ON THE OMITTED AREA PROBLEM

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1. Introduction. Let S denote the class of univalent functions f in $\Delta=\{z:|z|<1\}$ with $f(0)=0$, $f'(0)=1$. The omitted area problem was first posed by Goodman [6] in 1949, reiterated by MacGregor [9] in his survey article in 1972, then reposed in a more general setting by Brannan [1]. It appears in Bernardi's survey article [4] and in several open problem sets since then ([5], [10]). The problem is to find the maximum area of the region omitted from Δ by $f(\Delta)$ as f varies over the family S. If we let

$$\beta=\sup_{f\in S}\operatorname{area}[\Delta-f(\Delta)], \tag{1.1}$$

then it is known that

$$.24\pi\le\beta<.38\pi.$$

The lower bound was recently obtained by Barnard and Pearce [3], who "rounded the corners" in certain gearlike domains to obtain a close approximation to the extremal functions suggested by the authors in [2] and [8]. The upper bound is conceptually harder since it requires an estimate on the omitted area of each function in S. Thus it is perhaps not surprising that the upper bound of $.38\pi$ obtained by Goodman and Reich [7] in 1955 has not been improved upon over the years. Unfortunately, it appears difficult to use the geometric description of the extremal function given in [2] and [8] to calculate β directly. However, we use an easily obtained generalization of the second author's major result in [8] to obtain the upper bound $\beta<.31\pi$ found in Corollary 1 to our Theorem 1.

We first state the result in [8].

THEOREM A. *There is an f in S with $\beta=\operatorname{area}[\Delta-f(\Delta)]$, where β is as in (1.1), and for which $f(\Delta)$ is circularly symmetric with respect to the positive real axis. Moreover, there exist θ_0, θ_1, $0<\theta_0<\theta_1<\pi$, and $\alpha>0$ such that if $E_j=\{e^{i\theta}:\theta_j<\theta<2\pi-\theta_j\}$, $j=0,1$, then*

(1.2) *f' has a non-zero continuous extension to $E_0\cup\Delta$ which is Hölder-continuous with exponent $\frac{1}{2}$ on E_0,*

(1.3) $f(E_1)=\partial[f(\Delta)\cap\Delta]=\Gamma$, $f(e^{i\theta_1})=e^{i\psi}, 0<\psi<\pi$, *and* $\partial f(\Delta)=(-\infty,-1)\cup\{e^{i\theta}:\psi\le\theta\le2\pi-\psi\}\cup\Gamma$,

(1.4) $|f'(e^{i\theta})|=\alpha$, $e^{i\theta}\in E_1-\{e^{i\phi}:\operatorname{Im} f(e^{i\phi})=0\}$,

(1.5) $\{e^{i\phi}:\operatorname{Im} f(e^{i\phi})=0\}$ *consists of at most a finite number of arcs or points.*

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We remark that the proof of Theorem A in [8] is rather long and complicated. However, the main thrust of the argument in [8] is to show that a suitable generalization of the Julia variational formula applies in this case, even though $\partial f(\Delta) \cap \Delta$ is not known *a priori* to be smooth. For the reader interested in only the intuitive idea behind the proof of Theorem A, we note that if the boundary of the extremal domain is assumed sufficiently smooth, then the key equality (1.4) follows from using the Julia variational formula as in the proof of (2.1).

Next, for $w = u + iv$, we let

$$\beta_1 = \sup_{f \in S} \iint_{\Delta - f(\Delta)} |w|^{-2}\, du\, dv. \tag{1.6}$$

The proof in [8] of Theorem A is easily adapted to prove the following theorem.

THEOREM B. *There is an F in S with*

$$\beta_1 = \iint_{\Delta - F(\Delta)} |w|^{-2}\, du\, dv,$$

where β_1 is as in (1.6), *and for which $F(\Delta)$ is circularly symmetric with respect to the positive real axis. Moreover, there exist θ_0, θ_1, $0 < \theta_0 < \theta_1 < \pi$, and $\alpha > 0$ such that if $E_j = \{e^{i\theta} : \theta_j < \theta < 2\pi - \theta_j\}$, $j = 0, 1$, then* (1.2), (1.3), *and* (1.5) *are valid with f replaced by F. Also, F satisfies*

$$|F'(e^{i\theta})| = \alpha |F(e^{i\theta})|, \quad e^{i\theta} \in E_1 - \{e^{i\phi} : \operatorname{Im} F(e^{i\phi}) = 0\}. \tag{1.7}$$

In order to modify the proof of Theorem A to obtain Theorem B, essentially all that is necessary is to replace "area" in the arguments by the "weighted area" in (1.6). We omit the details. Finally we note that in [8] the analogue of Theorems A and B was obtained for the class S^* of starlike univalent functions. In this case, the extremal function in each theorem is unique and the exact values of β and β_1 (with S replaced by S^*) are known (see [8, Theorems 2 and 3]).

In this paper we use Theorem B to prove the following.

THEOREM 1. *Let F be as in Theorem* B. *Then F is unique and*

$$\beta_1 = (.37\ldots)\pi.$$

As mentioned previously, Theorem 1 implies the following.

COROLLARY 1. *Let β be as in* (1.1). *Then*

$$\beta \le .31\pi.$$

Theorem 1 will be proved in Section 2. In Section 3 we point out how Corollary 1 can be deduced from Theorem 1.

2. Proof of Theorem 1. Let F be as in Theorem B and put $P(z) = zF'(z)/F(z)$. Let $K = \{e^{i\phi} : \operatorname{Im} F(e^{i\phi}) = 0\} \cap E_1$ and $K_1 = E_1 - K$. We first show that K does not contain an arc I. Indeed, otherwise from an application of the Julia variational formula in the usual way (see e.g. [2]) and the fact that F is extremal, it would follow that

(2.1) $$|P(e^{i\theta})| \le \alpha, \quad e^{i\theta} \in K.$$

A sketch of the argument necessary to prove (2.1) is as follows: Let $\Gamma = \partial F(\Delta) \cap \Delta$, and let $n(w)$ be the outer normal to $F(\Delta)$ when $w \in \Gamma$ and $\operatorname{Im} w \neq 0$. Suppose that $\gamma_1 \subseteq K \cap \{z: \operatorname{Im} z > 0\}$, $\gamma_2 \subseteq K_1 \cap \{z: \operatorname{Im} z > 0\}$ are two arcs and put $\Gamma_1 = F(\gamma_1)$, $\Gamma_2 = F(\gamma_2)$. We note that Γ_1 is an interval on the negative real axis. We regard Γ_1 as having two sides, a top half (denoted Γ_1^+) and a bottom half. Let ψ be a non-negative infinitely differentiable function in Δ which vanishes at the endpoints of Γ_1, Γ_2. If $w \in \Gamma_2$, we displace Γ_2 at w an amount $\epsilon\psi(w)$ in the direction of $n(w)$. We also displace $w \in \Gamma_1^+$ an amount $\epsilon\psi(w)$ in the positive v direction. Suppose the resulting domain is simply connected and let $\hat{F}$ be the Riemann mapping function from Δ to this domain with $\hat{F}(0) = 0$, $\hat{F}'(0) > 0$. Then from the Julia variational formula we see that, up to $o(\epsilon)$ terms,

$$\hat{F}(z) = F(z) + \frac{\epsilon z F'(z)}{2\pi}\left[\int_{\gamma_2} \frac{\zeta+z}{\zeta-z}\psi(F(\zeta))\frac{|d\zeta|}{|F'(\zeta)|} - \int_{\gamma_1} \frac{\zeta+z}{\zeta-z}\psi(F(\zeta))\frac{|d\zeta|}{|F'(\zeta)|}\right].$$

It follows that, up to $o(\epsilon)$ terms, the change in the mapping radius is given by

$$\Delta_1 = \frac{\epsilon}{2\pi}\left[\int_{\gamma_2} \psi(F(\zeta))\frac{|d\zeta|}{|F'(\zeta)|} - \int_{\gamma_1} \psi(F(\zeta))\frac{|d\zeta|}{|F'(\zeta)|}\right].$$

Now the change in the weighted area in (1.6) is, up to $o(\epsilon)$ terms, given by

$$\begin{aligned}\Delta_2 &= \epsilon\left[\int_{\Gamma_1} \frac{\psi(w)}{|w|^2}\,|dw| - \int_{\Gamma_2} \frac{\psi(w)}{|w|^2}\,|dw|\right] \\ &= \epsilon\left[\int_{\gamma_1} \frac{\psi(F(\zeta))}{|F'(\zeta)|}|P(\zeta)|^2\,|d\zeta| - \int_{\gamma_2} \frac{\psi(F(\zeta))}{|F'(\zeta)|}|P(\zeta)|^2\,|d\zeta|\right].\end{aligned}$$

Since $|P| = \alpha$ on γ_2, we see that if $|P| > \alpha$ on γ_1 then $\Delta_2 > 2\pi\alpha^2\Delta_1$. If (2.1) were false, we could choose ψ, γ_1, γ_2 so that $\hat{F}$ exists and $\Delta_1 = 0$. Then $\Delta_2 > 0$, which contradicts the extremality of F for sufficiently small $\epsilon > 0$. Thus (2.1) is valid.

To continue the proof that K does not contain an arc I, we next note from Theorem B that

(2.2) $$P(\partial\Delta - \{1\}) \subseteq (-\infty, \infty) \cup \{it: -\infty < t < \infty\} \cup \{w: |w| = \alpha\}.$$

Also, from (1.2) it follows that

(2.3) $$0 \notin P(\partial\Delta - \{1\}).$$

Since the winding number of a point with respect to $P(\partial\Delta - \{1\})$ is constant on components of the complement of $P(\partial\Delta - \{1\})$, it follows from (2.1)–(2.3) that

(2.4) $$|P| = \alpha \quad \text{on } E_1.$$

In fact, otherwise we could show that $P = 0$ somewhere in Δ, which would imply that F is not univalent. Finally, from (2.4) we deduce that $P = i\alpha$ on I. It then follows from the identity theorem for analytic functions that $P \equiv i\alpha$ in Δ. We conclude from this contradiction that K contains no arcs. From this statement, (1.2), and (1.7), we see that (2.4) continues to hold.

Next we deduce, from the mapping properties of F in Theorem B, (2.2), (2.4), and the above remark on winding numbers, that P maps Δ univalently onto a domain with

$$P[\partial\Delta-\{1\}]=(-\infty,-\alpha)\cup\{w:|w|=\alpha\}\cup\{\pm it:t\geq\rho\}, \tag{2.5}$$

for some $\rho>\alpha$. Using (2.5), we can calculate the explicit form of P either by a Schwarz–Cristoffel type argument or by recognizing P as a composition of elementary functions. We prefer the latter method.

Let $-1<r<1$, $M>1$, $R=2r/(1+r^2)$, and $\operatorname{Im} w>0$. In order to obtain somewhat simpler forms and because of their repeated use, we let

$$\begin{aligned}
M_1&=\sqrt{M/(M-1)}, & M_2&=(M-1)/(M+1), & M_3&=(R-M)/(M+1),\\
w_1&=\sqrt{w-M_2}, & w_2&=\sqrt{w+M_2}, & w_3&=(M+1)w+M,\\
R_1&=\sqrt{2M-R+1}, & R_2&=\sqrt{2M-R-1}, & R_3&=\sqrt{(1+R)/(1-R)}.
\end{aligned}$$

Here, as in the sequel, $\sqrt{w+a}$ (a real, $\operatorname{Im} w>0$) is chosen so that $0\leq\arg\sqrt{w+a}\leq\pi/2$. Let

$$\sigma(w)=-\frac{w_1w_2}{2\sqrt{M}/(1+M)+i\sqrt{w-1}\sqrt{w+1}},\quad \operatorname{Im} w>0.$$

We note that σ maps $(-\infty,-1]\cup[1,\infty]$ onto $\{\xi:|\xi|=1,\ \operatorname{Im}\xi\geq 0\}$. Also, $[-1,-M_2)$ is mapped onto $[1,\infty)$ while $(-M_2,M_2)$ is mapped onto a line segment on the imaginary axis joining a point is, $s>1$, to ∞. Finally, $(M_2,1]$ is mapped onto $(-\infty,-1]$.

Next, put

$$\begin{aligned}
w(\zeta)&=\left[\frac{2\zeta}{1+\zeta^2}-M\right]\Big/(M+1),\quad |\zeta|<1,\ \operatorname{Im}\zeta>0,\\
\zeta(z)&=(z+r)/(1+rz),\quad |z|<1,\ \operatorname{Im} z>0.
\end{aligned}\tag{2.6}$$

We note that $w(\zeta(z))$, $|z|<1$, $\operatorname{Im} z>0$, maps the unit upper semidisk univalently onto the upper half-plane in such a way that $[-1,1]$ is mapped onto $[-1,-M_2]$. Thus

$$H(z)=\sigma[w(\zeta(z))],\quad |z|<1,\ \operatorname{Im} z>0,$$

maps this semidisk onto

$$\{\xi:\operatorname{Im}\xi>0\}-[\{\xi:|\xi|\leq 1\}\cup\{it:t\geq s\}]$$

in such a way that $[-1,1)$ is mapped onto $[1,\infty)$. We now extend H univalently to Δ using the Schwarz reflection principle. From the above discussion, we conclude for some choice of r and M that $H/H(1)=P$. To find F we solve (2.6) for z and ζ in terms of w. We obtain

$$\begin{aligned}
\frac{dz}{z}&=\frac{(1-r^2)d\zeta}{(\zeta-r)(1-r\zeta)}=\frac{(1-r^2)w_3\,d\zeta}{\zeta[-2r+(1+r^2)w_3]},\\
-i\frac{d\zeta}{\zeta}&=\frac{dw}{\sqrt{w+1}\,w_2w_3}.
\end{aligned}\tag{2.7}$$

Using (2.7) and changing variables, we obtain

$$\begin{aligned}\int H(z)\frac{dz}{z} &= (1-r^2)i\int\frac{\sigma(w)\,dw}{\sqrt{w+1}\,w_2[-2r+(1+r^2)w_3]}\\ &= -(1-r^2)i\int\frac{w_1[2\sqrt{M}/(M+1)-i\sqrt{w^2-1}\,]\,dw}{\sqrt{w+1}\,w_1^2w_2^2[-2r+(1+r^2)w_3]}\\ &= \frac{-2(1-r^2)i\sqrt{M}}{(M+1)}\int\frac{dw}{\sqrt{w+1}\,w_1w_2^2[-2r+(1+r^2)w_3]}\\ &\quad -(1-r^2)\int\frac{(w-1)\,dw}{\sqrt{w-1}\,w_1w_2^2[-2r+(1+r^2)w_3]}\\ &= I+J.\end{aligned}$$

I and J can be integrated using the trigonometric substitutions

$$(M+1)w = M\sec\psi - 1 \quad\text{and}\quad (M+1)w = M+\sec\psi,$$

respectively. Doing this, we get locally for $\operatorname{Im} z>0$, $|z|<1$,

$$\operatorname{Log} F(z) = \lambda_1(I+J)(w(\zeta(z)))+\lambda_2, \tag{2.8}$$

where

$$\begin{aligned}I(w) &= R_3\left\{-M_1\operatorname{Log}\left[\frac{A(w)-\sqrt{M-1}\,i}{A(w)+\sqrt{M-1}\,i}\right]\right.\\ &\quad\left.+\frac{2\sqrt{M}}{\sqrt{1+R}\,R_2}\operatorname{Log}\left[\frac{\sqrt{1+R}\,A(w)-R_2 i}{\sqrt{1+R}\,A(w)+R_2 i}\right]\right\},\\ J(w) &= R_3\left\{-M_1\operatorname{Log}\left[\frac{M_1-B(w)}{M_1+B(w)}\right]+\frac{R_1}{R_2}\operatorname{Log}\left[\frac{R_1-R_2B(w)}{R_1+R_2B(w)}\right]\right\},\end{aligned} \tag{2.9}$$

and

$$A(w) = w_1/\sqrt{w+1}, \qquad B(w) = \sqrt{w-1}/w_1.$$

Here λ_1 and λ_2 are constants to be determined so that $F\in S$. Also all logarithms in (2.9) can be chosen so that $-\pi\le\arg(\cdot)\le 0$, as follows from mapping properties of A, B and the fact that $\operatorname{Im} w>0$. To determine λ_1, λ_2, note that $w(\zeta(0)) = M_3$, so if $w=M_3+\epsilon$ then, for $\epsilon\to 0$,

$$\sqrt{1+R}\,(R_2)^{-1}A(w) = i\left[1-\frac{M(M+1)\epsilon}{(R+1)R_2^2}\right]+O(\epsilon^2),$$

$$R_2B(w)/R_1 = 1+\frac{(M+1)\epsilon}{R_1^2R_2^2}+O(\epsilon^2).$$

Letting

$$R_4 = \frac{R_2-\sqrt{M-1}\,\sqrt{R+1}}{R_2+\sqrt{M-1}\,\sqrt{R+1}}, \qquad R_5 = \frac{M_1R_2-R_1}{M_1R_2+R_1},$$

and using the above estimates in (2.9), we obtain

$$I(M_3+\epsilon) = R_3\left\{-M_1\operatorname{Log}(R_4)+\frac{2\sqrt{M}}{\sqrt{1+R}\,R_2}\operatorname{Log}\left[\frac{-M(M+1)\epsilon}{2(R+1)R_2^2}\right]\right\}+O(\epsilon), \tag{2.10}$$

$$J(M_3+\epsilon)=R_3\left\{-M_1\operatorname{Log}(R_5)+\frac{R_1}{R_2}\operatorname{Log}\left[\frac{-\epsilon(M+1)}{2R_1^2R_2^2}\right]\right\}+O(\epsilon). \tag{2.11}$$

Next observe from (2.6) that

$$\epsilon(\zeta(z))=\frac{2(1-r^2)^2z}{(1+r^2)^2(M+1)}+O(|z|^2),\quad |z|<1,\ \operatorname{Im} z>0.$$

Using this equality in (2.10)-(2.11) and (2.9), we obtain

$$\begin{aligned}(\lambda_1)^{-1}&(\operatorname{Log}F(z)-\lambda_2)/R_3\\ &=\left[\frac{2\sqrt{M}}{\sqrt{1+R}R_2}+\frac{R_1}{R_2}\right]\operatorname{Log}z-M_1\operatorname{Log}(R_4R_5)\\ &\quad+\frac{2\sqrt{M}}{\sqrt{1+R}R_2}\operatorname{Log}\left[\frac{-M(1-R)}{R_2^2}\right]+\frac{R_1}{R_2}\operatorname{Log}\left[\frac{-(1-R^2)}{R_1^2R_2^2}\right]+O(|z|).\end{aligned} \tag{2.12}$$

Thus

$$(\lambda_1)^{-1}=R_3\left[\frac{2\sqrt{M}}{\sqrt{1+R}R_2}+\frac{R_1}{R_2}\right],$$

$$\begin{aligned}\lambda_2=-\lambda_1R_3\bigg\{&-M_1\operatorname{Log}(R_4R_5)+\frac{2\sqrt{M}}{\sqrt{1+R}R_2}\operatorname{Log}\left[\frac{M(1-R)}{R_2^2}\right]\\ &+\frac{R_1}{R_2}\operatorname{Log}\left[\frac{(1-R^2)}{R_1^2R_2^2}\right]-\frac{i\pi}{R_2}\left[R_1+\frac{2\sqrt{M}}{\sqrt{1+R}}\right]\bigg\}=\mu+i\pi.\end{aligned} \tag{2.13}$$

Let $e^{i\theta}$ be such that $w(\zeta(e^{i\theta}))=M_2$. Then from our construction we see that $F(e^{i\theta})=-1$. Using (2.9) with $w=M_2$ and the above equalities, it follows that

$$\begin{aligned}\lambda_1^{-1}(\log F(e^{i\theta})-\lambda_2)/R_3&=\lambda_1^{-1}(i\pi-\lambda_2)/R_3\\ &=\left[2M_1-\frac{2\sqrt{M}}{\sqrt{1+R}R_2}-\frac{R_1}{R_2}\right]i\pi.\end{aligned} \tag{2.14}$$

Hence, from (2.14),

$$\mu=0, \tag{2.15}$$

$$2M_1-\frac{2\sqrt{M}}{\sqrt{1+R}R_2}-\frac{R_1}{R_2}=0. \tag{2.16}$$

We note for fixed R, $-1<R<1$, that the left-hand side of (2.16) considered as a function of M is decreasing on $(1,\infty)$, as is easily shown. It follows that (2.16) defines M as a function of R with $\lim_{R\to-1}M(R)=1$. Using a computer, we calculated M as a function of R for varying values of R. We then put these numbers into the expression for μ. It turns out that (2.15) and (2.16) have a unique solution for $-1<R<1$, $M>1$, when $R=-0.69051\cdots$ and $M=1.31846\cdots$. Uniqueness was proved by first showing that the derivative of the expression for μ considered as a function of R has a negative derivative on $[-.9,-\frac{1}{2}]$. We then estimated μ on $(-1,-.9)$ and $(-\frac{1}{2},1)$.

With R, M as above, we now calculate β_1 in Theorem 1. Put $P(\xi)=(\log|\xi|)^2$, $\xi\in\mathbf{C}-\{0\}$, and let $f(w)=F(z)$ where w and z are related by (2.6). Using Green's theorem, symmetry, and conformal mapping, we deduce for

$$N=\Delta\cap\partial F(\Delta)\cap\{z: \operatorname{Im} z>0\}$$

that

$$\begin{aligned}\beta_1&=2\int_{\Delta-F(\Delta)} P_{\xi\bar{\xi}}\,dA\\&=\frac{1}{2}\int_{\partial F(\Delta)\cap\Delta}\frac{\partial P}{\partial n}\,ds=\int_N\frac{\partial P}{\partial n}\,ds\\&=\int_{|u|>1}\frac{d}{dv}(\log|f(w)|)^2\,du,\end{aligned}$$

where dA is Lebesgue two-dimensional measure. Here $w=u+iv$ and the above integral is taken over a portion of the real axis. Observe from (2.9) that I is pure imaginary and J is real when $w=u+iv$, $v=0$, $|u|>1$. Using this fact and the Cauchy–Riemann equations, we find that

$$\beta_1=2i\lambda_1^2\int_{|u|>1} J\frac{dI}{du}\,du.$$

Since J and dI/dw are analytic in $\operatorname{Im} w>0$, the above integral can be evaluated by residues. To do this for given small $\epsilon_i>0$, $1\le i\le 6$, let $C=\sum_{i=1}^{12} C_i$, where

$$\begin{aligned}
C_1&=\{e^{i\theta}/\epsilon_1: 0\le\theta\le\pi\},\\
C_2&=\{u: -1/\epsilon_1\le u\le -\epsilon_2-1\},\\
C_3&=\{-\epsilon_2 e^{-i\theta}-1: 0\le\theta\le\pi\},\\
C_4&=\{w: -1+\epsilon_2\le u\le M_3-\epsilon_3,\ v=0\},\\
C_5&=\{M_3-\epsilon_3 e^{-i\theta}: 0\le\theta\le\pi\},\\
C_6&=\{w: M_3+\epsilon_3\le u\le -M_2-\epsilon_4,\ v=0\},\\
C_7&=\{-M_2-\epsilon_4 e^{-i\theta}: 0\le\theta\le\pi\},\\
C_8&=\{w: -M_2+\epsilon_4\le u\le M_2-\epsilon_5,\ v=0\},\\
C_9&=\{M_2-\epsilon_5 e^{-i\theta}: 0\le\theta\le\pi\},\\
C_{10}&=\{w: M_2+\epsilon_5\le u\le 1-\epsilon_6,\ v=0\},\\
C_{11}&=\{1-\epsilon_6 e^{-i\theta}: 0\le\theta\le\pi\},\\
C_{12}&=\{w: 1+\epsilon_6\le u\le 1/\epsilon_1,\ v=0\},
\end{aligned}$$

(see Figure 1). From Cauchy's theorem we see that

$$\int_C J\frac{dI}{dw}\,dw=0.$$

We note that

$$\frac{dI}{dw}=\frac{-2\sqrt{1-R^2}\sqrt{M}\,i}{(1+M)^2\sqrt{w+1}\,w_1(w-M_3)w_2^2}. \tag{2.17}$$

Using (2.17) and (2.9) we deduce, as $\epsilon_1,\epsilon_2\to 0$, that

$$\int_{C_1+C_2+C_3} J\frac{dI}{dw}\,dw\to\int_{-\infty}^{-1} J\frac{dI}{dw}\,dw.$$

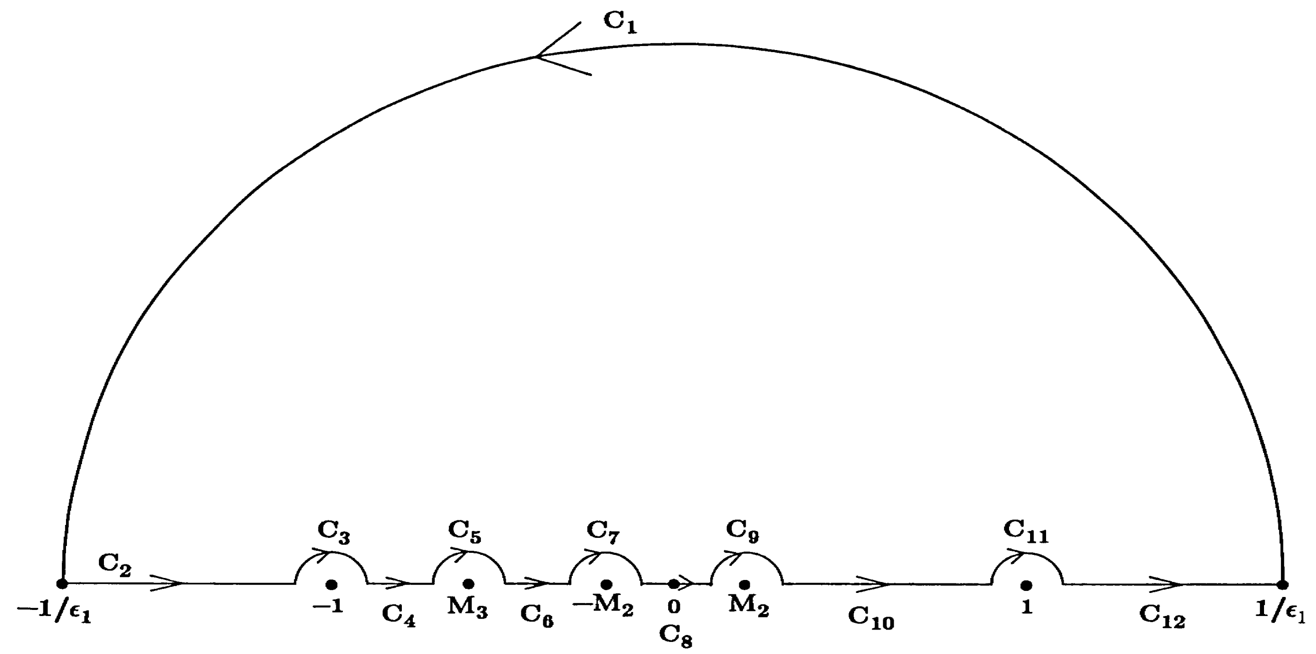

Figure 1

Similarly, as ϵ_1, ϵ_6, and $\epsilon_5 \to 0$,

$$\int_{C_9+C_{11}+C_{12}} J\frac{dI}{dw}\,dw \to \int_1^\infty J\frac{dI}{dw}\,dw.$$

Using the fact that $J(dI/dw)$ is real on C_{10}, C_4 and the above observations, we get

$$\frac{1}{2}\lambda_1^{-2}\beta_1 = \operatorname{Im}\left[\sum_{i=5}^{8}\int_{C_i} J\frac{dI}{dw}\,dw\right]. \tag{2.18}$$

Now on C_5, it follows from (2.17) and (2.11) that

$$\operatorname{Im}\left[\int_{C_5} J\frac{dI}{dw}\,dw\right] = -2\pi\left[\frac{\sqrt{M}}{R_2\sqrt{1-R}} + O(\epsilon_3)\right]J[M_3+\epsilon_3]. \tag{2.19}$$

Also on C_6, dI/dw is real and $\operatorname{Im} J(w) = -\pi R_1 R_3/R_2$. Hence

$$\operatorname{Im}\int_{C_6} J\frac{dI}{dw}\,dw = -\pi\frac{R_1R_3}{R_2}[\operatorname{Re} I(-M_2-\epsilon_4) - \operatorname{Re} I(M_3+\epsilon_3)]. \tag{2.20}$$

Again from (2.17),

$$\operatorname{Im}\int_{C_7} J\frac{dI}{dw}\,dw = [\pi M_1R_3 + O(\epsilon_4)]\operatorname{Re} J[-M_2+\epsilon_4]. \tag{2.21}$$

Finally, since dI/dw is real and $\operatorname{Im} J = \pi R_3[M_1 - R_1/R_2] = \delta$ on C_8, we find that

$$\begin{aligned}\operatorname{Im}\int_{C_8} J\frac{dI}{dw}\,dw &= \delta[\operatorname{Re} I(M_2-\epsilon_5) - \operatorname{Re} I(-M_2+\epsilon_4)]\\ &= -\delta\operatorname{Re} I(-M_2+\epsilon_4) + O(\sqrt{\epsilon_5}),\end{aligned} \tag{2.22}$$

since $\operatorname{Re} I(M_2) = 0$.

We note that if $w=-M_2+\epsilon$, then

$$\frac{A(w)}{\sqrt{M-1}}=i\left[1-\frac{M}{4M_2}\epsilon\right]+O(\epsilon^2),$$

$$\frac{B(w)}{M_1}=1+\frac{\epsilon}{(4MM_2)}+O(\epsilon^2).$$

Thus,

$$I(-M_2+\epsilon)=R_3\left\{-M_1\operatorname{Log}[-M\epsilon/(8M_2)]+\frac{2\sqrt{M}}{\sqrt{1+RR_2}}\operatorname{Log}(-R_4)\right\}+O(\epsilon) \tag{2.23}$$

and

$$J(-M_2+\epsilon)=R_3\{-M_1\operatorname{Log}[-\epsilon/(8MM_2)]+(R_1/R_2)\operatorname{Log}(-R_5)\}+O(\epsilon). \tag{2.24}$$

Using (2.10)-(2.11) and (2.23)-(2.24) in (2.19)-(2.22), we obtain, after summing (2.19)-(2.22),

$$\lim_{\epsilon\to 0}\operatorname{Im}\sum_{i=5}^{8}\int_{C_i}J\frac{dI}{dw}\,dw=\pi\lambda_1^{-1}M_1R_3\operatorname{Log}(R_5/R_4)+2\pi M_1^2R_3^2\operatorname{Log}M+\frac{2\pi\sqrt{M}\sqrt{1+RR_1}}{R_2^2(1-R)}\operatorname{Log}\left(\frac{MR_1^2}{R+1}\right),$$

where $\epsilon=\max_{1\le j\le 6}\{\epsilon_j\}$. Evaluating this equality and (2.13) at the values of R and M mentioned earlier, and using (2.18), we obtain Theorem 1. □

3. Proof of Corollary 1. Let f be as in Theorem A. That is, $\beta=\operatorname{area}[\Delta-f(\Delta)]$, where β is as in (1.1). Corollary 1 is an easy consequence of the following assertion: Let r_0 be such that $\pi(1-r_0^2)=\beta$. If $w=u+iv$, then

$$\iint_{\Delta-f(\Delta)}|w|^{-2}\,du\,dv\ge\iint_{\{r_0<|w|<1\}}|w|^{-2}\,du\,dv. \tag{3.1}$$

To prove this assertion, note that if

$$G_1=[\Delta-f(\Delta)]-\{w:r_0<|w|<1\},$$
$$G_2=\{w:r_0<|w|<1\}-[\Delta-f(\Delta)],$$

then

$$\max_{w\in G_2}(|w|^{-2})\le\min_{w\in G_1}(|w|^{-2}),$$

since $r\to r^{-2}$ is decreasing on $(0,\infty)$. This inequality clearly implies (3.1). From (3.1) and Theorem 1 we get $-2\pi\log(r_0)\le .37\pi$, which implies that $\beta=\pi(1-r_0^2)\le .31\pi$. This completes the proof of Corollary 1. □

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