ON SOME GENERALIZATIONS OF THE TRANSFINITE DIAMETER

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In this paper we will establish a useful form of the Golusin inequalities [3] for univalent functions in a multiply-connected domain. We will utilize them to obtain a generalization of the concept of the transfinite diameter of a bounded closed set, which in turn can be used to obtain some interesting distortion theorems in conformal mapping.

1. Let Δ be a domain in the complex ζ-plane which contains the point at infinity and is bounded by N smooth curves Γ_ν $(\nu=1,\dots,N)$. The orientation of Γ_ν is chosen so that Δ lies to the left. Let $f(\zeta)$ be univalent and regular analytic in Δ, except for a simple pole at infinity, such that $f'(\infty)=1$. Define the functional

$$\phi[f]=\sum_{i,k=1}^{n} x_i x_k \log\left|\frac{f(\zeta_i)-f(\zeta_k)}{\zeta_i-\zeta_k}\right| \tag{1}$$

for n given points $\zeta_i\in\Delta$ and n fixed real constants x_i. Here

$$\log|(f(\zeta_i)-f(\zeta_k))/(\zeta_i-\zeta_k)|$$

is to be replaced by $\log|f'(\zeta_i)|$ for $i=k$. We ask for the functions $f(\zeta)$ of the above type which maximize the functional.

The existence of such extremum functions is ensured by the compactness of the function class considered. Let then $f(\zeta)$ be such an extremum function and let $D=f(\Delta)$ be the image domain of Δ in the z-plane. We can characterize the extremum function and the extremum domain by the method of variations. We choose a point z_0 on ∂D and consider a function

$$z^*=z+\frac{a\rho^2}{z-z_0}+o(\rho^2), \quad |a|=1, \tag{2}$$

which is univalent in the whole z-plane except for a small subcontinuum of ∂D around z_0. The existence of such variation is well known [5].

The function

$$f^*(\zeta)=f(\zeta)+\frac{a\rho^2}{f(\zeta)-z_0}+o(\rho^2) \tag{3}$$

is then in the same family of functions and competes with $f(\zeta)$. We easily calculate

$$\phi[f^*]=\phi[f]-\operatorname{Re}\left\{a\rho^2\left(\sum_{i=1}^{n}\frac{x_i}{z_0-f(\zeta_i)}\right)^2\right\}+o(\rho^2), \tag{4}$$

and because of the extremality of $f(\zeta)$, we obtain the inequality

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$$\text{(5)} \qquad \operatorname{Re}\left\{a\rho^2\left(\sum_{i=1}^{n}\frac{x_i}{z_0-f(\zeta_i)}\right)^2\right\}+o(\rho^2)\geq 0$$

for all admissible variations (2). By Schiffer's fundamental lemma [5] we conclude that D is a slit domain bounded by analytic arcs with the differential equation

$$\text{(6)} \qquad z'(\tau)^2\left(\sum_{i=1}^{n}\frac{x_i}{z-f(\zeta_i)}\right)^2+1=0,$$

where τ is a properly chosen real parameter. This leads to

$$\text{(7)} \qquad \frac{d}{d\tau}\sum_{i=1}^{n}x_i\log|z-f(\zeta_i)|=0.$$

That is,

$$\text{(8)} \qquad \sum_{i=1}^{n}x_i|\log f(\zeta)-f(\zeta_i)|=k_\nu \quad \text{on each } \Gamma_\nu.$$

Equations (7) and (8) characterize the extremum function $f(\zeta)$ and the extremum domain.

To evaluate these results, we introduce the Green's function $g(\zeta,\eta)$ of Δ, which is harmonic in Δ except for the logarithmic pole at the point η:

$$\text{(9)} \qquad g(\zeta,\eta)=\log\frac{1}{|\zeta-\eta|}+\gamma(\zeta,\eta),$$

and which vanishes if $\zeta\to\partial\Delta$. We define

$$\text{(10)} \qquad g(\zeta)=g(\zeta,\infty)=\log|\zeta|+\rho+O\left(\frac{1}{|\zeta|}\right),$$

where ρ is the Robin constant of Δ. We need also the harmonic measures of the various curves Γ_ν, defined by

$$\text{(11)} \qquad \omega_\nu(\zeta)=\frac{1}{2\pi}\int_{\Gamma_\nu}\frac{\partial g(\eta,\zeta)}{\partial n}\,ds$$

and their periods

$$\text{(12)} \qquad p_{\nu\mu}=\frac{1}{2\pi}\int_{\Gamma_\mu}\frac{\partial\omega_\nu}{\partial n}\,ds,$$

where n denotes the interior normal of $\partial\Delta$. It is well known that the matrix $((p_{\mu\nu}))_{1,\dots,N-1}$ is negative definite and hence possesses a negative-definite inverse matrix $((\pi_{\mu\nu}))_{1,\dots,N-1}$.

Now we can conclude that (8) leads to the identity

$$\text{(13)} \qquad \sum_{i=1}^{n}x_i\log|f(\zeta)-f(\zeta_i)|=sg(\zeta)-\sum_{i=1}^{n}x_i g(\zeta,\zeta_i)+\sum_{\nu=1}^{N-1}{}' k_\nu^*\omega_\nu(\zeta)+k_N,$$

with $s=\sum_{i=1}^{n}x_i$ and $k_\nu^*=k_\nu-k_N$.

We still have to determine the constants k_ν^*. Observe that

$$\text{(14)} \qquad \log|f(\zeta)-f(\zeta_i)|=\log|\zeta|+O\left(\frac{1}{|\zeta|}\right) \quad \text{as } \zeta\to\infty.$$

Hence, letting $\zeta \to \infty$ in (13), we find

$$0 = s\rho - \sum_{i=1}^{n} x_i g(\zeta_i) + \sum_{\nu=1}^{N-1} k_\nu^* \omega_\nu(\infty) + k_N. \tag{15}$$

Next, we use the fact that $\log(f(\zeta) - f(\zeta_i))$ does not change if ζ circulates around each closed curve Γ_ν; that is,

$$\frac{1}{2\pi} \int_{\Gamma_\nu} \frac{\partial}{\partial n} \log|f(\zeta) - f(\zeta_i)|\, ds = 0. \tag{16}$$

Hence, by Definitions (11) and (12), we have from (13) the N equations

$$0 = s\omega_\nu(\infty) - \sum_{i=1}^{n} x_i \omega_\nu(\zeta_i) + \sum_{\mu=1}^{N-1} k_\mu^* p_{\nu\mu}, \quad \nu = 1, \dots, N. \tag{17}$$

Using the inverse matrix $((\pi_{\mu\nu}))_{1,\dots,N-1}$, we find

$$k_\mu^* = \sum_{\alpha=1}^{N-1} \pi_{\mu\alpha} \sum_{i=1}^{n} x_i(\omega_\alpha(\zeta_i) - \omega_\alpha(\infty)), \quad \mu = 1, \dots, N-1, \tag{18}$$

while from (15)

$$k_N = -s\rho + \sum_{i=1}^{n} x_i g(\zeta_i) - \sum_{\mu,\nu=1}^{N-1} \pi_{\nu\mu} \omega_\nu(\infty) \sum_{i=1}^{n} x_i(\omega_\mu(\zeta_i) - \omega_\mu(\infty)). \tag{18'}$$

We combine (13) with (18) and (18′) to obtain

$$\begin{aligned}\sum_{i=1}^{n} x_i \log\left|\frac{f(\zeta) - f(\zeta_i)}{\zeta - \zeta_i}\right| &= s(g(\zeta) - \rho) + \sum_{i=1}^{n} x_i g(\zeta_i) - \sum_{i=1}^{n} x_i \gamma(\zeta, \zeta_i) \\ &\quad + \sum_{\mu,\nu=1}^{N-1} \pi_{\nu\mu}(\omega_\nu(\zeta) - \omega_\nu(\infty)) \sum_{i=1}^{n} x_i(\omega_\mu(\zeta_i) - \omega_\mu(\infty)).\end{aligned} \tag{19}$$

By writing (19) for $\zeta = \zeta_k$, multiplying by x_k and summing over k, we obtain

$$\begin{aligned}\phi[f] &= 2s \sum_{i=1}^{n} x_i g(\zeta_i) - s^2\rho - \sum_{i,k=1}^{n} x_i x_k \gamma(\zeta_i, \zeta_k) \\ &\quad + \sum_{\mu,\nu=1}^{N-1} \pi_{\nu\mu}\left(\sum_{k=1}^{n} x_k \omega_\mu(\zeta_k) - s\omega_\mu(\infty)\right)\left(\sum_{k=1}^{n} x_k \omega_\nu(\zeta_k) - s\omega_\nu(\infty)\right).\end{aligned} \tag{20}$$

Because f is extremal, we have the general estimate

$$\begin{aligned}\phi[f] &\le 2s \sum_{i=1}^{n} x_i g(\zeta_i) - s^2\rho - \sum_{i,k=1}^{n} x_i x_k \gamma(\zeta_i, \zeta_k) \\ &\quad + \sum_{\mu,\nu=1}^{N-1} \pi_{\nu\mu}\left(\sum_{k=1}^{n} x_k \omega_\mu(\zeta_k) - s\omega_\mu(\infty)\right)\left(\sum_{k=1}^{n} x_k \omega_\nu(\zeta_k) - s\omega_\nu(\infty)\right)\end{aligned} \tag{21}$$

for every admissible function $f(\zeta)$. Since the matrix $((\pi_{\mu\nu}))_{1,\dots,N-1}$ is negative definite, we have

$$\phi[f] \le 2s \sum_{i=1}^{n} x_i g(\zeta_i) - s^2\rho - \sum_{i,k=1}^{n} x_i x_k \gamma(\zeta_i, \zeta_k). \tag{21'}$$

2. In 1923, Fekete [2] introduced an important set measure in the complex plane defined as follows. Let Γ be a bounded closed set in the ζ-plane. Define the nth diameter of Γ by

$$d_n = \max_{\zeta_i \in \Gamma} \left(\prod_{1 \le i < k \le n} |\zeta_i - \zeta_k| \right)^{1/\binom{n}{2}}. \tag{22}$$

It is easily seen that $d_{n+1} \le d_n$ and therefore

$$d = \lim_{n \to \infty} d_n \tag{23}$$

exists. Fekete called d the transfinite diameter of the set Γ and made interesting applications of this concept to algebraic problems.

Szegö [6] pointed out the potential theoretic significance of this measure. Indeed, let $\Gamma = \partial\Delta = \bigcup_{\nu=1}^{N} \Gamma_\nu$. Consider a positive mass distribution μ on the compact set Γ with $\mu(\Gamma) = 1$. The logarithmic potential of μ and its energy integral are defined by

$$p(z) = \int_\Gamma \log \frac{1}{|\zeta - z|} \, d\mu(\zeta) \tag{24}$$

and

$$I[\mu] = \int_\Gamma \int_\Gamma \log \frac{1}{|z - \zeta|} \, d\mu(z) \, d\mu(\zeta), \tag{25}$$

respectively. Let

$$d\hat{\mu} = \frac{1}{2\pi} \frac{\partial g}{\partial n} \, ds, \tag{26}$$

where g is the Green's function of Δ with a pole at ∞. It is obvious that

$$\hat{\mu}(\Gamma) = \frac{1}{2\pi} \int_\Gamma \frac{\partial g}{\partial n} \, ds = 1. \tag{27}$$

It is well known that

$$I[\hat{\mu}] = \min_\mu I[\mu] = \rho, \tag{28}$$

where ρ is the Robin constant of Δ.

Now take n points ζ_i on Γ and write (22) in the form

$$-\binom{n}{2} \log d_n \le \frac{1}{2} \sum_{\substack{i,k=1 \\ i \ne k}}^{n} \log \frac{1}{|\zeta_i - \zeta_k|}. \tag{29}$$

Multiplying (29) by $\prod_{i=1}^{n} d\hat{\mu}(\zeta_i) \ge 0$ and integrating over all ζ_i, we find that in view of (27) and (28)

$$-\binom{n}{2} \log d_n \le \binom{n}{2} \rho \tag{30}$$

and in the limit, as $n \to \infty$,

$$\log \frac{1}{d} \le \rho. \tag{31}$$

To prove the simple relation between the Robin constant and the transfinite diameter

$$\rho = \log \frac{1}{d}, \tag{32}$$

one has to prove the inverse inequality to (31). This was achieved by an ingenious application of the theory of Chebyshev polynomials. We will now apply the Golusin inequality (21) to achieve the same result, and this method will allow us to deal analogously with many generalizations of the Fekete concept.

3. We define the continua $\Gamma_\nu(\epsilon)$ in Δ which are homotopic and close to the Γ_ν and on which $g(\zeta) \le \epsilon$. We apply the inequality (21′) to the identity function $f(\zeta) = \zeta$, the constants $x_i = 1/n$ (so that $s = 1$), and choose the points ζ_i on $\Gamma(\epsilon) = \bigcup_{\nu=1}^N \Gamma_\nu(\epsilon)$ such that

$$d_n(\epsilon)^{\binom{n}{2}} = \prod_{1 \le i < k \le n} |\zeta_i - \zeta_k|. \tag{33}$$

Since $\phi[f] = 0$ for $f(\zeta) = \zeta$ and $g(\zeta, \eta) = -\log|\zeta - \eta| + \gamma(\zeta, \eta)$, we have, by (21′),

$$\frac{2}{n^2} \sum_{1 \le i < k \le n} \log|\zeta_i - \zeta_k| \le 2\epsilon - \rho - \frac{2}{n^2} \sum_{1 \le i < k \le n} g(\zeta_i, \zeta_k) - \frac{1}{n^2} \sum_{i=1}^n \gamma(\zeta_i, \zeta_i). \tag{34}$$

Let $\max_{\zeta \in \Gamma(\epsilon)} |\gamma(\zeta, \zeta)| = M$. Thus,

$$\begin{aligned} \frac{n-1}{n} \log d_n(\epsilon) &\le 2\epsilon - \rho - \frac{2}{n^2} \sum_{1 \le i < k \le n} g(\zeta_i, \zeta_k) + \frac{M}{n} \\ &\le 2\epsilon - \rho + \frac{M}{n}, \end{aligned} \tag{35}$$

since $g(\zeta, \eta) > 0$ in Δ. We find in the limit, as $n \to \infty$,

$$\log d(\epsilon) \le 2\epsilon - \rho. \tag{36}$$

It is easy to show that $\lim_{\epsilon \to 0} d(\epsilon) = d$, and so we obtain

$$\log \frac{1}{d} \ge \rho. \tag{37}$$

This is the sought opposite inequality which allows us to infer the identity (32).

We have proved the identity in the case that the point set considered consists of N smooth curves. In the usual way, the same identity can be extended to the most general set.

4. We can now extend our treatment to more general set measures. Suppose, for example, the bounded closed set Γ can be decomposed into two disjoint closed sets A and B. We define the nth modulus of A and B as follows:

$$M_n(A, B) = \max_{\zeta_i \in A, \eta_i \in B} \left(\frac{\prod_{i<k} |\zeta_i - \zeta_k| \prod_{i<k} |\eta_i - \eta_k|}{\prod_{i,k} |\zeta_i - \eta_k|} \right)^{1/\binom{n}{2}}, \tag{38}$$

where $i, k \in \{1, \dots, n\}$. We also define

$$M_n^*(A, B) = \max_{\zeta_i \in A, \eta_i \in B} \left(\frac{\prod_{i<k} |\zeta_i - \zeta_k| \prod_{i<k} |\eta_i - \eta_k|}{\prod_{i \ne k} |\zeta_i - \eta_k|} \right)^{1/\binom{n}{2}}. \tag{38'}$$

By the same reasoning as in the case of the transfinite diameter used by Fekete, one finds that $M^*_{n+1}(A,B) \le M^*_n(A,B)$. Hence the limit of $M^*_n(A,B)$, as $n \to \infty$, exists. It is trivial to show that

$$M^*_n(A,B)d^{-2/(n-1)} \le M_n(A,B) \le M^*_n(A,B)\rho^{-2/(n-1)}, \tag{39}$$

where $d = \max_{\zeta \in A, \eta \in B}|\zeta - \eta|$ and $\rho = \min_{\zeta \in A, \eta \in B}|\zeta - \eta|$. It is obvious that

$$M(A,B) = \lim_{n \to \infty} M_n(A,B) \tag{40}$$

exists. We call the limit $M(A,B)$ the modulus of A and B.

To determine the potential theoretic significance of the modulus, we define the functional

$$\begin{aligned} I[\mu_1,\mu_2] = {} & \int_A \int_A \log \frac{1}{|z-\zeta|}\, d\mu_1(z)\, d\mu_1(\zeta) - 2\int_A \int_B \log \frac{1}{|z-\zeta|}\, d\mu_1(z)\, d\mu_2(\zeta) \\ & + \int_B \int_B \log \frac{1}{|z-\zeta|}\, d\mu_2(z)\, d\mu_2(\zeta), \end{aligned} \tag{41}$$

where μ_1 and μ_2 are positive mass distributions on A and B, respectively, with $\mu_1(A) = 1$ and $\mu_2(B) = 1$.

Now let

$$\omega(\zeta) = \sum_{\alpha=1}^{\lambda} \omega_\alpha(\zeta), \tag{42}$$

where $\omega_\alpha(\zeta)$ are the harmonic measures of Γ_ν of which the set A consists. Let

$$S = -\left(\sum_{\alpha,\beta=1}^{\lambda} p_{\alpha\beta}\right)^{-1}. \tag{43}$$

Define

$$\begin{aligned} d\hat{\mu}_1 &= -\frac{S}{2\pi}\frac{\partial\omega}{\partial n}\, ds \quad \text{on } A, \\ d\hat{\mu}_2 &= \frac{S}{2\pi}\frac{\partial\omega}{\partial n}\, ds \quad \text{on } B. \end{aligned} \tag{44}$$

It is clear that $\hat{\mu}_1(A) = 1$ and $\hat{\mu}_2(B) = 1$. Let

$$h(z) = \int_A \log \frac{1}{|\zeta - z|}\, d\hat{\mu}_1(\zeta) - \int_B \log \frac{1}{|\zeta - z|}\, d\hat{\mu}_2(\zeta). \tag{45}$$

It is easy to show that

$$h(z) = S\omega(z) \quad \text{for } z \in \Delta. \tag{46}$$

Since by (41) and (45) $I[\hat{\mu}_1,\hat{\mu}_2] = \int_A h(z)\, d\hat{\mu}_1(z) - \int_B h(z)\, d\hat{\mu}_2(z)$, and since $h(z) = S$ on A and $h(z) = 0$ on B, we have

$$I[\hat{\mu}_1,\hat{\mu}_2] = S. \tag{47}$$

Next we show that

$$I[\hat{\mu}_1,\hat{\mu}_2] = \min_{\mu_1,\mu_2} I[\mu_1,\mu_2]. \tag{48}$$

Indeed, let $\mu_1 = \hat{\mu}_1 + \nu_1$, $\mu_2 = \hat{\mu}_2 + \nu_2$. Then $\nu_1(A) = \nu_2(B) = 0$. It can be seen that

$$(49) \qquad I[\mu_1, \mu_2] = I[\hat{\mu}_1, \hat{\mu}_2] + I[\nu_1, \nu_2] + 2\int_A h(z)\,d\nu_1(z) - 2\int_B h(z)\,d\nu_2(z).$$

By equation (2-3) of [1], we find that

$$(50) \qquad I[\nu_1, \nu_2] = \frac{1}{2\pi}\iint \left[\int_A \frac{d\nu_1(\zeta)}{|\zeta - z|} - \int_B \frac{d\nu_2(\zeta)}{|\zeta - z|}\right]^2 dx\,dy \geq 0.$$

Observe that

$$(51) \qquad \int_A h(z)\,d\nu_1(z) = 0, \qquad \int_B h(z)\,d\nu_2(z) = 0,$$

since $h(z) = S\omega(z) = S$ on A, $h(z) = \omega(z) = 0$ on B, and $\nu_1(A) = 0$. Thus, it follows from (49), (50), and (51) that

$$(52) \qquad I[\mu_1, \mu_2] \geq I[\hat{\mu}_1, \hat{\mu}_2]$$

for any positive mass distributions μ_1, μ_2 with $\mu_1(A) = 1$ and $\mu_2(B) = 1$.

We proceed now as in the case of the transfinite diameter. We take n points $\zeta_i \in A$ and n points $\eta_i \in B$ and write, in view of (38),

$$(53) \qquad \sum_{i<k} \log \frac{1}{|\zeta_i - \zeta_k|} + \sum_{i<k} \log \frac{1}{|\eta_i - \eta_k|} - \sum_{i,k} \log \frac{1}{|\zeta_i - \eta_k|} \geq -\binom{n}{2} \log M_n(A, B).$$

By multiplying (53) with $\prod_{i=1}^n d\hat{\mu}_1(\zeta_i) \prod_{k=1}^n d\hat{\mu}_2(\eta_k) \geq 0$ and integrating over all ζ_i and η_k, we arrive at

$$(54) \qquad -\binom{n}{2} \log M_n(A, B) \leq \binom{n}{2} I[\hat{\mu}_1, \hat{\mu}_2] + n\int_A \int_B \log \frac{1}{|\zeta - \eta|}\,d\hat{\mu}_1(\zeta)\,d\hat{\mu}_2(\eta).$$

Thus, because of (47), we have

$$(55) \qquad \log M(A, B) \geq -S \quad \text{as } n \to \infty.$$

5. To obtain the opposite inequality of (55), we apply the inequality (21) to the identity function $f(\zeta) = \zeta$. Consider the approximating curves $\Gamma_\nu(\epsilon)$ on which $\max|\gamma(\zeta, \zeta)| = M$. We choose the points $\zeta_i \in A_\epsilon$, $\eta_i \in B_\epsilon$ such that

$$M_n(A_\epsilon, B_\epsilon) = \left(\frac{\prod_{i<k}|\zeta_i - \zeta_k| \prod_{i<k}|\eta_i - \eta_k|}{\prod_{i,k}|\zeta_i - \eta_k|}\right)^{1/\binom{n}{2}}.$$

We take $x_i = 1/n$ for $1 \leq i \leq n$ and $x_i = -1/n$ for $n+1 \leq i \leq 2n$, so that

$$s = \sum_{i=1}^{2n} x_i = 0.$$

Let $\zeta_1, \ldots, \zeta_n$ correspond to $x_1, \ldots, x_n$ and $\eta_1, \ldots, \eta_n$ to $x_{n+1}, \ldots, x_{2n}$. Since $\phi[f] = 0$, (21) can be written as

$$\frac{1}{n^2}\sum_{i,k} \gamma(\zeta_i, \zeta_k) + \frac{1}{n^2}\sum_{i,k} \gamma(\eta_i, \eta_k) - \frac{2}{n^2}\sum_{i,k} \gamma(\zeta_i, \eta_k) \leq$$

$$\le \frac{1}{n^2} \sum_{\mu,\nu=1}^{N-1} \pi_{\nu\mu} \sum_k (\omega_\nu(\zeta_k)-\omega_\nu(\eta_k)) \sum_k (\omega_\mu(\zeta_k)-\omega_\mu(\eta_k)).$$

This leads to

$$\frac{n-1}{n} \log M_n(A_\epsilon, B_\epsilon)$$

$$\le \frac{1}{n^2} \sum_{\mu,\nu=1}^{N-1} \pi_{\nu\mu} \sum_{k=1}^{n} (\omega_\nu(\zeta_k)-\omega_\nu(\eta_k)) \sum_{k=1}^{n} (\omega_\mu(\zeta_k)-\omega_\mu(\eta_k)) + \frac{2M}{n} + O(\epsilon), \tag{56}$$

where the facts that $g(\zeta,\eta)>0$ in Δ and $g(\zeta,\eta)=\log(1/|\zeta-\eta|)+\gamma(\zeta,\eta)$ have been used again, as well as the estimate $g(\zeta_j,\eta_k)=O(\epsilon)$. In order to evaluate the sum in (56), we turn to the following extremum problem. Let e_ν $(\nu=1,\dots,N)$ be real numbers such that $e_\nu>0$ for $\nu\le\lambda$, $e_\nu<0$ for $\nu\ge\lambda+1$, and

$$\sum_{\nu=1}^{\lambda} e_\nu = 1, \qquad \sum_{\nu=\lambda+1}^{N} e_\nu = -1. \tag{57}$$

We now ask for the maximum of $\sum_{\nu,\mu=1}^{N-1} \pi_{\nu\mu} e_\nu e_\mu$ under the side conditions (57). We use the Lagrange multiplier 2ℓ and discuss the maximum of $\sum_{\nu,\mu=1}^{N-1} \pi_{\nu\mu} e_\nu e_\mu + 2\ell \sum_{\nu=1}^{\lambda} e_\nu$. We find the conditions

$$\begin{aligned} &\sum_{\mu=1}^{N-1} \pi_{\nu\mu} e_\mu + \ell = 0 \quad \text{for } \nu=1,\dots,\lambda, \\ &\sum_{\mu=1}^{N-1} \pi_{\nu\mu} e_\mu = 0 \qquad\ \ \text{for } \nu=\lambda+1,\dots,N-1. \end{aligned} \tag{58}$$

Multiplying the νth equation by $p_{\alpha\nu}$ and summing over ν, we obtain

$$e_\alpha = -\ell \sum_{\nu=1}^{\lambda} p_{\alpha\nu}, \quad \alpha=1,\dots,N-1. \tag{59}$$

By (57) and (59) we have

$$\ell = -\left(\sum_{\alpha,\beta=1}^{\lambda} p_{\alpha\beta}\right)^{-1} = S. \tag{60}$$

Insert (59) in $\sum_{\nu,\mu=1}^{N-1} \pi_{\nu\mu} e_\nu e_\mu$ and find

$$\sum_{\nu,\mu=1}^{N-1} \pi_{\nu\mu} e_\nu e_\mu = \ell^2 \sum_{\nu,\mu=1}^{N-1} \pi_{\nu\mu} \sum_{\alpha,\beta=1}^{\lambda} p_{\nu\alpha} p_{\mu\beta} = \left(\sum_{\alpha,\beta=1}^{\lambda} p_{\alpha\beta}\right)^{-1} = -S. \tag{61}$$

Now let us return to (56). Since $\omega_\nu=0$ on Γ_μ for $\mu\ne\nu$ and $\omega_\nu=1$ on Γ_ν, we have

$$\sum_{k=1}^{n} (\omega_\nu(\zeta_k)-\omega_\nu(\eta_k)) = n_\nu + nO(\epsilon) \quad \text{for } \nu\le\lambda, \tag{62}$$

where n_ν is the number of the points $\zeta_k \in \Gamma_\nu(\epsilon)$, and

$$\sum_{k=1}^{n} (\omega_\nu(\zeta_k)-\omega_\nu(\eta_k)) = -n_\nu + nO(\epsilon) \quad \text{for } \nu\ge\lambda+1, \tag{63}$$

where n_ν is the number of the points $\eta_k \in \Gamma_\nu(\epsilon)$.

Let $e_\nu = n_\nu/n$ for $1 \le \nu \le \lambda$, $e_\nu = -n_\nu/n$ for $\lambda+1 \le \nu \le N$. Then $\sum_{\nu=1}^{\lambda} e_\nu = 1$ and $\sum_{\nu=\lambda+1}^{N} e_\nu = -1$. Thus we obtain, from (56) and (61),

$$\begin{aligned} \frac{n-1}{n} \log M_n(A_\epsilon, B_\epsilon) &\le \frac{1}{n^2} \sum_{\mu,\nu=1}^{N-1} \pi_{\nu\mu}(ne_\nu + nO(\epsilon))(ne_\mu + nO(\epsilon)) + \frac{2M}{n} + O(\epsilon) \\ &= \sum_{\mu,\nu=1}^{N-1} \pi_{\nu\mu} e_\nu e_\mu + \frac{2M}{n} + O(\epsilon) \qquad (64) \\ &\le -S + \frac{2M}{n} + O(\epsilon). \end{aligned}$$

This implies that

$$\log M(A,B) \le -S. \tag{65}$$

This combines with (55) to yield

$$M(A,B) = \exp\{-S\} = \exp\left\{1 \Big/ \sum_{\alpha,\beta=1}^{\lambda} p_{\alpha\beta}\right\}. \tag{66}$$

The result (66) characterizes the modulus $M(A,B)$. It is well known that

$$p_{\alpha\beta} = \frac{1}{2\pi} \int_{\Gamma_\alpha} (\partial\omega_\beta/\partial n)\, ds$$

is conformally invariant. Therefore the modulus $M(A,B)$ is invariant under conformal mapping.

6. Further generalizations are obvious. We consider again the N smooth curves Γ_ν and assign to each one a fixed real number e_ν such that

$$\sum_{\nu=1}^{N} e_\nu = 0. \tag{67}$$

Consider then the expression

$$\binom{n}{2} \log \pi_n(e_\nu) = \frac{1}{2} \sum_{\substack{i,k=1 \\ i\ne k}}^{nN} e_i e_k \log|\zeta_i - \zeta_k| \tag{68}$$

where on each Γ_ν, n points ζ_i are located and the coefficients e_i are chosen according to the continuum Γ_ν, on which each ζ_i is located. The points are to be chosen in such a way that $\pi_n(e_\nu) = \max$. It can be seen that $\pi_n(e_i)$ forms a convergent sequence and that

$$\lim_{n\to\infty} \pi_n(e_\nu) = \pi(e_\nu) \tag{69}$$

exists. We can also characterize $\pi(e_\nu)$ using the same argument as before and get

$$\pi(e_\nu) = \exp\left\{\sum_{\mu,\nu=1}^{N-1} \pi_{\mu\nu} e_\nu e_\mu\right\}, \tag{70}$$

where $((\pi_{\mu\nu}))_{1,\dots,N-1}$ is the inverse of $((p_{\mu\nu}))_{1,\dots,N-1}$. This shows in particular that $\pi(e_\nu)$ is a conformal invariant since the $\pi_{\mu\nu}$ are.

7. We can now combine the various definitions of potential theoretical quantities by means of Fekete limits to obtain distortion theorems in conformal mapping in the unit disk. We follow here a device used by Pommerenke [4] in the case of the original transfinite diameter and exemplify it for the case of the modulus.

Let

$$f(\zeta)=\zeta+\sum_{\nu=0}^{\infty} b_\nu \zeta^{-\nu} \tag{71}$$

be univalent and regular analytic in $|\zeta|>1$. For all such functions we have the Golusin inequality

$$\sum_{i,k} x_i x_k \log\left|\frac{f(\zeta_i)-f(\zeta_k)}{\zeta_i-\zeta_k}\right| \geq \sum_{i,k} x_i x_k \log\left(1-\frac{1}{\zeta_i \bar{\zeta}_k}\right) \tag{72}$$

for real constants x_i and $|\zeta_i|>1$. We choose now on $|\zeta|=1$ two disjoint closed sets A and B and project them onto the circle $|\zeta|=1+\epsilon$ by the definition

$$\zeta_i^*=(1+\epsilon)\zeta_i. \tag{73}$$

Thus the sets A_ϵ, B_ϵ are defined. We then choose the points $\zeta_i^* \in A_\epsilon$, $\eta_i^* \in B_\epsilon$ such that they yield exactly the nth modulus $M_n(A_\epsilon, B_\epsilon)$. Hence

$$\binom{n}{2}\log M_n(A_\epsilon, B_\epsilon)=\log\frac{\prod_{i<k}|\zeta_i^*-\zeta_k^*|\prod_{i<k}|\eta_i^*-\eta_k^*|}{\prod_{i,k}|\zeta_i^*-\eta_k^*|}. \tag{74}$$

Let $z_i^*=f(\zeta_i^*)$ and $w_i^*=f(\eta_i^*)$. Applying (72) with $x_i=1$ for ζ_i^* and $x_i=-1$ for η_i^*, we find

(75)

$$\sum_{i=1}^{n}\log|f'(\zeta_i^*)f'(\eta_i^*)|+2\log\frac{\prod_{i<k}|z_i^*-z_k^*|\prod_{i<k}|w_i^*-w_k^*|}{\prod_{i,k}|z_i^*-w_k^*|}$$
$$\geq \sum_{i=1}^{n}\log\left(1-\frac{1}{|\zeta_i^*|^2}\right)\left(1-\frac{1}{|\eta_i^*|^2}\right)+4\log\frac{\prod_{i<k}|\zeta_i^*-\zeta_k^*|\prod_{i<k}|\eta_i^*-\eta_k^*|}{\prod_{i,k}|\zeta_i^*-\eta_k^*|}+n^2O(\epsilon).$$

Here we have made use of the fact that

$$\left|\frac{\zeta_i^*\bar{\zeta}_k^*-1}{\zeta_i^*-\zeta_k^*}\right|>1,\quad \left|\frac{\eta_i^*\bar{\eta}_k^*-1}{\eta_i^*-\eta_k^*}\right|>1,\quad \text{and}\quad \left|\frac{\zeta_i^*\bar{\eta}_k^*-1}{\zeta_i^*-\eta_k^*}\right|=1+O(\epsilon). \tag{76}$$

Replacing the second term in the left-hand side of (75) by

$$2\binom{n}{2}\log M_n(f(A_\epsilon), f(B_\epsilon))$$

and using (74), we have

$$\log M_n(f(A_\epsilon), f(B_\epsilon)) \geq 2\log M_n(A_\epsilon, B_\epsilon)-\frac{1}{n(n-1)}\sum_i \log|f'(\zeta_i^*)f'(\eta_i^*)|$$

(77)

$$+\frac{1}{n(n-1)}\sum_i \log\left(1-\frac{1}{|\zeta_i^*|^2}\right)\left(1-\frac{1}{|\eta_i^*|^2}\right)+\frac{n}{n-1}O(\epsilon).$$

Observe that $f'(\zeta^*)$, $|\zeta^*|=1+\epsilon$, is bounded uniformly for fixed $\epsilon>0$. It is clear that

$$\lim_{n\to\infty}\left\{\frac{1}{n(n-1)}\sum_i \log|f'(\zeta_i^*)f'(\eta_i^*)|+\frac{1}{n(n-1)}\sum_i \log\left(1-\frac{1}{|\zeta_i^*|^2}\right)\left(1-\frac{1}{|\eta_i^*|^2}\right)\right\}=0$$

for fixed ϵ. Hence it follows from (77) that

$$\log M(f(A_\epsilon),f(B_\epsilon))\geq 2\log M(A_\epsilon,B_\epsilon)+O(\epsilon). \tag{78}$$

Because of the continuity of the modulus, we arrive at the elegant inequality

$$M(f(A),f(B))\geq M(A,B)^2 \tag{79}$$

as $\epsilon\to 0$.

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