TAUBERIAN THEOREMS FOR PLURIHARMONIC FUNCTIONS WHICH ARE BMO OR BLOCH

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0. Introduction. Suppose f is a bounded pluriharmonic function in the unit ball of $\mathbf{C}^n$. It is a corollary to Theorem 3 of [5] that f has a radial limit at a given boundary point if and only if the (a.e.) boundary values of f have a certain "derivative" at that point. The main result of the present paper is an analogous result for pluriharmonic functions satisfying a Bloch condition: see Theorem 1 below. Note that since Bloch functions need not have radial limits a.e., the statement of Theorem 1 involves instead certain linear functionals on the Bloch space which reduce to the average of the boundary values over certain sets, *if* these boundary values exist. Thus if f is Bloch and equals the Poisson–Szegö integral of a measure, the existence of a radial limit is equivalent to the existence of a "derivative" of the boundary measure (Corollary 1). In particular, in case f is both pluriharmonic and the Poisson–Szegö integral of a BMO function, we obtain Corollary 2. (The present Corollary 2 was the main result in the original version of this paper. Peter Jones, in collaboration with Carl Sundberg, suggested that exactly the same proof would yield Corollary 1, a stronger result.)

Theorem 1 will follow from Theorem 2, concerning Bloch functions in the unit *disc*. The averages in Theorem 2 are taken over open subsets of the disc, so that the non-existence of boundary values is no longer a problem. This reduction from a subset of the boundary of the unit ball in $\mathbf{C}^n$ to an open subset of $\mathbf{C}$ is available only if $n \geq 2$; this is the reason for the hypothesis "$n \geq 2$" in Theorem 1. (The statement of Theorem 1 is still true for $n=1$, but the proof is very much different and will appear elsewhere. Note that the case $n=1$ of Corollary 2 is contained in [6].)

Theorem 2, in turn, will follow from Theorem 3, which may be regarded as a quantitative version of results implicit in [5]; Theorem 3 is possibly of some interest in itself.

This paper had its origin in conversations and joint work with Wade Ramey; I wish to thank him.

1. Statement of results. Let $n \geq 2$. Let B denote the unit ball of $\mathbf{C}^n$, $S = \partial B$; let σ denote the rotation-invariant probability measure on S. Let $\mathcal{B} = \mathcal{B}(B)$ be the Bloch space, the space of all *pluriharmonic* functions $f: B \to \mathbf{C}$ such that the quantity

$$\frac{1-|z|^2}{n+1} \sum_{i,j=1}^{n} (\delta_{i,j} - z_i \bar{z}_j) \left(\frac{\partial f}{\partial z_i} \frac{\partial \bar{f}}{\partial \bar{z}_j} + \frac{\partial \bar{f}}{\partial z_i} \frac{\partial f}{\partial \bar{z}_j} \right)$$

is bounded in B. (This is simply the square of the norm on covectors dual to the Bergman metric, applied to the gradient of f. Various other characterizations of

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$\mathcal{B}(\Omega)$ for strictly pseudoconvex Ω are given in [2].) Note that "Bloch" usually entails "holomorphic", but we shall find it convenient to allow pluriharmonic functions to be Bloch. For $f \in \mathcal{B}$ let

$$\|f\|_{\mathcal{B}}^2 = |f(0)|^2 + \sup_{z \in B} \frac{1-|z|^2}{n+1} \sum_{i,j=1}^{n} (\delta_{i,j} - z_i \bar{z}_j) \left(\frac{\partial f}{\partial z_i} \frac{\partial \bar{f}}{\partial \bar{z}_j} + \frac{\partial \bar{f}}{\partial z_i} \frac{\partial f}{\partial \bar{z}_j} \right).$$

Then $\mathcal{B}$ is a Banach space.

Let $\mathfrak{M}$ denote the group of biholomorphic automorphisms of B; note that for f pluriharmonic in B, $f \in \mathcal{B}$ if and only if $\{f \circ \psi - f((\psi(0)) : \psi \in \mathfrak{M}\}$ is uniformly bounded on compact subsets of B.

Define a metric d on S by $d(\zeta, \zeta') = |1 - \langle \zeta, \zeta' \rangle|^{1/2}$ (see [7, p. 65]). For $\zeta \in S$, $\delta > 0$, let $Q_\delta(\zeta) = \{\zeta' \in S : d(\zeta, \zeta') < \delta\}$. Define BMO$(S)$ with respect to these "balls" $Q_\delta(\zeta)$ (so that this is "BMO_2" in [4]). Let P denote the Poisson–Szegö integral in B (as defined on p. 41 of [7], where it is called the "Poisson integral"). Our seminorm on BMO is equivalent to a "Garsia norm" in terms of P:

$$\|g\|_{\text{BMO}} \approx \sup_{z \in B} P[|g - P[g](z)|](z).$$

(Imitate pp. 224–5 of [3], using Lemma 5.4.5 of [4].) It follows from the $\mathfrak{M}$-invariance of P that there exists an absolute contant c such that for all $g \in \text{BMO}$ and $\psi \in \mathfrak{M}$, $\|g \circ \psi\|_{\text{BMO}} \le c \|g\|_{\text{BMO}}$. This shows that if $g \in \text{BMO}(S)$ and $f = P[g]$ happens to be pluriharmonic then $f \in \mathcal{B}$. This fact in turn explains why Corollary 2 below is a corollary to Corollary 1.

Let $e_1 = (1, 0, \ldots, 0)$. We shall prove the following.

PROPOSITION 1. *Let $n \ge 2$. For any $\delta > 0$ there exists a bounded linear functional A_δ on $\mathcal{B}(B)$ such that for any $f \in \mathcal{B}$,*

$$A_\delta f = \lim_{r \to 1} \frac{1}{\sigma(Q_\delta)} \int_{Q_\delta(e_1)} f(r\zeta)\, d\sigma(\zeta).$$

(In fact, if we let $g(\lambda) = f(\lambda e_1)$ for $\lambda = x + iy$ in the unit disc, then

$$A_\delta f = \frac{1}{\alpha(\delta)} \int_{\mathcal{V}_\delta} g(\lambda)(1 - |\lambda|^2)^{n-2}\, dx\, dy,$$

where $\mathcal{V}_\delta = \{\lambda : |\lambda| < 1,\ |1-\lambda|^{1/2} < \delta\}$ and $\alpha(\delta) = \int_{\mathcal{V}_\delta} (1 - |\lambda|^2)^{n-2}\, dx\, dy$.)

That is, $A_\delta f$ *would* be the average of the boundary values of f over $Q_\delta(e_1)$, if only f *had* boundary values. Note that the formula for $A_\delta f$ is nonsense if $n = 1$. Our main result is the following.

THEOREM 1. *Let $n \ge 2$. For $f \in \mathcal{B}(B)$ we have $\lim_{r \to 1^-} f(re_1) = 0$ if and only if $\lim_{\delta \to 0^+} A_\delta f = 0$.*

COROLLARY 1. *Let $n \ge 2$. Suppose $f \in \mathcal{B}(B)$; suppose $f = P[\mu]$ for some measure μ on S. Then $\lim_{r \to 1^-} f(re_1) = 0$ if and only if*

$$\lim_{\delta \to 0^+} \frac{\mu(Q_\delta(e_1))}{\sigma(Q_\delta(e_1))} = 0.$$

COROLLARY 2. *Suppose* $g \in \mathrm{BMO}(S)$ *and* $f = P[g]$ *is pluriharmonic. Then* $\lim_{r\to 1^-} f(re_1) = 0$ *if and only if*

$$\lim_{\delta\to 0^+} \frac{1}{\sigma(Q_\delta)} \int_{Q_\delta(e_1)} g\, d\sigma = 0.$$

Note that the hypothesis that $P[g]$ be pluriharmonic in Corollary 2 is essential; see Theorem 2 of [5].

Let us say a word about why Corollary 1 follows from Theorem 1. Combining Proposition 1 above with Lemma 4.2 in [5] shows that if $f = P[\mu]$ is pluriharmonic, then

$$A_\delta f = \frac{\mu(Q_\delta(e_1))}{\sigma(Q_\delta(e_1))}.$$

(The hypothesis "$\mu \geq 0$" was not essential in Lemma 4.2 of [5].) We have explained above why Corollary 2 follows from Corollary 1.

Let $D \subseteq \mathbf{C}$ be the unit disc; let $\alpha(\delta)$ and $\mathcal{V}_\delta$ be as in Proposition 1 above. Theorem 1 will follow directly from Theorem 2.

THEOREM 2. *Let* $n \geq 2$. *Suppose* $g \in \mathcal{B}(D)$. *Then* $\lim_{r\to 1^-} g(r) = 0$ *if and only if*

$$\lim_{\delta\to 0^+} \frac{1}{\alpha(\delta)} \int_{\mathcal{V}_\delta} g(\lambda)(1-|\lambda|^2)^{n-2}\, dx\, dy = 0.$$

Theorem 2 will, in turn, follow from Theorem 3: Let $\Pi^+ \subseteq \mathbf{C}$ be the upper half plane. For $\delta > 0$, let $D_\delta^+ = \{\lambda \in \Pi^+ : |\lambda| < \delta\}$; let

$$\beta(\delta) = \int_{D_\delta^+} y^{n-2}\, dx\, dy.$$

Let $h^\infty(\Pi^+) = \{$bounded harmonic functions in $\Pi^+\}$. For g integrable on bounded subsets of Π^+, $\delta > 0$, let

$$L_\delta(g) = \frac{1}{\beta(\delta)} \int_{D_\delta^+} g(x+iy) y^{n-2}\, dx\, dy.$$

THEOREM 3. *Let* $n \geq 2$. *There exists a constant c such that if* $g \in h^\infty(\Pi^+)$ *and for all* $\delta > 0$ *we have* $|L_\delta(g)| \leq \gamma$, *then for all* $y > 0$

$$|g(iy)| \leq c\gamma^{1/(n+2)} \|g\|_\infty^{(n+1)/(n+2)}.$$

In particular, if $L_\delta(g) = 0$ for all $\delta > 0$, then $g(iy) = 0$ for all $y > 0$. This fact is implicit in the proof of Proposition 4.4 of [5]. The results in [5] followed from Wiener's Tauberian Theorem; the present result will follow from Proposition 2 in Section 4, which may be regarded as a quantitative version of Wiener's Tauberian Theorem.

2. Proof that Theorem 2 implies Theorem 1. First let us prove Proposition 1.

Suppose $f \in \mathcal{B}(B)$, $n \geq 2$. For $\lambda \in D$ (i.e., $\lambda \in \mathbf{C}$, $|\lambda| < 1$) let $g(\lambda) = f(\lambda e_1)$. It is immediate from the definition that $g \in \mathcal{B}(D)$. For $0 < r < 1$, let $f_r(z) = f(rz)$, $g_r(\lambda) = g(r\lambda)$. Since f_r is continuous on $\bar{B}$ and pluriharmonic in B, application of Lemma 3.2 of [5] to the function $f_r \chi_{Q_\delta(e_1)}$ shows that

$$\frac{1}{\sigma(Q_\delta)}\int_{Q_\delta(e_1)} f(r\zeta)\,d\sigma(\zeta) = \frac{1}{\alpha(\delta)}\int_{\nabla_\delta} g_r(\lambda)(1-|\lambda|^2)^{n-2}\,dx\,dy \qquad (\lambda = x+iy).$$

Since $g \in \mathcal{B}(D)$, g blows up at most logarithmically at the boundary of D, so $g \in L^1(D)$. Thus $g_r \to g$ in $L^1(D)$ as $r \to 1$, so that

$$\lim_{r\to 1}\frac{1}{\sigma(Q_\delta)}\int_{Q_\delta(e_1)} f(r\zeta)\,d\sigma(\zeta) = \frac{1}{\alpha(\delta)}\int_{\nabla_\delta} g(\lambda)(1-|\lambda|^2)^{n-2}\,dx\,dy,$$

giving Proposition 1. (If one keeps track of the sizes of things here, one sees that $|A_\delta f| \le c_\delta \|f\|_{\mathcal{B}}$.)

With Proposition 1 proved, it is evident that Theorem 2 implies Theorem 1.

3. Proof that Theorem 3 implies Theorem 2.

LEMMA 1. *Suppose $\phi \in L^1((0,\pi))$ and, for some integer $m \ge 0$,*

$$\int_0^\pi \phi(\theta)(\sin\theta)^m\,d\theta = 0.$$

Then

$$\left|\int_0^\pi \phi\left(\frac{\pi}{2} + t\left(\theta - \frac{\pi}{2}\right)\right)(\sin\theta)^m\,d\theta\right| \le c\|\phi\|_1(1-t)$$

for $0 < t \le 1$.

Proof. Let

$$\chi(\theta) = \begin{cases} (\sin\theta)^m, & \theta \in (0,\pi), \\ 0, & \theta \notin (0,\pi). \end{cases}$$

Then χ is Lipschitz and has compact support, so that if

$$\chi_t(\theta) = \frac{1}{t}\chi\left(\frac{\pi}{2} + \frac{1}{t}\left(\theta - \frac{\pi}{2}\right)\right),$$

then $\|\chi - \chi_t\|_\infty \le c(1-t)$ for $0 < t \le 1$. Since $\int \phi\chi = 0$, the lemma follows:

$$\int_0^\pi \phi\left(\frac{\pi}{2} + t\left(\theta - \frac{\pi}{2}\right)\right)(\sin\theta)^m\,d\theta = \left|\int \phi\chi_t\right| = \left|\int \phi(\chi_t - \chi)\right| \le c\|\phi\|_1(1-t). \qquad \square$$

Let $\mathcal{B}(\Pi^+)$ denote the space of functions g harmonic in Π^+ for which $|\nabla g(x+iy)| \le c/y$; let

$$\|g\|_{B(\Pi^+)} = |g(i)| + \sup\{y|\nabla g(x+iy)| : x+iy \in \Pi^+\}.$$

LEMMA 2. *Suppose $g \in \mathcal{B}(\Pi^+)$, $\delta > 0$; suppose $n \ge 2$. There exists c independent of δ such that:*

(i) *if $0 < y \le \delta$, $|x| \le \delta$, then $|g(i\delta) - g(x-iy)| \le c\|v\|_{\mathcal{B}}(1+\log(\delta/y))$;*
(ii) $(1/\beta(\delta))\int_{D_\delta^+} |g(i\delta) - g(x+iy)|y^{n-2}\,dx\,dy \le c\|g\|_{\mathcal{B}}$; *and*
(iii) $|g(i\delta) - L_\delta(g)| \le c\|g\|_{\mathcal{B}}$.

The proof of (i) is an exercise; (i) implies (ii) and (ii) implies (iii).

Now suppose Theorem 3 is known. As a first step towards proving Theorem 2 we prove the following.

THEOREM 1.9. *Suppose* $g \in \mathcal{B}(\Pi^+)$. *Then* $g(i\delta)=0$ *for all* $\delta>0$ *if and only if* $L_\delta(g)=0$ *for all* $\delta>0$.

Proof. Suppose $g(i\delta)\equiv 0$. The Schwarz reflection principle implies that $g(x+iy)+g(-x+iy)\equiv 0$, so that $L_\delta(g)\equiv 0$.

Suppose, on the other hand, that $L_\delta(g)\equiv 0$. Lemma 2(iii) implies that g is bounded on the imaginary axis: $|g(i\delta)|\le c\|g\|_{\mathcal{B}}$ for all $\delta>0$. For $z\in\Pi^+$ and $0<t<1$ define $g_t(z)=g(i(z/i)^t)$. (Here z/i lies in the right half-plane; the principal branch of $(z/i)^t$ is intended.) Lemma 2(i) shows that $g_t\in h^\infty(\Pi^+)$ and that, in fact, $\|g_t\|_\infty\le c\|g\|_{\mathcal{B}}(1+|\log(1-t)|)$. Our hypothesis implies that for almost every $r>0$ (hence for every $r>0$) we have $\int_0^\pi g(re^{i\theta})(\sin\theta)^{n-2}\,d\theta=0$. Hence an integration in polar coordinates shows that for any $\delta>0$, $|L_\delta(g_t)|\le c\|g\|_{\mathcal{B}}(1-t)$, by Lemma 1. Now Theorem 3 shows that

$$|g_t(iy)|\le c\|g\|_{\mathcal{B}}(1-t)^{1/(n+2)}(1+|\log(1-t)|)^{(n+1)/(n+2)}.$$

Let t approach 1: we obtain $g(iy)=0$. □

A normal families argument which we omit (see, e.g., [6] for analogous arguments) leads from Theorem 1.9 to the following Theorem 1.99.

THEOREM 1.99. *Let* $n\ge 2$. *Suppose* $g\in\mathcal{B}(\Pi^+)$. *Then* $\lim_{\delta\to 0^+} g(i\delta)=0$ *if and only if* $\lim_{\delta\to 0^+} L_\delta(g)=0$.

Now the Cayley transform, with a bit of care, transfers Theorem 1.99 from Π^+ to D, where it becomes Theorem 2. (Note that if $\Phi:\Pi^+\to D$ is the Cayley transform, then $f\circ\Phi\in\mathcal{B}(\Pi^+)$ if and only if $f\in\mathcal{B}(D)$.)

4. A version of Wiener's Tauberian Theorem. The present section is devoted to the proof of the following proposition, which may be regarded as a quantitative version of Wiener's Tauberian Theorem (albeit with extra hypotheses).

PROPOSITION 2. *Suppose* $K\in L^1(R)$ *and the Fourier transform* $\hat K$ *has no zero on* R. *Suppose* $\hat K$ *is continuously differentiable; let* $\psi=(\hat K)^{-1}$ *and let*

$$N(R)=|\psi(0)|+R^{1/2}\left(\int_{-R}^{R}|\psi'|^2\right)^{1/2}.$$

Suppose $u\in L^\infty(R)$ *and* $\|K*u\|_\infty\le\delta$. *Then if* $F\in L^1(R)$ *is absolutely continuous (i.e.,* $F'\in L^1(R)$*) we have*

$$\|F*u\|_\infty\le c\inf_{\epsilon>0}(\epsilon\|F'\|_1\|u\|_\infty+\delta\|F\|_1 N(1/\epsilon)).$$

Note that this shows $F*u=0$ if $K*u=0$, which is Wiener's theorem. The content of Proposition 2 is that $F*u$ is "small" if $K*u$ is "small". We begin with a few lemmas.

LEMMA 3. *Suppose* $\psi\in C^1(R)$. *Let*

$$N(R)=|\psi(0)|+R^{1/2}\left(\int_{-R}^{R}|\psi'|^2\right)^{1/2}.$$

Given $R>0$, *there exists* $\phi\in C_c^1(R)$ *such that*

$$\phi\,|_{[-R,R]}=\psi\,|_{[-R,R]}\quad \text{and}\quad \|\phi\|_1^{1/3}\|\phi'\|_2^{2/3}\le cN(R).$$

Proof. By a dilation, we may assume $R=1$. Note that

$$\int_{-1}^{1}|\psi(x)|\,dx\le 2|\psi(0)|+\int_{-1}^{1}(1-|x|)\,|\psi'(x)|\,dx\le cN(1),$$

so that

$$\left(\int_{-1}^{1}|\psi|\right)^{1/3}\left(\int_{-1}^{1}|\psi'|^2\right)^{1/3}\le cN(1).$$

Similarly $|\psi(\pm 1)|\le cN(1)$; thus $\psi\,|_{[-1,1]}$ may be extended to a ϕ having support in $[-2,2]$, having the required properties. □

LEMMA 4. *For* $\phi\in C_c^1(R)$, $\|\hat\phi\|_1\le c\|\phi\|_1^{1/3}\|\phi'\|_2^{2/3}$. .

Proof. Splitting the integral into two pieces $\int_{|x|\le R}$ and $\int_{|x|>R}$, standard estimates show that

$$\|\hat\phi\|_1\le c(R\|\phi\|_1+R^{-1/2}\|\phi'\|_2)$$

for any $R>0$. Let $R=\|\phi\|_1^{-2/3}\|\phi'\|_2^{2/3}$. □

Lemmas 3 and 4 immediately imply the following.

LEMMA 5. *Suppose* $\psi\in C^1(R)$; *let*

$$N(R)=|\psi(0)|+R^{1/2}\left(\int_{-R}^{R}|\psi'|^2\right)^{1/2}.$$

Given $R>0$, *there exists* $g\in L^1(R)$ *such that*

$$\hat g\,|_{[-R,R]}=\psi\,|_{[-R,R]}\quad \text{and}\quad \|g\|_1\le cN(R).$$

For $\phi\in L^1(R)$ and $\epsilon>0$ define $\phi_\epsilon(x)=(1/\epsilon)\phi(x/\epsilon)$.

LEMMA 6. *Suppose* $K\in L^1(R)$, $\hat K$ *has no zero on* R, $\hat K\in C^1(R)$. *Define* $N(R)$ *as in Proposition* 2 *above. Suppose* $\phi\in L^1(R)$ *and the support of* $\hat\phi$ *is contained in* $[-1,1]$. *Then for any* $F\in L^1(R)$ *and* $\epsilon>0$ *there exists* $g\in L^1(R)$ *such that*

$$F*\phi_\epsilon=g*K\quad \text{and}\quad \|g\|_1\le c\|F\|_1\|\phi\|_1 N(1/\epsilon).$$

Proof. By Lemma 5 we may find $h\in L^1(R)$ such that $\hat h\,|_{[-1/\epsilon,1/\epsilon]}=\psi\,|_{[-1/\epsilon,1/\epsilon]}$ and $\|h\|_1\le cN(1/\epsilon)$. (Here $\psi=(\hat K)^{-1}$, as above.) Let $g=F*\phi_\epsilon*h$. Then $\|g\|_1\le \|F\|_1\|\phi_\epsilon\|_1\|h\|_1\le c\|F\|_1\|\phi\|_1N(1/\epsilon)$. Note that $\hat\phi_\epsilon$ is supported on $[-1/\epsilon,1/\epsilon]$, on which interval $\hat h=(\hat K)^{-1}$. Thus $\hat g\hat K=\hat F\hat\phi_\epsilon\hat h\hat K=\hat F\hat\phi_\epsilon$, so that $g*K=F*\phi_\epsilon$. □

LEMMA 7. *Suppose* $F\in L^1(R)$ *is absolutely continuous; suppose* $\phi\in L^1(R)$, $\int\phi=1$, *and* $\int_{-\infty}^{\infty}|x|\,|\phi(x)|\,dx<\infty$. *Then for any* $\epsilon>0$,

$$\|F-F*\phi_\epsilon\|_1\le\epsilon\|F'\|_1\int_{-\infty}^{\infty}|x|\,|\phi(x)|\,dx.$$

Proof. By Fubini's theorem,

$$\int |F(x)-F(x-y)|\,dx \le |y|\,\|F'\|_1.$$

Since $\int \phi = 1$, we see that

$$F(x)-F*\phi_\epsilon(x) = \int (F(x)-F(x-y))\phi_\epsilon(y)\,dy,$$

so that

$$\begin{aligned}\|F-F*\phi_\epsilon\|_1 &\le \iint |F(x)-F(x-y)|\,|\phi_\epsilon(y)|\,dx\,dy\\ &\le \|F'\|_1 \int |y|\,|\phi_\epsilon(y)|\,dy\\ &= \epsilon\|F'\|_1 \int |y|\,|\phi(y)|\,dy.\end{aligned}$$

□

Proof of Proposition 2. Suppose K, F, and u are as in the statement of Proposition 2. Pick $\phi \in L^1(R)$ such that $\hat{\phi}$ is supported in $[-1,1]$, $\int \phi = 1$, and

$$\int_{-\infty}^{\infty} |x|\,|\phi(x)|\,dx < \infty.$$

Let $\epsilon > 0$. By Lemma 6 there exists $g \in L^1(R)$ with $\|g\|_1 \le cN(1/\epsilon)\|F\|_1$ and $g*K = F*\phi_\epsilon$. Now Lemma 7 implies $\|F-g*K\|_1 = \|F-F*\phi_\epsilon\|_1 \le c\epsilon\|F'\|_1$. Thus

$$\begin{aligned}\|F*u\|_\infty &\le \|(F-g*K)*u\|_\infty + \|g*(K*u)\|_\infty\\ &\le \|F-g*K\|_1\|u\|_\infty + \|g\|_1\|K*u\|_\infty\\ &\le c(\epsilon\|F'\|_1\|u\|_\infty + \delta\|F\|_1 N(1/\epsilon)).\end{aligned}$$

□

5. Proof of Theorem 3. Our application of Proposition 2 requires a bit of preliminary set-up.

DEFINITION. For $m = 0, 1, \ldots$ and $\xi \in \mathbf{R}$, $I_m(\xi) = \int_0^\pi (\sin\theta)^m e^{-\xi\theta}\,d\theta$.

LEMMA 8. *For $m = 0, 1, \ldots$ there exists $c_m > 0$ such that*

$$I_m(\xi) \ge c_m \frac{1+e^{-\pi\xi}}{(1+|\xi|)^{m+1}}$$

for all $\xi \in \mathbf{R}$.

Proof. By induction on m. First

$$I_0(\xi) = \frac{1-e^{-\pi\xi}}{\xi} \quad (\xi \ne 0);$$

integration by parts a few times shows that

$$I_1(\xi) = \frac{1+e^{-\pi\xi}}{1+\xi^2} \quad \text{and} \quad I_{m+2}(\xi) = \frac{(m+1)(m+2)I_m(\xi)}{(m+2)^2+\xi^2}.$$

□

PROPOSITION 3. *Let $n \ge 2$. There exist $K, F \in L^1(R)$ such that if $g \in h^\infty(\Pi^+)$ and $u(t) = g(e^t) + g(-e^t)$, then*

(i) $g(iy)=u*F(\log y)$ $(y>0)$ *and*
(ii) $L_\delta(g)=u*K(\log\delta)$ $(\delta>0)$.
Further: F is absolutely continuous, $\hat{K}$ has no zero on R, $\hat{K}\in C^1(R)$, and if $N(R)$ is as in Proposition 2 then $N(R)\le c(1+R)^{n+1}$ $(R>0)$.

Proof. Let $s=\log y$. The Poisson formula for Π^+ shows that

$$\begin{aligned} g(iy) &= \frac{1}{\pi}\int_0^\infty (g(t)+g(-t))\frac{y}{t^2+y^2}\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^\infty u(x)\frac{1}{e^{x-s}+e^{s-x}}\,dx \\ &= u*F(s) \end{aligned}$$

if $F(x)=[\pi(e^x+e^{-x})]^{-1}$. This gives (i) and shows that F is absolutely continuous.

Similarly the Poisson formula shows that (ii) holds for *some* $K\in L^1(R)$. As in [5], we may use (ii) to calculate $\hat{K}(\xi)$ as follows.

Fix $\xi\in\mathbf{R}$ and let $g(z)=[1+e^{-\xi\pi}]^{-1}e^{i\xi\log z}$ (here "log" denotes the principal branch). Then $g\in h^\infty(\Pi^+)$ (in fact, $g\in H^\infty(\Pi^+)$) and $u(t)=g(e^t)+g(-e^t)=e^{i\xi t}$. Thus (ii) and an integration in polar coordinates show that

$$\begin{aligned} \hat{K}(\xi) &= \int_{-\infty}^\infty K(t)e^{-i\xi t}\,dt \\ &= K*u(0)=K*u(\log 1)=L_1(g) \\ &= \frac{1}{\alpha(1)}\frac{1}{1+e^{-\xi\pi}}\frac{1}{n+i\xi}I_{n-2}(\xi). \end{aligned}$$

This shows $\hat{K}$ has no (real) zeros and $\hat{K}\in C^1(R)$. Lemma 8 shows that $|\hat{K}(\xi)|\ge c/(1+|\xi|)^n$. A bit of calculus shows that

$$\left|\frac{d}{d\xi}\hat{K}(\xi)\right|\le c|\hat{K}(\xi)|,$$

so that if $\psi=(\hat{K})^{-1}$ then

$$|\psi'(\xi)|=\left|\frac{(\hat{K})'(\xi)}{(\hat{K}(\xi))^2}\right|\le\frac{c}{|\hat{K}(\xi)|}\le c(1+|\xi|)^n.$$

This shows that $N(R)\le c(1+R)^{n+1}$. □

We can now prove Theorem 3. Suppose $g\in h^\infty(\Pi^+)$ and $|L_\delta(g)|\le\gamma$ for all $\delta>0$. Pick K,F and define u as in Proposition 3. The fact that $|L_\delta(\lambda)|\le\gamma$ for all δ means $\|u*K\|_\infty\le\gamma$. So Propositions 2 and 3 show that for any $y>0$,

$$\begin{aligned} |g(iy)| &\le \|u*F\|_\infty \\ &\le c\inf_{\epsilon>0}(\epsilon\|u\|_\infty+\gamma N(1/\epsilon)) \\ &\le c\inf_{\epsilon>0}(\epsilon\|g\|_\infty+\gamma(1+1/\epsilon)^{n+1}) \\ &\le c\gamma^{1/(n+2)}\|g\|_\infty^{(n+1)/(n+2)}. \end{aligned}$$

(Let $\epsilon = (\gamma/\|g\|)^{1/(n+2)}$.) □

Added in Proof: Lemma 1 above is not true in the case $m = 0$. (In this case the function χ appearing in the proof of Lemma 1 is not even continuous, much less Lipschitz.) One may revise this lemma by adding the hypothesis that ϕ blows up at most logarithmically at 0 and π; one then sees that

$$\left|\int_0^\pi \phi\left(\frac{\pi}{2} + t\left(\theta - \frac{\pi}{2}\right)\right)(\sin\theta)^m \, d\theta\right| \le c\|\phi\|_1 (1-t)(1 + |\log(1-t)|).$$

This is sufficient for the application in the proof of Theorem 1.9.

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