v(x))\,dx$. For $R\ge 0$ let

$$S_R=\{v\in V\mid a(v)\le R^2\}$$
$$\gamma_R=\sup_{S_R} I(v)$$
$$\rho_R=\inf_{S_R} F(v).$$

We will prove the following.

LEMMA 3.1. *Under the hypotheses of Theorem* 2.2, $G(v)$ *has a minimum and* $I(v)$ *has a maximum on* S_R.

Postponing the proof of Lemma 3.1 until later, we show how it can be used in the proof of Theorem 2.2. First we note that $u+\varphi$ is in V when $u\in V$ and $\varphi\in L^\infty$ with compact support in Ω. For by (2.19) and (2.11),

$$g(x,u)-\sum_1^N c_k(\varphi)W_k(x)|u|^{\sigma_k}-C_0(\varphi)|g(x,u)|\le g(x,u+\varphi)\le g(x,0)+B(x,u+\varphi) \tag{3.1}$$

for x in some domain $G\subset\subset\Omega$ containing the support of φ. By Lemma 2.1, the functions on the right and left in (3.1) are in $L^1(G)$. Thus the same is true of $g(x,u+\varphi)$. Since

$$\int_\Omega g(x,u+\varphi)\,dx=\int_G g(x,u+\varphi)\,dx+\int_{\Omega\setminus G} g(x,u)\,dx,$$

we see that $g(x, u+\varphi) \in L^1(\Omega)$. Suppose u is an interior point of S_R and G attains its minimum on S_R at u. We shall show that u is a solution of (2.11). Let Y be the set of those $\varphi \in W \cap L^\infty$ having compact supports in Ω. If $\varphi \in Y$ and $a(\varphi)$ is sufficiently small, then $G(u+\varphi) \geq G(u)$ and consequently

$$\int_\Omega [g(x, u+\varphi) - g(x, u)]\, dx \leq 2a(u, \varphi) + a(\varphi). \tag{3.2}$$

Let φ be any function in Y and let $G \subset\subset \Omega$ contain its support. Put

$$h(x, u, \varphi) = g(x, u+\varphi) - g(x, u) + \sum_1^N c_k(\varphi) W_k(x) |u|^{\sigma_k} + c_0(\varphi) |g(x, u)|.$$

Then $h(x, u, \varphi) \geq 0$ by (2.19), and

$$t^{-1} h(x, u, t\varphi) \to \varphi f(x, u) + \sum_1^N e_k(\varphi) W_k(x) |u|^{\sigma_k} + e_0(\varphi) |g(x, u)| \quad \text{a.e. in } G$$

by (2.16). Moreover, by (3.2),

$$\begin{aligned} t^{-1} \int_G h(x, u, t\varphi)\, dx &\leq \lambda^{-1} [a(u, \varphi) + \tfrac{1}{2} t a(\varphi)] \\ &\quad + t^{-1} \int_G \left(\sum_1^N c_k(t\varphi) W_k(x) |u(x)|^{\sigma_k} + c_0(t\varphi) |g(x, u)| \right) dx \\ &\quad \to \lambda^{-1} a(u, \varphi) + \int_G \left(\sum_1^N e_k(\varphi) W_k |u|^{\sigma_k} + e_0(\varphi) |g(x, u)| \right) dx \end{aligned}$$

as $t \to 0$. Thus, by Fatou's lemma,

$$\int \varphi(x) f(x, u)\, dx \leq \lambda^{-1} a(u, \varphi).$$

Replacing φ by $-\varphi$, we see that

$$a(u, \varphi) = \lambda(f(x, u), \varphi), \quad \varphi \in Y. \tag{3.3}$$

Thus u is a solution of (2.21). It thus remains only to show that G indeed has an interior minimum in S_R for some $R > 0$. Suppose, to the contrary, that G has no interior minimum on S_R for any $R > 0$. Then we must have $a(v) = R^2$ for every $v \in S_R$ such that $G(v) = \rho_R$. This implies that $\gamma_R = I(v)$ for each such v as well. For if there were a $w \in S_R$ such that $I(v) < I(w)$, then we would have

$$G(w) = a(w) - 2\lambda I(w) < R^2 - 2\lambda I(v) = G(v),$$

contradicting the fact that $G(v) = \rho_R$. Hence

$$\rho_R = G(v) = a(v) - 2\lambda I(v) = R^2 - 2\lambda \gamma_R.$$

Thus

$$R^2 - 2\lambda \gamma_R = \rho_R \leq \rho_0 = 2\lambda I(0) = -2\lambda \gamma_0 \tag{3.4}$$

for each $R > 0$. By (2.11) and Lemma 2.1,

$$\int [g(x,u)-g(x,0)]\,dx \le \int B(x,u)\,dx \tag{3.5}$$
$$\le \sum_1^\infty C_k M_{\alpha_k,r_k,t_k}(V_k)\|u\|_{m,2}^{q_k} = M(\|u\|_{m,2}).$$

Thus by (2.7)

$$\gamma_R - \gamma_0 \le M(K_1 R). \tag{3.6}$$

Combining (3.4) and (3.6) we see that $\lambda^{-1} \le 2M(K_1R)/R^2$ for all $R>0$. Thus $\lambda_0 \le \lambda$ by (2.20). This contradicts the assumption that $\lambda < \lambda_0$. □

We now give the following.

Proof of Lemma 3.1. Let $b \ge 0$ be fixed, and put $H(v) = ba(v) - 2\lambda I(v)$. We must show that H has a minimum on S_R. Let $\{v_j\}$ be a minimizing sequence. Since W is a Hilbert space, we can extract a subsequence (also denoted by $\{v_j\}$) converging weakly to some element in W. By Lemma 2.1,

$$\int_\Omega B(x, v_j(x))\,dx \to \int_\Omega B(x, v(x))\,dx. \tag{3.7}$$

Since $B(x,v)-g(x,v) \ge 0$, by Fatou's lemma we have

$$\int_\Omega [B(x,v)-g(x,v)]\,dx \le \liminf \int_\Omega [B(x,v_j)-g(x,v_j)]\,dx.$$

Since the left-hand side is bounded, this shows that $g(x,v(x))$ is in $L^1(\Omega)$. This means that $v \in S_R$. We also have

$$ba(v)+2\lambda \int_\Omega [B(x,v_j)-g(x,v_j)]\,dx = b[a(v)-a(v_j)]+2\lambda\int_\Omega B(x,v_j)\,dx + H(v_j).$$

Since $H(v_j)$ converges to its glb ρ_R in S_R and since $a(v) \le \liminf a(v_j)$, we see that $H(v) \le \rho_R$. This proves the lemma. □

We note that assumptions (2.19), (2.22), and (2.23) are all unnecessary when $n < 2m$, since the functions in W are bounded. It then follows automatically that $u+\varphi$ is in V when $u \in V$ and $\varphi \in L^\infty$ has compact support. For if G is the support of φ, then

$$\int_\Omega g(x, u+\varphi)\,dx = \int_\Omega g(x,u)\,dx + \int_G \varphi f(x, u+\theta\varphi)\,dx.$$

Both integrals on the right exist (the second in view of (2.10)). Note that one can always take the function $\theta(x)$ to be measurable (cf. [3, p. 177]). Next we note that

$$t^{-1}\int [g(x,u+t\varphi)-g(x,u)]\,dx = \int_G \varphi(x) f(x, u+t\theta\varphi)\,dx.$$

The integral on the right converges to a limit as $t \to 0$, since the integrand converges a.e. and is majorized by a function in $L^1(G)$ by (2.10). Thus

$$t^{-1}[G(u+t\varphi)-G(u)] \to 2a(u,\varphi)+2\lambda\int \varphi f(x,u)\,dx$$

as $t \to 0$ for each φ in Y. Since $G(u)$ is an interior minimum in S_R and the limit is independent of the sign of t, it follows that (3.3) holds.

4. An alternate approach. Now we turn to the proof of Theorem 2.3. Let $\psi(t)$ be an infinitely differentiable function on $\mathbf{R}$ such that $\psi(t)=t$ for $t<-1$, $\psi(t)=0$ for $t>1$, and $0\le\psi'(t)\le 1$. For each $k>1$ put

$$\psi_k(t)=\begin{cases} k+\psi(t-k), & t\ge 0,\\ -k-\psi(-t-k), & t<0.\end{cases}$$

Note that $\psi_k(t)$ is infinitely differentiable,

$$\min(|t|,k-1)\le|\psi_k(t)|\le\min(|t|,k), \tag{4.1}$$

$$0\le\psi_k'(1)\le 1,\quad \psi_k(t)/t\ge 0, \tag{4.2}$$

$$\psi_k'(t)=0 \quad \text{for } |t|\ge k+1, \tag{4.3}$$

and

$$0\le t\psi_k'(t)/\psi_k(t)\le(k+1)/(k-1). \tag{4.4}$$

Put

$$g_k(x,u)=g(x,\psi_k(u)),$$
$$f_k(x,u)=\partial g_k(x,u)/\partial u=f(x,\psi_k(u))\psi_k'(u),$$
$$I_k(u)=\int_\Omega g_k(x,u)\,dx,$$
$$G_k(u)=a(u)-2\lambda I_k(u).$$

Note that

$$g_k(x,u)-g(x,0)\le B(x,\psi_k(u))\le B(x,u)$$

and

$$|g_k(x,u+v)-g_k(x,u)|=|vf(x,\varphi_k(u+\theta v))\psi_k'(u+\theta v)|$$
$$\le|v|V(x),\quad x\in G\subset\subset\Omega$$

by (2.10). It therefore follows that g_k satisfies all of the hypotheses of Theorem 2.2. Therefore, by that theorem we can conclude that there is a $u_k\in S_R$ such that

$$a(u_k,v)=\lambda(f_k(u_k),v),\quad v\in W. \tag{4.5}$$

Put

$$h_\pm(x,u)=\begin{cases}[uf(x,u)]_\pm/u, & u\ne 0,\\ f(x,0)_\pm, & u=0.\end{cases}$$

Then $uh_\pm(x,u)\ge 0$ and

$$f(x,u)=h_+(x,u)-h_-(x,u). \tag{4.6}$$

Set $h_{\pm k}(x,u)=h_\pm(x,\psi_k(u))\psi_k'(u)$. Then

$$f_k(x,u)=[h_+(x,\psi_k(u))-h_-(x,\psi_k(u))]\psi_k'(u) \\ =h_{+k}(x,u)-h_{-k}(x,u). \tag{4.7}$$

Consequently, by (4.5),

$$a(u_k,\varphi u_k)=\lambda(f_k(u_k),\varphi u_k) \\ =\lambda(h_{+k}(u_k),\varphi u_k)-(h_{-k}(u_k),\varphi u_k)] \tag{4.8}$$

for all $\varphi\in C_0^\infty(\Omega)$. Assume (2.22) holds.

Note that, by (2.22), (4.2), and (4.4),

$$vh_{+k}(v)=[\psi_k(v)f(x,\psi_k(x))]_+\psi_k'(v)v/\psi_k(v) \\ \le 2b(x,\psi_k(v))\le 2b(x,v),\quad k\ge 3. \tag{4.9}$$

Thus

$$(h_{+k}(u_i),\varphi u_i)\le 2\|\varphi b(u_i)\|_1 \\ \le C\Sigma N_j\|u_i\|_{m,2}^{\sigma_k}\le C_1(R), \tag{4.10}$$

and consequently, by (4.8),

$$\lambda(h_{-k}(u_i),\varphi u_i)\le\lambda C_1(R)+a(u_i)^{1/2}a(\varphi u_i)^{1/2} \\ \le C_2(R). \tag{4.11}$$

Take $\varphi\ge 0$ and let $G\subset\subset\Omega$ contain the support of φ. Then

$$\int\varphi|h_{+k}(x,u_k)-h_{+j}(x,u_j)|\,dx\le\int\varphi|h_{+k}(x,u_k)-h_{+\ell}(x,u_k)|\,dx \\ +\int\varphi|h_{+\ell}(x,u_k)-h_{+\ell}(x,u_j)|\,dx \\ +\int\varphi|h_{+\ell}(x,u_j)-h_{+j}(x,u_j)|\,dx \\ =I_1+I_2+I_3. \tag{4.12}$$

Take $\ell<j,k$ and let $G_{k\ell}$ be the set of all $x\in G$ such that $|u_k(x)|>\ell-1$. Since $h_{+k}(x,v)=h_{+\ell}(x,v)=h_+(x,v)$ for $|v|<\ell-1$, the first integral and on the right in (4.12) vanishes outside $G_{k\ell}$. Hence

$$\begin{aligned} I_1 &\le\int_{G_{k\ell}}\varphi(|h_{+k}(x,u_k)|+|h_{+\ell}(x,u_k)|)\,dx \\ &\le\frac{1}{\ell-1}\int_{G_{k\ell}}\varphi[u_kh_{+k}(x,u_k)+u_kh_{+\ell}(x,u_k)]\,dx \\ &=\frac{1}{\ell-1}[(h_{+k}(u_k),\varphi u_k)+(h_{+\ell}(u_k),\varphi u_k)] \\ &\le C_1(R)/(\ell-1) \end{aligned}$$

by (4.10), since $vh_{+k}(x,v)\ge 0$ for all k. Let $\epsilon>0$ be given, and take ℓ so large that $C_1(R)<\epsilon(\ell-1)$. Then $I_1<\epsilon$ and similarly $I_3<\epsilon$. By (2.10) there is a function $V(x)\in L^1(G)$ such that

$$|h_{+\ell}(x,v)|\le V(x),\quad x\in G,\ v\in \mathbf{R}.$$

Since $h_{+\ell}(x,u_k)-h_{+\ell}(x,u_j)\to 0$ a.e. in G as $j,k\to\infty$, we have

$$I_2=\int_G \varphi|h_{+\ell}(x,u_k)-h_{+\ell}(x,u_j)|\,dx\to 0,\quad j,k\to\infty.$$

This shows that the left-hand side of (4.12) does likewise. Hence $\varphi h_{+k}(x,u_k)$ converges in $L^1(G)$ to a limit. Since the u_k are in S_R, there is a subsequence converging weakly in W to an element u in W. Another subsequence will converge a.e. to u. For this subsequence $\varphi h_{+k}(x,u_k)$ will converge a.e. to $\varphi h_+(x,u)$. Thus $\varphi h_{+k}(x,u_k)$ converges in $L^1(G)$ to $\varphi h_+(x,u)$. Since φ was arbitrary, we see that $h_{+k}(x,u_k)$ converges in $L^1(G)$ to $h_+(x,u)$ for each $G\subset\subset\Omega$. The same reasoning applies to $h_{-k}(x,u_k)$ (all we need is (4.11) in place of (4.10)). Hence $f_k(x,u_k)\to f(x,u)$ in $L^1(G)$ for each $G\subset\subset\Omega$. Thus, if $\varphi\in Y$, then

$$a(u_k,\varphi)\to a(u,\varphi),\ (f_k(u_k),\varphi)\to(f(u),\varphi).$$

Thus

$$a(u,\varphi)=\lambda(f(u),\varphi),\quad \varphi\in Y \tag{4.13}$$

by (4.4). This completes the proof when (2.22) holds. If (2.23) holds we have, in place of (4.11),

$$(h_{-k}(u_i),\varphi\psi_k(u_i))\le\|\varphi b(u_i)\|_1\le C_1(R),$$

and in place of (4.10) we have

$$\lambda(h_{+k}(u_i),\varphi\psi_k(u_i))\le\lambda C_1(R)+a(u_i)^{1/2}a(\varphi\psi_k(u_i))^{1/2}\le C_2(R).$$

The proof then proceeds as before. □

In proving Remark 2.5 we shall make use of the following simple lemma.

LEMMA 4.1. *If* $w(x)\in L^1(\Omega_0)$, $0<d<\infty$, *and*

$$\int_{\Omega_0} w(x)\varphi(x)^d\,dx=0\quad(\varphi\in C_0^\infty(\Omega),\ \varphi(x)\ge 0) \tag{4.14}$$

then $w(x)=0$ a.e.

Proof. Let $j_k(x)=k^n\exp\{-(1-k^2|x|^2)^{-1}\}$ and let y be any point in Ω_0. Then for k sufficiently large, the function $\varphi(x)=j_k(x-y)^{1/d}$ is in $C_0^\infty(\Omega_0)$ and is ≥ 0. Thus, by (4.14),

$$\int w(x)j_k(x-y)\,dx=0. \tag{4.15}$$

It is well known that this implies that $w(x)=0$ a.e. □

We now give the following.

Proof of Theorem 2.4. It was shown in the proof of Theorem 2.2 that G has an interior minimum in S_R for some $R>0$, and that this minimum is a solution of (2.21). We shall show that under hypotheses (2.24) and (2.25), the minimum cannot be at 0. To see this, we note that by hypothesis there is a $\psi \in C_0^\infty(\Omega_0)$ such that $a(\psi)=1$ and

$$b(t)=\int_{\Omega_0} w(x)\psi(x)|\psi(x)|\alpha(t\psi(x))\,dx \to \infty \tag{4.16}$$

as $t\to 0$. Moreover, for $t>0$ sufficiently small,

$$b(t)t^2-\Sigma b_k(t)\le I(t\psi)-I(0) \tag{4.17}$$

by (2.25), where

$$b_k(t)=\int_{\Omega_0} w_k(x)\beta_k(t\psi(x))\,dx\le C_k t^2. \tag{4.18}$$

Thus there is a $t<R$ such that

$$G(t\psi)-G(0)\le t^2-2\lambda(b(t)t^2-\Sigma b_k(t))<0. \tag{4.19}$$

This shows that the minimum of G in S_R is not $G(0)$. Thus the solution given by Theorem 2.2 is not 0.

Next let us turn our attention to Theorem 2.3. It follows from (4.19) and (2.24) that there is a $t<R$ and a $\delta>0$ such that $G(t\psi)\le G(0)-\delta$. Since ψ is bounded, $G_k(t\psi)=G(t\psi)$ for k sufficiently large. The solution u_k of (4.5) can be taken as the point of S_R where G_k attains its minimum. Thus

$$G_k(u_k)\le G(0)-\delta. \tag{4.20}$$

As shown in the proof of Theorem 2.3, $\{u_k\}$ has a subsequence converging a.e. and weakly in W to an element u. We have

$$a(u)\le \liminf a(u_k). \tag{4.21}$$

Moreover, by (2.11) and (4.1),

$$g_k(x,u)-g(x,0)\le B(x,\psi_k(u))\le B(x,u).$$

Thus $h_k(x,u)=B(x,u)+g(x,0)-g_k(x,u)\ge 0$, and

$$\begin{aligned}&\int_\Omega [B(x,u)+g(x,0)-g(x,u)]\,dx\\ &\le \liminf \int_\Omega h_k(x,u_k)\,dx \\ &\le \int_\Omega [B(x,u)+g(x,0)]\,dx-\limsup\int_\Omega g_k(x,u_k)\,dx\end{aligned} \tag{4.22}$$

by (3.4) and Fatou's lemma. Thus by (4.20)–(4.22),

$$G(u)\le \liminf G(u_k)\le G(0)-\delta.$$

This shows that $u\ne 0$, and the proof is complete. □

Proof of Remark 2.5. By Lemma 4.1 there is a $\psi \in W$ such that $a(\psi)=1$ and

$$b=\int w(x)\psi(x)\,|\psi(x)|^{1-\theta}\,dx>0.$$

We merely take $b(t)=t^{-\theta}b$ and continue as before. □

Proof of Theorem 2.6. By (2.24) there is a $\psi \in W$ such that $a(\psi)=1$ and (4.16) holds as $t\to\infty$. By (4.17) and (4.18), we see that (4.19) holds for t sufficiently large. By hypothesis, there is an $R>t$ such that

$$2M(K_1R)/R^2<\lambda^{-1}.$$

By the argument given in the proof of Theorem 2.2, G does not attain its minimum on the boundary of S_R. On the other hand, (4.19) shows that the minimum is not $G(0)$. Thus the solution given by Theorem 2.2 is not 0. In the case of Theorem 2.3 we follow the proof of Theorem 2.4. □

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