AN EXPLICIT KOPPELMAN TYPE INTEGRAL FORMULA ON ANALYTIC VARIETIES

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Introduction. The purpose of this paper is to prove an explicit Koppelman type integral formula on analytic varieties, thus generalizing the Koppelman integral formula on domains in $\mathbf{C}^n$ as contained in Øvrelid [2]. This generalization is the content of Theorem 1. Roughly speaking, if V is an analytic variety in $\mathbf{C}^n$ defined by m holomorphic functions and M is domain on V, then we construct explicit kernels K_q so that for every $(0,q)$-form u with $\mathbf{C}^1$-coefficients on $\bar{M}$ we have:

$$u=\int_{\partial M} u\wedge K_q-\int_M \bar\partial u\wedge K_q-\bar\partial\left(\int_M u\wedge K_{q-1}\right).$$

The variety is assumed to have no singular point on $\bar{M}$. We also assume that ∂M, the boundary of M, is smooth. (See the next paragraph for a precise description of the setting and statement of the result.)

The construction of the kernels K_q and the proof of the integral formula are based on some results from [1]. These results generalized earlier results of Stout [4]. We also follow ideas from Øvrelid [2]. We will use the standard notation and terminology for differential forms (see, e.g., Rudin [3] and Wells [5]; see also Øvrelid [2]).

Description of the setting. Let $D\subset\mathbf{C}^n$ be a bounded domain with smooth boundary and let $\gamma_j(\zeta,z)$, $j=1,\dots,n$, be smooth functions defined for $\zeta\in\bar{D}$, $z\in D$ such that:

(i) $(\zeta-z,\gamma(\zeta,z))=:\sum_{j=1}^n(\zeta_j-z_j)\gamma_j(\zeta,z)\neq 0$ for $\zeta\neq z$;
(ii) $\gamma_j(\zeta,z)=\bar\zeta_j-\bar z_j$ for $|\zeta-z|<$ small constant.

Let $h_1,\dots,h_m$ be m, $m<n$, holomorphic functions in a domain Ω with $\Omega\supset\bar{D}$. Let $h_{ij}(\zeta,z)$ be holomorphic functions (in $(\zeta,z)\in\Omega\times\Omega$) so that

$$h_i(\zeta)-h_i(z)=\sum_{j=1}^n h_{ij}(\zeta,z)(\zeta_j-z_j),\quad i=1,\dots,m,\ (\zeta,z)\in\Omega\times\Omega.$$

Set $V=:\{z\in\Omega: h_1(z)=\dots=h_m(z)=0\}$ and set $M=:V\cap D$ and $\partial M=:V\cap(\partial D)$. Define

$$|\nabla(h_1,\dots,h_m)(\zeta)|^2=:\sum_{1\le j_1<\dots<j_m\le n}\left|\frac{\partial(h_1,\dots,h_m)}{\partial(\zeta_{j_1},\dots,\zeta_{j_m})}(\zeta)\right|^2.$$

Our assumptions are that $|\nabla(h_1,\dots,h_m)|\neq 0$ on $\bar{M}$, that is, that variety V has no singular point on $\bar{M}$ and that V meets ∂D transversally. Thus M is a complex manifold of (real-) dimension $2n-2m$ and ∂M is a smooth manifold of dimension $2n-2m-1$.

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We introduce now the differential forms:

$$A(\zeta,z)=:\sum_{1\le j_0<\cdots<j_m\le n}(-1)^{j_0+\cdots+j_m}\begin{vmatrix}\gamma_{j_0} & \cdots & \gamma_{j_m}\\ h_{1j_0} & \cdots & h_{1j_m}\\ \vdots & & \vdots\\ h_{mj_0} & \cdots & h_{mj_m}\end{vmatrix}\bigwedge_{k\ne j_0,\dots,j_m}[\bar\partial_z\gamma_k+\bar\partial_\zeta\gamma_k],$$

$\beta(\zeta)=:$

$$\frac{1}{|\nabla(h_1,\dots,h_m)(\zeta)|^2}\sum_{1\le j_1<\cdots<j_m\le n}(-1)^{j_1+\cdots+j_m}\overline{\frac{\partial(h_1,\dots,h_m)}{\partial(\zeta_{j_1},\dots,\zeta_{j_m})}}\bigwedge_{k\ne j_1,\dots,j_m}d\zeta_k,$$

and

$$K(\zeta,z)=:c(n,m)\frac{A(\zeta,z)\wedge\beta(\zeta)}{(\zeta-z,\gamma(\zeta,z))^{n-m}},$$

where

$$c(n,m)=:(-1)^{[(n-m)(n-m-1)]/2+1}\cdot\frac{(n-m-1)!}{(2\pi i)^{n-m}}.$$

Then $K(\zeta,z)$ can be decomposed in the following way:

$$K(\zeta,z)=\sum_{q=0}^{n-m-1}(-1)^qK_q(\zeta,z),$$

where $K_q(\zeta,z)$ is a $(0,q)$-form in z and a $(n-m,n-m-q-1)$-form in ζ. The above relation defines $K_q(\zeta,z)$, $q=0,\dots,n-m-1$, completely. Also define $K_{-1}=K_{n-m}=0$.

With the above notation and assumptions we will prove the following theorem.

THEOREM 1. *Fix a q with $0\le q\le n-m$. Then for every $u(z)\in \mathbf{C}^1_{(0,q)}(\bar M)$ (i.e., $u(z)$ is a $(0,q)$-form with $\mathbf{C}^1$-coefficients in $\bar M$), the following integral formula holds for $z\in M$:*

$$u(z)=\int_{\zeta\in\partial M}u(\zeta)\wedge K_q(\zeta,z)-\int_{\zeta\in M}\bar\partial u(\zeta)\wedge K_q(\zeta,z)-\bar\partial_z\left[\int_{\zeta\in M}u(\zeta)\wedge K_{q-1}(\zeta,z)\right].$$

In order to prove Theorem 1 we need two lemmas.

LEMMA 1. *$d_{\zeta,z}K(\zeta,z)=\bar\partial_{\zeta,z}K(\zeta,z)=0$ for $(\zeta,z)\in M\times M-\{\zeta=z\}$, and therefore $\bar\partial_\zeta K_q=\bar\partial_z K_{q-1}$. (The differential form $K(\zeta,z)$ is considered restricted to the manifold $M\times M-\{\zeta=z\}$.)*

Proof. Consider the differential form

$$\nu=\nu(\zeta,z,\xi)=:\theta(\zeta,z,\xi)\wedge\beta(\zeta)\wedge\beta(z),$$

where

$$\theta(\zeta,z,\xi)=:(\zeta-z,\xi)^{m-n}\sum_{j_0<\cdots<j_m}(-1)^{j_0+\cdots+j_m}\begin{vmatrix}\xi_{j_0} & \cdots & \xi_{j_m}\\ h_{1j_0} & \cdots & h_{1j_m}\\ \vdots & & \vdots\\ h_{mj_0} & \cdots & h_{mj_m}\end{vmatrix}\bigwedge_{k\ne j_0,\dots,j_m}d\xi_k,$$

defined for $(\zeta, z, \xi) \in M \times M \times \mathbf{C}^n$, so that $(\zeta - z, \xi) = \sum_{j=1}^n (\zeta_j - z_j)\xi_j \neq 0$. A computation shows that

$$dv = d_{\zeta, z, \xi} v = 0 \tag{1.1}$$

(this computation is carried out in [1]; see Propositions I.1 and I.2). Now consider the map

$$M \times M \ni (\zeta, z) \to (\zeta, z, \gamma_1(\zeta, z), \ldots, \gamma_n(\zeta, z)) \in M \times M \times \mathbf{C}^n. \tag{1.2}$$

If v^* is the pull-back of v via the map (1.2) then it follows from (1.1) that

$$d_{\zeta, z} v^* = 0. \tag{1.3}$$

But it is easy to check that

$$d_{\zeta, z} v^* = c[\bar{\partial}_{\zeta, z} K(\zeta, z)] \wedge \beta(z) \tag{1.4}$$

for some non-zero constant c. Now (1.3) and (1.4) imply the lemma. (The local representation of $\beta(z)$, $z \in M$, as given in [1, Prop. I.1], was also used.) □

LEMMA 2. *For a smooth function* $f(\zeta)$, $\zeta \in M$ (*not necessarily holomorphic*), *we have*

$$\lim_{\epsilon \to 0} \int_{\{\zeta \in M: |\zeta - z| = \epsilon\}} f(\zeta) K_0(\zeta, z) = f(z) \quad (z \in M).$$

Moreover, the convergence is uniform in z *on compact sets of* M. (In the above limit, ϵ goes to 0 through points so that $\{\zeta \in M: |\zeta - z| = \epsilon\}$ is a smooth manifold; the existence of such ϵ's which go to 0 is guaranteed by Sard's theorem.)

Proof. Fix a $z \in M$ and assume without loss of generality that $W(z) \neq 0$, where

$$W(\zeta) =: \frac{\partial(h_1, \ldots, h_m)}{\partial(\zeta_{n-m+1}, \ldots, \zeta_n)}(\zeta).$$

It follows by the implicit function theorem that the equations $h_1(\zeta) = \cdots = h_m(\zeta) = 0$ can be solved for $\zeta_{n-m+1}, \ldots, \zeta_n$ locally at z, giving (say)

$$\zeta_{n-m+1} = \tilde{h}_1(\zeta'), \ldots, \zeta_n = \tilde{h}_m(\zeta')$$

for some holomorphic functions $\tilde{h}_1, \ldots, \tilde{h}_m$ of ζ' ($\zeta' =: (\zeta_1, \ldots, \zeta_{n-m})$).

Now consider the domain $G_\epsilon \subset \mathbf{C}^{n-m}$ defined as follows:

$$G_\epsilon =: \left\{ \zeta' \in \mathbf{C}^{n-m}: \sum_{j=1}^{n-m} |\zeta_j - z_j|^2 + \sum_{k=1}^{m} |\tilde{h}_k(\zeta') - \tilde{h}_k(z')|^2 < \epsilon^2 \right\}$$

and the map

$$G_\epsilon \ni \zeta' \xrightarrow{\eta} (\zeta', \tilde{h}_1(\zeta'), \ldots, \tilde{h}_m(\zeta')) \in \{\zeta \in M: |\zeta - z| < \epsilon\}.$$

We have

$$\int_{\{\zeta \in M: |\zeta - z| = \epsilon\}} f(\zeta) K_0(\zeta, z) = \int_{\zeta' \in \partial G_\epsilon} \eta^*[f(\zeta) K_0(\zeta, z)], \tag{2.1}$$

where $\eta^*[f(\zeta)K_0(\zeta,z)]$ denotes the pull-back of the differential form $f(\zeta)K_0(\zeta,z)$ via the map η.

A computation shows that

$$\eta^*[f(\zeta)K_0(\zeta,z)]=c_{(n-m,0)}g(\zeta)\frac{\omega^*(\delta)\wedge\omega(\zeta')}{(\sum_{j=1}^{n-m}(\zeta_j-z_j)\delta_j)^{n-m}}, \tag{2.2}$$

where

(i) $g(\zeta)=:f(\zeta)\cdot[W(\zeta)]^{-1}\cdot T(\zeta,z)$;

(ii) $T(\zeta,z)=:\det[h_{i,n-m+j}(\zeta,z)]_{1\le i,j\le m}$;

(iii) $\omega^*(\delta)=:\sum_{j=1}^{n-m}(-1)^j\delta_j\bar\partial\delta_1\wedge\cdots\wedge\widehat{\bar\partial\delta_j}\wedge\cdots\wedge\bar\partial\delta_{n-m}$ (here $\bar\partial=\bar\partial_{\zeta'}$);

(iv) $\delta_j=:(\bar\zeta_j-\bar z_j)-\frac{1}{T}\sum_{k=1}^{m}(\bar\zeta_{n-m+k}-\bar z_{n-m+k})T_k^j,\ j=1,\dots,n-m$

(recall that $\gamma_j=\bar\zeta_j-\bar z_j$ for $|\zeta-z|<$ small constant);

(v) T_k^j is the determinant of the matrix obtained from the matrix $\begin{bmatrix} h_{1,n-m+1} & \cdots & h_{1n}\\ \vdots & & \vdots \\ h_{m,n-m+1} & \cdots & h_{mn}\end{bmatrix}$ by substituting its kth column by $\begin{bmatrix} h_{1j}\\ \vdots \\ h_{mj}\end{bmatrix}$; and

(vi) $\omega(\zeta')=:d\zeta_1\wedge\cdots\wedge d\zeta_{n-m}$.

The computation which proves (2.2) is carried out in [1] (in the proof of Prop. I.6). We point out that ζ in the right-hand side of (2.2) means

$$\zeta=(\zeta',\tilde h_1(\zeta'),\dots,\tilde h_m(\zeta'))$$

and that the right-hand side of (2.2) is a differential form in $\mathbf{C}^{n-m}$ in the variable $\zeta'=(\zeta_1,\dots,\zeta_{n-m})$. The same is true for ζ in (i)–(v). For example, ζ_{n-m+k} in (iv) means $\zeta_{n-m+k}=\tilde h_k(\zeta')$. Thus δ_j is considered as a function of ζ'. We will not need the explicit form of δ_j as given by (iv) but only the fact that

$$\Delta=:(\zeta'-z',\delta)=\sum_{j=1}^{n-m}(\zeta_j-z_j)\delta_j=\sum_{j=1}^{n}|\zeta_j-z_j|^2>0 \quad \text{if } \zeta'\ne z'.$$

Now let $r(\epsilon)>0$ be so that $\bar B_\epsilon\subset G_\epsilon$, where $B_\epsilon=:\{\zeta'\in\mathbf{C}^{n-m}: |\zeta'-z'|<r(\epsilon)\}$. Let $\chi_\epsilon(\zeta')$ be a smooth function of ζ', $0\le\chi_\epsilon\le 1$, with compact support in G_ϵ and identically 1 in a neighborhood of $\bar B_\epsilon$. Define

$$\theta_j^{(\epsilon)}(\zeta',z')=:\chi_\epsilon(\zeta')(\zeta_j-z_j)+(1-\chi_\epsilon(\zeta'))\delta_j(\zeta',z'),\quad j=1,\dots,n-m.$$

We have

$$(\zeta'-z',\theta^{(\epsilon)})=:\sum_{j=1}^{n-m}(\zeta_j-z_j)\theta_j^{(\epsilon)}=\chi_\epsilon|\zeta'-z'|^2+(1-\chi_\epsilon)\Delta>0 \quad \text{if } \zeta'\ne z'.$$

Next, applying Koppelman's integral formula for the domain $G_\epsilon\subset\mathbf{C}^{n-m}$ and the function $g(\zeta)$, we obtain

$$c_{(n-m,0)}\int_{\zeta'\in\partial G_\epsilon} g(\zeta)\frac{\omega^*(\delta)\wedge\omega(\zeta')}{\Delta^{n-m}} = g(z)+I_\epsilon, \tag{2.3}$$

where

$$I_\epsilon = c\int_{\zeta'\in G_\epsilon}(\bar\partial_{\zeta'}g)\wedge\frac{\omega^*(\theta^{(\epsilon)})\wedge\omega(\zeta')}{(\zeta'-z',\theta^{(\epsilon)})^{n-m}}$$

for some constant c. (Recall that the integrands in the integrals of (2.3) are considered differential forms in ζ'.) But

$$g(z)=f(z) \tag{2.4}$$

(since $W(z)=T(z,z)$; see [1, Prop. I.5]). We claim that

$$\lim_{\epsilon\to 0} I_\epsilon = 0. \tag{2.5}$$

This follows from the following estimate,

$$\begin{aligned}\int_{G_\epsilon}\frac{|\zeta'-z'|\,dV(\zeta')}{(\zeta'-z',\theta^{(\epsilon)})^{n-m}} &= \int_{G_\epsilon}\frac{|\zeta'-z'|\,dV(\zeta')}{[\chi_\epsilon|\zeta'-z'|^2+(1-\chi_\epsilon)\Delta]^{n-m}}\\ &\le \int_{G_\epsilon\cap\{\chi_\epsilon>1/3\}}+\int_{G_\epsilon\cap\{1-\chi_\epsilon>1/3\}}\\ &\le 2\cdot 3^{n-m}\int_{G_\epsilon}\frac{|\zeta'-z'|\,dV(\zeta')}{|\zeta'-z'|^{2n-2m}},\end{aligned}$$

together with the fact that $|\zeta'-z'|^{2m+1-2n}$ is locally integrable (dV is the Lebesgue measure in $\mathbf{C}^{n-m}$).

Now (2.1)–(2.5) imply the lemma in the case z is fixed. Finally, an inspection of the proof shows that the convergence is uniform on compact sets of M. This completes the proof of Lemma 2. □

Now we turn to the proof of Theorem 1. This proof is analogous to the proof of Koppelman's integral formula given by Øvrelid [2] and we will give only the necessary modifications.

Proof of Theorem 1. Exactly as in Øvrelid [2], the problem is reduced to showing that

$$\lim_{\epsilon\to 0}\int_{\{(\zeta,z)\in M\times M:\,|\zeta-z|=\epsilon\}} u(\zeta)\wedge K(\zeta,z)\wedge v(z) = \int_{z\in M} u(z)\wedge v(z) \tag{3.1}$$

for every $v(z)\in(C_0^\infty(M))_{(n-m,n-m-q)}$. (In this reduction Lemma 1 is to be used too.) Let us recall also that $\gamma_j(\zeta,z)=\bar\zeta_j-\bar z_j$ for $|\zeta-z|<$ small constant. Following Øvrelid, to prove (3.1) we consider the map

$$T\colon \mathbf{C}^n\times\mathbf{C}^n\to\mathbf{C}^n\times\mathbf{C}^n$$

$$(z,w)\to T(z,w)=:(\zeta,z)=:(z+w,z).$$

T^{-1} maps the set $E=:\{(\zeta,z)\in M\times M:\,|\zeta-z|=\epsilon\}$ onto

$$T^{-1}(E)=\{(z,w)\in\mathbf{C}^n\times\mathbf{C}^n: z\in M,\ z+w\in M,\ |w|=\epsilon\}.$$

We have

$$\begin{aligned}\int_E u(\zeta)\wedge K(\zeta,z)\wedge v(z)&=\int_{T^{-1}(E)}T^*[u(\zeta)\wedge K(\zeta,z)\wedge v(z)]\\ &=\int_{z\in M}\left(\int_{\{w:z+w\in M,\ |w|=\epsilon\}}T^*[u(\zeta)\wedge K(\zeta,z)\wedge v(z)]\right).\end{aligned}\tag{3.2}$$

Now we compute:

$$T^*[u(\zeta)\wedge K(\zeta,z)\wedge v(z)]=u(z+w)\wedge K(z+w,z)\wedge v(z).$$

Suppose $u(\zeta)=\sum_{I\sim q}u_I(\zeta)\,d\bar{\zeta}_I$, where the summation $\sum_{I\sim q}$ is extended over all ordered multiindices $I=\{1\le i_1<\cdots<i_q\le n\}$, $u_I(\zeta)$ is a function, and $d\bar{\zeta}_I=d\bar{\zeta}_{i_1}\wedge\cdots\wedge d\bar{\zeta}_{i_q}$. Then

$$u(\zeta)\wedge K(\zeta,z)\wedge v(z)=\sum_{I\sim q}u_I(\zeta)\,d\bar{\zeta}_I\wedge K(\zeta,z)\wedge v(z).\tag{3.3}$$

For degree reasons ($z\in M$) we have:

$$\begin{aligned}&d\bar{\zeta}_I\wedge A(\zeta,z)\wedge\beta(\zeta)\wedge v(z)=d\bar{\zeta}_I\wedge A(\zeta,z)\\ &\quad\wedge\left[\sum_{j_1<\cdots<j_m}(-1)^{j_1+\cdots+j_m}\overline{\frac{\partial(h_1,\ldots,h_m)}{\partial(\zeta_{j_1},\ldots,\zeta_{j_m})}}(\zeta)\bigwedge_{k\neq j_1,\ldots,j_m}[d(\zeta_k-z_k)]\right]\wedge v(z).\end{aligned}$$

Therefore, setting

$$\tilde{\alpha}(z,w)=:\sum_{j_0<\cdots<j_m}(-1)^{j_0+\cdots+j_m}\begin{vmatrix}\bar{w}_{j_0}&\cdots&\bar{w}_{j_m}\\ h_{1j_0}&\cdots&h_{1j_m}\\ \vdots&&\vdots\\ h_{mj_0}&\cdots&h_{mj_m}\end{vmatrix}\bigwedge_{k\neq j_0,\ldots,j_m}d\bar{w}_k$$

(here $h_{ij}=h_{ij}(z+w,z)$) and

$$\begin{aligned}\tilde{\beta}(z,w)=:&\frac{1}{|\nabla(h_1,\ldots,h_m)(w+z)|^2}\\ &\times\sum_{j_1<\cdots<j_m}(-1)^{j_1+\cdots+j_m}\overline{\frac{\partial(h_1,\ldots,h_m)}{\partial(\zeta_{j_1},\ldots,\zeta_{j_m})}}(z+w)\bigwedge_{k\neq j_1,\ldots,j_m}dw_k,\end{aligned}$$

we have

$$T^*[d\bar{\zeta}_I\wedge A(\zeta,z)\wedge\beta(\zeta)\wedge v(z)]=d\overline{(z+w)}_I\wedge\tilde{\alpha}(z,w)\wedge\tilde{\beta}(z,w)\wedge v(z)$$

and consequently, for a fixed $z\in M$, we obtain

$$\begin{aligned}&\int_{\{w:z+w\in M,\ |w|=\epsilon\}}T^*[|\zeta-z|^{2m-2n}u_I(\zeta)\,d\bar{\zeta}_I\wedge A(\zeta,z)\wedge\beta(\zeta)\wedge v(z)]\\ &=\frac{1}{\epsilon^{2n-2m}}\int_{\{w:z+w\in M,\ |w|=\epsilon\}}u_I(z+w)\,d\bar{z}_I\wedge\tilde{\alpha}(z,w)\wedge\tilde{\beta}(z,w)\wedge v(z).\end{aligned}\tag{3.4}$$

Now let $\epsilon\to 0$ in (3.4). By Lemma 2 we obtain

$$(3.5)\qquad \lim_{\epsilon\to 0}\int_{\{w:\, z+w\in M,\, |w|=\epsilon\}} T^*[u_I(\zeta)\, d\bar{\zeta}_I\wedge K(\zeta,z)\wedge v(z)] = u_I(z)\, d\bar{z}_I\wedge v(z).$$

Moreover, the convergence is uniform in z on compact sets of M. Now (3.2), (3.3) and (3.5) prove (3.1) and complete the proof of Theorem 1. □

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