CHARACTERIZING CERTAIN INCOMPLETE INFINITE-DIMENSIONAL ABSOLUTE RETRACTS

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0. Introduction and preliminaries. The study of infinite-dimensional manifolds modeled on $Q=[-1,1]^\infty$ and $s=(-1,1)^\infty$ reached a climax when H. Torunczyk gave a topological characterization theorem for these spaces: A locally compact ANR is a Q-manifold if and only if any map $f: C\to X$ of a compact (metric) space can be approximated by a closed embedding. Similarly, a complete ANR X is an s-manifold if and only if any map $f: C\to X$ of a complete (metric) space can be approximated by a closed embedding.

The second author has characterized manifolds modeled on $\sigma=\{(t_1,t_2,\dots)\in[-1,1]^\infty: t_i=0$ for all but finitely many $i\}$ and $\Sigma=\{(t_1,t_2,\dots)\in Q^\infty: t_i=0$ for all but finitely many $i\}$ in the same spirit [20]: An ANR X is a σ-manifold if and only if X can be represented as a countable union of finite-dimensional compacta, each of which is a strong Z-set in X, and any map $f: C\to X$ of a finite-dimensional compactum C, that is a Z-embedding when restricted to a closed subset $D\subseteq C$, can be approximated by a Z-embedding $g: C\to X$ so that $g\mid D=f\mid D$. (The characterization theorem for Σ-manifolds is obtained by deleting the words "finite-dimensional.") Although the resemblance with the characterization theorems for Q-manifolds and s-manifolds is obvious, one cannot avoid observing the much cleaner structure of Torunczyk's theorems. However, the mention of strong Z-sets is necessary, since examples of fake s-manifolds constructed in [4] lead to a straightforward construction of an AR X that can be represented as $\sigma\cup\{\text{point}\}$, such that $X\not\approx\sigma$, but X satisfies the hypotheses of the characterization theorem for σ, after deleting the word "strong." Similarly, if we replace the relative approximation condition by an absolute one (i.e., requiring $D=\varnothing$), then a counterexample is constructed by J. P. Henderson and J. J. Walsh [18].

In this paper we introduce a notion of strong $\mathcal{C}$-universality for a class $\mathcal{C}$ of (separable, metric) spaces. In the case that $\mathcal{C}=\{$(finite-dimensional) compacta$\}$ this is precisely the property stated in the characterization theorem for Σ (respectively σ).

The key idea that allows one to prove the characterization theorem for Σ and σ is the notion of an (f.d.) cap set (finite-dimensional compact absorption set), due to R. D. Anderson [2]. Loosely speaking, $\Sigma\cong Q-s\subset Q$ is a cap set, since it is strongly $\mathcal{C}$-universal ($\mathcal{C}=\{\text{compacta}\}$) and there are small maps $Q\to\Sigma\subset Q$. This notion has been subsequently generalized by different authors (cf. [5], [24], [27], [14]). In §3 we introduce the definition of a $\mathcal{C}$-absorbing set, which represents

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a slight modification of the above concept. It enables us to study the geometry of certain (incomplete, non-σ-compact) subsets of the Hilbert space. The lack of completeness of these spaces is remedied by two facts: (1) they embed nicely into $s=(-1,1)^\infty$ so that any G_δ-subset of s containing them must be a copy of s, and (2) they can be represented as a countable union of Z-sets.

A sufficient knowledge of geometry of the space, via a now routine decomposition theory argument due to R. D. Edwards ("Edwards strategy" [9]), leads to the topological characterization of the space.

Carrying out this program, we reprove the characterization theorems for Σ and σ, and obtain new characterization theorems in the same spirit. In particular, we characterize $\Sigma\times s$ (Corollary 6.3); the absolute Borel sets Ω_α, Λ_α (Theorem 6.5), which include $\Lambda_1=\Sigma$, $\Omega_1=\Sigma\times s$, $\Omega_2=\Sigma^\infty$; and ARs that have a form $W(T,*)=\{(t_1,t_2,\dots)\in T^\infty: t_i=*$ for all but finitely many $i\}$ (the weak product) for an AR T (Corollary 5.5).

All spaces in this paper are separable and topologized by a metric d.

For a subset $A\subset X$ and $x\in X$ we set $d(x,A)=\inf\{d(x,a): a\in A\}$. By definition, $d(x,\emptyset)=\infty$. For $\epsilon>0$ we set $N_\epsilon(A)=\{x\in X: d(x,A)<\epsilon\}$. As usual, $\operatorname{diam} A=\sup\{d(x,y): x\in A, y\in A\}$ and $\operatorname{Cl}_X A=\{x\in X: d(x,A)=0\}$. By $\operatorname{cov}(X)$ we denote the set of all open covers of X. For $\mathcal{U},\mathcal{V}\in\operatorname{cov}(X)$ we define

$$\operatorname{mesh}\mathcal{U}=\sup\{\operatorname{diam} U: U\in\mathcal{U}\}\quad\text{and}\quad \operatorname{St}(\mathcal{U},\mathcal{V})=\{\operatorname{St}(U,\mathcal{V}): U\in\mathcal{U}\},$$

where $\operatorname{St}(A,\mathcal{V})=\bigcup\{V\in\mathcal{V}: A\cap V\neq\emptyset\}$ for $A\subseteq X$. We use $\operatorname{St}\mathcal{U}$ to denote $\operatorname{St}(\mathcal{U},\mathcal{U})$ and inductively $\operatorname{St}^{n+1}\mathcal{U}=\operatorname{St}(\operatorname{St}^n\mathcal{U},\mathcal{U})$. For $\mathcal{U},\mathcal{V}\in\operatorname{cov}(X)$, $\mathcal{V}<\mathcal{U}$ means that $\mathcal{V}$ refines $\mathcal{U}$. For maps $f,g: X\to Y$ and for $\mathcal{U}\in\operatorname{cov}(Y)$ the symbol $(f,g)<\mathcal{U}$ means that for each $x\in X$ there is $U\in\mathcal{U}$ such that $\{f(x),g(x)\}\subseteq U$. For a map $\epsilon: Y\to(0,\infty)$ we say that g is ϵ-close to f provided $d(f(x),g(x))<\epsilon(f(x))$ for $x\in X$. It is well known that two topologies on the set of maps $X\to Y$ given by open covers and maps ϵ respectively coincide, so we use both concepts interchangeably. A homotopy $H: X\times[0,1]\to Y$ is said to be a $\mathcal{U}$-homotopy (ϵ-homotopy) if for each $x\in X$ there is $U\in\mathcal{U}$ with

$$H(\{x\}\times[0,1])\subseteq U\qquad (d(H(x,t),H(x,0))<\epsilon(H(x,0)).$$

A function $\epsilon: X\to(0,\infty)$ is said to be *Lipschitz* if $|\epsilon(x)-\epsilon(x')|\leq d(x,x')$ for $x,x'\in X$. For any $\epsilon: X\to(0,\infty)$ there is a Lipschitz function $\epsilon': X\to(0,\infty)$ such that $\epsilon'(x)<\epsilon(x)$, $x\in X$.

$X\in\mathrm{ANR}$ (AR) means that X is an ANR (AR) for the class of (separable, metric) spaces.

A map $f: X\to Y$ is a *near-homeomorphism* if for any $\mathcal{U}\in\operatorname{cov}(Y)$ there is a homeomorphism $h: X\to Y$ such that $(h,f)<\mathcal{U}$. For $X,Y\in\mathrm{ANR}$ a map $f: X\to Y$ is said to be a *fine homotopy equivalence* if for any $\mathcal{U}\in\operatorname{cov}(Y)$ there is a map $g: Y\to X$ such that fg is $\mathcal{U}$-homotopic to id_Y and gf is $f^{-1}(\mathcal{U})$-homotopic to id_X. A map $f: X\to Y$ between ANRs is a *UV^∞-map* provided for each $y\in Y$ and any neighborhood U of $y\in Y$ there is a neighborhood V of $y\in Y$ such that $V\subseteq U$ and the inclusion $f^{-1}(V)\to f^{-1}(U)$ is null homotopic. It is well known (cf. [15]) that f is UV^∞ if and only if f is a fine homotopy equivalence. If $f_n: X\to Y$ is a

UV^∞-map, $n=1,2,3,\ldots$ and $f_n \to f$ uniformly, then f is a UV^∞-map. In particular, near-homeomorphisms between ANRs are fine homotopy equivalences. The converse holds if X and Y are assumed to be Q-manifolds or s-manifolds; we expand this list by adding manifolds modeled on certain (incomplete) spaces.

We say that a map $f: X \to Y$ is *closed over a subset* $A \subseteq Y$ if for each $a \in A$ and each neighborhood U of $f^{-1}(a)$ (which might be empty) there exists a neighborhood V of a such that $f^{-1}(V) \subseteq U$.

If $f: X \to Y$ is a map, and $A \subseteq Y$ a closed subset, we can form the *adjunction space* $X \cup_f A$. As a set,

$$X \cup_f A = (X - f^{-1}(A)) \cup A;$$

the topology on $X \cup_f A$ is generated by the open sets in $X - f^{-1}(A)$ and by the sets of the form $f^{-1}(U-A) \cup (U \cap A)$ for open sets $U \subseteq Y$. Note that $f: X \to Y$ factors through $X \cup_f A$ via $p: X \to X \cup_f A$ and $q: X \cup_f A \to Y$ defined by

$$p(x) = \begin{cases} x, & x \in X - f^{-1}(A), \\ f(x) & x \in f^{-1}(A), \end{cases} \qquad q(x) = \begin{cases} f(x), & x \in X - f^{-1}(A), \\ x, & x \in A. \end{cases}$$

If f is a fine homotopy equivalence, then $X \cup_f A$ is an ANR and both p and q are fine homotopy equivalences.

Finally, $\mathbf{N}$ denotes the set of positive integers.

1. Strong Z-sets and the Strong Discrete Approximation Property. A closed subset A of $X \in$ ANR is a *Z-set in X* if for each $\mathcal{U} \in \operatorname{cov}(X)$ there is a map $f: X \to X$ with $(f, \mathrm{id}_X) < \mathcal{U}$ and $f(X) \cap A = \varnothing$ (cf. [1], [16]). Analogously, a closed subset A of $X \in$ ANR is a *strong Z-set* (*in X*) if for each $\mathcal{U} \in \operatorname{cov}(X)$ there is a map $f: X \to X$ with $(f, \mathrm{id}_X) < \mathcal{U}$ and $\mathrm{Cl}_X f(X) \cap A = \varnothing$. Although the distinction between these two notions was apparently known to D. W. Henderson, it has been rediscovered recently in [4], where an example of a planar, one-dimensional, complete ANR is constructed, containing a point x such that $\{x\}$ is a Z-set but not a strong Z-set.

It is much easier to detect Z-sets than strong Z-sets: a closed subset A of $X \in$ ANR is a Z-set if and only if for each open subset U of X the inclusion $U - A \to U$ is a (weak) homotopy equivalence [16]. The *nice* case occurs when $X \in$ ANR has the property that each Z-set in X is a strong Z-set in X. Examples of nice ANR's are: (1) locally compact ANR's; (2) manifolds modeled on metrizable locally convex topological vector spaces F with $F^\infty \cong F$ [16] (including $l_2 \cong s$); (3) ANR's that can be embedded into nice ANR's so that the complement is locally homotopy negligible (e.g., the pseudoboundary Σ of the Hilbert cube Q). A set $A \subset X$ is *locally homotopy negligible* if for every open set $U \subseteq X$ the inclusion $U - A \to U$ is a weak homotopy equivalence [22].

In this section we establish basic properties of strong Z-sets and give two more properties that imply "niceness."

LEMMA 1.1. *Let $X \in$ ANR, $A \subseteq X$ a strong Z-set, $\mathcal{U} \in \operatorname{cov}(X)$, and $f: C \to X$ a map from a space C. Suppose that $D \subseteq C$ is a closed subset such that $f \mid D: D \to X$ is a closed embedding. Then there is a map $g: C \to X$ such that $(f, g) < \mathcal{U}$, $g \mid D = f \mid D$, $g(C-D) \cap A = \varnothing$, and g is closed over A.*

Proof. We construct sequences $\{f_i: C \to X\}$, $\{C_i: C_i$ is a closed subset of $C\}$, and $\{\mathcal{W}_i: \mathcal{W}_i \in \operatorname{cov}(X)\}$ satisfying the following conditions:

(a) $f_i \mid D = f$;

(b) $C_i \cup N_{1/i}(D) = C$, $C_{i-1} \subseteq \operatorname{int} C_i$, and $C_i \cap D = \varnothing$;

(c) $f_i \mid C_{i-1} = f_{i-1} \mid C_{i-1}$ and $\operatorname{Cl}_X f_i(C_i) \cap A = \varnothing$;

(d) $\operatorname{St} \mathcal{W}_i < \mathcal{W}_{i-1}$, $\operatorname{diam}(f_{i-1}^{-1}(W) - C_i) < 1/i$ for each $W \in \operatorname{St}^2 \mathcal{W}_i$, mesh $\mathcal{W}_i < 2^{-i}$; and

(e) $(f_i, f_{i-1}) < \mathcal{W}_i$.

The construction starts with $C_0 = \varnothing$, $\mathcal{W}_0 \in \operatorname{cov}(X)$ with $\operatorname{St} \mathcal{W}_0 < \mathcal{U}$, and $f_0 = f$. Suppose that $\mathcal{W}_{i-1}$, f_{i-1}, C_{i-1} have been constructed. First choose a locally finite $\tilde{\mathcal{W}}_i \in \operatorname{cov}(X)$ such that $\tilde{W} \in \tilde{\mathcal{W}}_i$ implies $\operatorname{diam}(f_{i-1}^{-1}(\tilde{W}) \cap D) < 1/3i$ (this is possible since $f_{i-1} \mid D$ is a closed embedding). Let $\mathcal{W}_i \in \operatorname{cov}(X)$ be such that $\operatorname{St}^2 \mathcal{W}_i$ refines both $\tilde{\mathcal{W}}_i$ and $\mathcal{W}_{i-1}$ and mesh $\mathcal{W}_i < 2^{-i}$. Note that

$$U = \{c \in C: \text{ if } \tilde{W} \in \tilde{\mathcal{W}}_i \text{ and } f_{i-1}(c) \in \operatorname{Cl}_X \tilde{W}, \text{ then } d(c, f_{i-1}^{-1}(\operatorname{Cl}_X \tilde{W}) \cap D) < 1/3i\}$$

is an open subset of C containing D. Choose a closed set $C_i \subseteq C$ such that $C_i \cup U = C$, and such that (b) holds.

Let N be a closed neighborhood of A such that $\operatorname{Cl}_X f_{i-1}(C_{i-1}) \cap N = \varnothing$. Since A is a strong Z-set in $X \in \mathrm{ANR}$, there exists a small homotopy $H: C \times [0,1] \to X$ with $H_0 = f_{i-1}$, $\operatorname{Cl}_X H_1(C) \cap A = \varnothing$. The homotopy H and the neighborhood N can be chosen so small that $H(C_{i-1} \times [0,1]) \cap N = \varnothing$ and $H(C \times \{1\}) \cap N = \varnothing$.

Let $\alpha: C \to [0,1]$ be a map with $\alpha(D \cup C_{i-1}) = \{0\}$ such that the graph of $\alpha \mid C_i$ misses $H^{-1}(N)$ (which is a closed subset of $C \times [0,1]$ disjoint from $C_{i-1} \times [0,1] \cup C \times \{1\}$). Finally, let $f_i(c) = H(c, \alpha(c))$.

Setting $g = \lim_{i \to \infty} f_i$, we have $(f, g) < \mathcal{U}$, $g \mid D = f \mid D$, and $g(C - D) \cap A = \varnothing$. For $a \in A$, $g^{-1}(a) = f^{-1}(a) \cap D$ is a singleton, say $\{\tilde{a}\}$. Let $i \in \mathbf{N}$ be given and choose $W \in \operatorname{St}^2 \mathcal{W}_i$ with $a \in W$. By (d), $\tilde{a} \in f_{i-1}^{-1}(W) - C_i \subseteq N_{1/i}(\tilde{a})$, and thus $c \notin C_i$ and $c \notin N_{1/i}(\tilde{a})$ imply $f_{i-1}(c) \notin W$. Since $W \in \operatorname{St}^2 \mathcal{W}_i$ was arbitrary with $a \in W$, we conclude that $c \notin C_i$ and $c \notin N_{1/i}(\tilde{a})$ imply $f_{i-1}(c) \notin \operatorname{St}^2(a, \mathcal{W}_i)$. Finally, (e) implies that in such case $g(c)$ and a are not $\mathcal{W}_i$-close. If $W \in \mathcal{W}_i$ is any element containing a, we have $g^{-1}(W) \subseteq f_{i-1}^{-1}(W) - C_i \subseteq N_{1/i}(\tilde{a})$, and hence g is closed over A. □

COROLLARY 1.2 (cf. Theorem 2.4 in [22]). *A closed subset A of $X \in$ ANR is a strong Z-set if and only if there is a homotopy $H: X \times [0,1] \to X$ such that*

(1) $H(x, 0) = x$, $x \in X$,

(2) $H(X \times (0,1]) \cap A = \varnothing$, *and*

(3) *H is closed over A.*

Proof. Suppose that A is a strong Z-set. An application of Lemma 1.1 to $C = X \times [0,1]$ ($f: C \to X$ defined by $f(x,t) = x$) and $D = X \times \{0\} \subset C$ produces a map $g(=H): X \times [0,1] \to X$ with (1)–(3). (In addition, H can be chosen to be as small as we want.)

On the other hand, if $H: X \times [0,1] \to X$ satisfying (1)–(3) is given, and if $\mathcal{U} \in \operatorname{cov}(X)$, choose a map $\alpha: C \to (0,1]$ such that the map $g: X \to X$ defined by $g(x) = H(x, \alpha(x))$ is $\mathcal{U}$-close to id_X. The verification that $\operatorname{Cl}_X g(X) \cap A = \varnothing$ is left to the reader. □

The next goal is to establish that the property of being a strong Z-set is local.

LEMMA 1.3. *Let $X \in$ ANR, $A \subseteq X$ a strong Z-set in X, and $U \subseteq X$ an open subset. Then $A \cap U$ is a strong Z-set in U.*

Proof. By Corollary 1.2 there is a homotopy $H: X \times [0,1] \to X$ closed over A with $H_0 = \mathrm{id}_X$ and $H(X \times (0,1]) \cap A = \varnothing$. Choose a map $\alpha: U \to (0,1]$ such that $H\{(x,t): x \in U,\ 0 \le t \le \alpha(x)\} \subseteq U$ and let $G: U \times [0,1] \to U$ be defined by $G(x,t) = H(x, t \cdot \alpha(x))$. It is easy to verify that G is closed over $A \cap U$, $G(x,0) = x$, and $G(U \times (0,1]) \cap A = \varnothing$. Thus, by Corollary 1.2, $A \cap U$ is a strong Z-set in U. □

LEMMA 1.4. *Let $X \in$ ANR, $A \subseteq X$ a Z-set in X, $A_i \subseteq X$ a strong Z-set in X $(i = 1, 2, \dots)$, and $A = \bigcup_{i=1}^{\infty} A_i$. Then A is a strong Z-set in X.*

Proof. Without loss of generality, we assume that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. We can approximately factor $\mathrm{id}_X: X \to X$ through a σ-compact space P (say, a locally finite countable simplicial complex). We show that each map $f: P \to X$ can be approximated by a map $g: P \to X$ with $\mathrm{Cl}_X\, g(P) \cap A = \varnothing$.

Write $P = \bigcup_{i=1}^{\infty} P_i$, with each P_i compact and $P_1 \subseteq P_2 \subseteq P_3 \subseteq \cdots$. We define sequences $\{\mathfrak{U}_i: \mathfrak{U}_i \in \mathrm{cov}(X)\}$, $\{f_i: P \to X\}$, and $\{(V_i, W_i): V_i, W_i$ are open in X, $V_i \supseteq \mathrm{Cl}_X W_i \supseteq W_i \supseteq A_i\}$ with the following properties:

(4) $\mathrm{St}\, \mathfrak{U}_i < \mathfrak{U}_{i-1}$, $\mathrm{St}\, \mathfrak{U}_i < \{V_{i-1}, X - \mathrm{Cl}_X W_{i-1}\}$, $\mathrm{mesh}\, \mathfrak{U}_i < 2^{-i}$;

(5) $(f_i, f_{i-1}) < \mathfrak{U}_i$, $f_i \mid P_{i-1} = f_{i-1} \mid P_{i-1}$, $\mathrm{Cl}_X f_i(P) \cap A_i = \varnothing$, $f_i(P) \cap A = \varnothing$; and

(6) $f_i(P) \cap V_i = \varnothing$.

For the inductive construction, observe that A_i is a strong Z-set in $X - f_{i-1}(P_{i-1})$ (by Lemma 1.3), and hence we can let $f_i = h f_{i-1}$, where $h: X - f_{i-1}(P_{i-1}) \to X - f_{i-1}(P_{i-1})$ is a suitably chosen map that extends to $h: X \to X$ by $h(x) = x$, $x \in f_{i-1}(P_{i-1})$.

Setting $g = \lim_{i \to \infty} f_i$, we have $(f, g) < \mathrm{St}\, \mathfrak{U}_1$ and $\mathrm{Cl}_X\, g(P) \cap A = \varnothing$, since

$$g(P) \cap \left(\bigcup W_i\right) = \varnothing.$$

Thus, A is a strong Z-set in X. □

COROLLARY 1.5. *Let A be a closed subset of $X \in$ ANR. Then A is a strong Z-set in X if and only if there is a $\mathfrak{U} \in \mathrm{cov}(X)$ such that $A \cap U$ is a strong Z-set in U, for each $U \in \mathfrak{U}$.*

Proof. Necessity follows from Lemma 1.3. To prove sufficiency, we can assume without loss of generality that $\mathfrak{U} = \{U_1, U_2, \dots\}$ is countable. Write $A \cap U_i = \bigcup_{j=1}^{\infty} A_i^j$, where each A_i^j is closed in X. Then each A_i^j is a strong Z-set in X, $A = \bigcup_{i,j=1}^{\infty} A_i^j$, and A is a Z-set in X (the property of being a Z-set is local)! Consequently, by Lemma 1.4, A is a strong Z-set in X. □

COROLLARY 1.6. *If A is a topologically complete closed subset of $X \in$ ANR, and if $A = \bigcup_{i=1}^{\infty} A_i$, where each A_i is a strong Z-set in X, then A is a strong Z-set in X.*

Proof. By [7], A is a Z-set, hence (by Lemma 1.4) a strong Z-set. □

The following approximation property plays the crucial role in the theory of Hilbert space manifolds (see [23], [26], [4]): A (metric) space X has the *Strong Discrete Approximation Property* if for each map $f: \bigoplus_{i=1}^{\infty} Q_i \to X$ of a countable disjoint union of Hilbert cubes into X and for each $\mathcal{U} \in \operatorname{cov}(X)$ there exists a map $g: \bigoplus_{i=1}^{\infty} Q_i \to X$ $\mathcal{U}$-close to f, such that the collection $\{g(Q_i)\}_{i=1}^{\infty}$ is discrete in X.

The following proposition is proved in [4] for complete X.

PROPOSITION 1.7. *Let $X \in \mathrm{ANR}$ satisfy the Strong Discrete Approximation Property. Then every Z-set in X is a strong Z-set in X.*

Proof. As in the proof of Lemma 1.4, approximately factor id_X through a locally compact space P, and let $f: P \to X$ be a map. Write $P = P' \cup P''$, where $P' = \bigcup_{i=1}^{\infty} P_i'$, $P'' = \bigcup_{i=1}^{\infty} P_i''$, and $\{P_i'\}_{i=1}^{\infty}$, $\{P_i''\}_{i=1}^{\infty}$ are discrete families of compacta. For $\mathcal{U} \in \operatorname{cov}(X)$ choose $\mathcal{V} \in \operatorname{cov}(X)$ such that each map $\alpha: P' \to X$ that is $\mathcal{V}$-close to $f \mid P'$ extends to a map $\tilde{\alpha}: P \to X$ that is $\mathcal{U}$-close to f. Let A be a Z-set in X, and let $\alpha: P' \to X - A$ be a map such that $\{\alpha(P_i')\}_{i=1}^{\infty}$ is discrete in X, and α is $\mathcal{V}$-close to $f \mid P'$. Thus there is an extension $\tilde{\alpha}: P \to X$ of α such that $(\tilde{\alpha}, f) < \mathcal{U}$. Choose $\mathcal{W} \in \operatorname{cov}(X)$ such that if $\tilde{\beta}: P \to X$ is $\mathcal{W}$-close to $\tilde{\alpha}$, then $\{\tilde{\beta}(P_i')\}_{i=1}^{\infty}$ is discrete and $\tilde{\beta}(P') \cap A = \varnothing$. Repeating the construction for P'', we produce a map $\tilde{\beta}: P \to X$ such that $(\tilde{\alpha}, \tilde{\beta}) < \mathcal{U}$, $(\tilde{\alpha}, \tilde{\beta}) < \mathcal{W}$, $\tilde{\beta}(P'') \cap A = \varnothing$, and $\{\tilde{\beta}(P_i'')\}_{i=1}^{\infty}$ is discrete. For a small map $h: X \to X - A$ the collections $\{h\tilde{\beta}(P_i')\}_{i=1}^{\infty}$, $\{h\tilde{\beta}(P_i''\}_{i=1}^{\infty}$ are discrete, and consequently $\mathrm{Cl}_X\, h\tilde{\beta}(P) \cap A = \varnothing$, and $h\tilde{\beta}$ approximates f. □

COROLLARY 1.8. *Let $X \in \mathrm{ANR}$ satisfy the Strong Discrete Approximation Property. Then every compact subset of X is a strong Z-set in X.*

Proof. For a given map $f: Q \to X$, apply the Strong Discrete Approximation Property to $\bigoplus_{i=1}^{\infty} f_i: \bigoplus_{i=1}^{\infty} Q_i \to X$, with $f_i = f$. If $\bigoplus_{i=1}^{\infty} g_i$ approximates $\bigoplus_{i=1}^{\infty} f_i$, and if $\{g_i(Q_i)\}_{i=1}^{\infty}$ is discrete, then at least one $g_i(Q_i)$ misses a given compactum K, which is therefore a Z-set in X. By Proposition 1.7 it follows that K is a strong Z-set in X. □

There are many interesting spaces that can be written as countable unions of strong Z-sets. The next lemma establishes that such spaces are "nice."

LEMMA 1.9. *Suppose $X \in \mathrm{ANR}$ can be represented as $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is a strong Z-set in X. Then X satisfies the Strong Discrete Approximation Property.*

Proof. By Corollary 1.6, each compact subset K of X is a strong Z-set in X. Let $f: \bigoplus_{i=1}^{\infty} Q_i \to X$ be a map and let $\mathcal{U} \in \operatorname{cov}(X)$. We construct sequences $\{g_i: X \to X\}$, $\{\mathcal{U}_i: \mathcal{U}_i \in \operatorname{cov}(X)\}$ and $\{(V_i, W_i): V_i, W_i$ are open sets in X, $V_i \supseteq \mathrm{Cl}_X W_i \supseteq W_i \supseteq X_i \cup \bigcup_{j=1}^{i} g_j g_{j-1} \cdots g_1 f(\bigoplus_{k=1}^{j} Q_k)\}$ such that

(7) $g_i(X) \cap V_i = \varnothing$,

(8) $\mathrm{St}\, \mathcal{U}_i < \mathcal{U}_{i-1}$, $\mathrm{St}\, \mathcal{U}_i < \{V_{i-1}, X - \mathrm{Cl}_X W_{i-1}\}$, and

(9) $(g_i, \mathrm{id}_X) < \mathcal{U}_i$.

Define $f': \bigoplus_{i=1}^{\infty} Q_i \to X$ by

$$f'(q) = g_i g_{i-1} \cdots g_1 f(q), \quad \text{for } q \in Q_i.$$

Then f' is $\operatorname{St} \mathcal{U}_1$-close to f, and $\{f'(Q_i)\}_{i=1}^{\infty}$ is a discrete family in X (since $f'(\bigoplus_{j=i+1}^{\infty} Q_j) \cap W_i = \varnothing$, $i = 1, 2, \ldots$). □

The next lemma will be needed in the sequel.

LEMMA 1.10. *Let $f: X \to Y$ be a fine homotopy equivalence between* ANR's, *and let A be a closed subset of Y. Then A is a strong Z-set in $X \cup_f A$ if and only if A is a strong Z-set in Y.*

Proof. Assuming that A is a strong Z-set in $X \cup_f A$, we can construct a small map $h: Y \to Y$ with $\operatorname{Cl}_Y h(Y) \cap A = \varnothing$ by choosing an approximate right inverse $g: Y \to X \cup_f A$ to the induced fine homotopy equivalence $p: X \cup_f A \to Y$, and picking a map $h': X \cup_f A \to X \cup_f A$ close to id with $\operatorname{Cl}_{X \cup_f A}(h'(X \cup_f A)) \cap A = \varnothing$. Then $h = ph'g: Y \to Y$ has the desired properties.

Now suppose that A is a strong Z-set in Y, and let $\mathcal{U} \in \operatorname{cov}(X \cup_f A)$. Cover $A \subseteq Y$ by a family $\mathcal{V}$ of open sets in Y such that $p^{-1}(\mathcal{V})$ refines $\mathcal{U}$. Let $V = \bigcup \mathcal{V}$ and let $\mathcal{W} \in \operatorname{cov}(V)$ such that $\operatorname{St} \mathcal{W} < \mathcal{V}$ and, for each $W \in \mathcal{W}$, $\operatorname{diam} W < \inf\{d(w, y): w \in W,\ y \in Y - V\}$. Let F, G, H be open subsets of Y such that $A \subseteq F \subseteq \operatorname{Cl}_Y F \subseteq G \subseteq \operatorname{Cl}_Y G \subseteq H \subseteq \operatorname{Cl}_Y H \subseteq V$. Since A is a strong Z-set in F (by Lemma 1.3) there is a map $\varphi: Y \to Y$ such that $\varphi \mid Y - F = \text{id}$, $\varphi \mid V: V \to V$ is $\mathcal{W}$-close to id_V, and $\varphi(Y) \cap M = \varnothing$ for some closed-in Y neighborhood $M \subseteq F$ of A. Since $p \mid p^{-1}(V - M) \to V - M$ is a fine homotopy equivalence, there is a map $q: V - M \to p^{-1}(V - M)$ and a $p^{-1}(\mathcal{W})$-homotopy $h_t: p^{-1}(V - M) \to p^{-1}(V - M)$ such that $h_0 = \text{id}$, $h_1 = qp \mid p^{-1}(V - M)$. Let $\lambda: p^{-1}(V) \to [0, 1]$ be a function such that $\lambda^{-1}(0) = p^{-1}(V - H)$ and $\lambda^{-1}(1) = p^{-1}(\operatorname{Cl}_Y G)$. Define $g: X \cup_f A \to X \cup_f A$ by

$$g(x) = \begin{cases} x & \text{for } x \in p^{-1}(Y - H) \\ h_{\lambda(x)}(x) & \text{for } x \in p^{-1}(V - F) \\ q\varphi p(x) & \text{for } x \in p^{-1}(G) \end{cases}.$$

Then $(g, \text{id}_{X \cup_f A}) < \mathcal{U}$, and $g(X \cup_f A) \cap p_A^{-1}(M) = \varnothing$. Thus A is a strong Z-set in $X \cup_f A$. □

In [22] it was shown that every ANR X can be embedded into a complete ANR $\tilde{X}$ so that $\tilde{X} - X$ is locally homotopy negligible in $\tilde{X}$.

LEMMA 1.11. *Suppose an* ANR *X is embedded into a complete metric space T, and A is a strong Z-set in X. Then there is a G_δ-subset $\tilde{X}$ of T such that*

(i) *$\tilde{X}$ is an* ANR,
(ii) *$X \subseteq \tilde{X}$ and $\tilde{X} - X$ is locally homotopy negligible in $\tilde{X}$, and*
(iii) *$\tilde{A} = \operatorname{Cl}_{\tilde{X}}(A)$ is a strong Z-set in $\tilde{X}$.*

Proof. By Corollary 1.2, there is a homotopy $H: X \times [0, 1] \to X$ so that (1)-(3) holds. By [22], there is a G_δ-subset $\tilde{X}_1$ of T such that $X \subseteq \tilde{X}_1$, $\tilde{X}_1$ is an ANR, and $\tilde{X}_1 - X$ is locally homotopy negligible in $\tilde{X}_1$. Note that, by [22], any G_δ-subset $\tilde{X}$ of $\tilde{X}_1$ with $X \subseteq \tilde{X} \subseteq \tilde{X}_1$ is an ANR (and $\tilde{X} - X$ is locally homotopy negligible in $\tilde{X}$). Then $H: X\times[0,1]\to\tilde{X}_1$ extends to a G_δ-set containing $X\times[0,1]$ (see [10]). Trimming it down, we obtain a G_δ-subset $\tilde{X}_2$ of $\tilde{X}_1$, and an extension $\tilde{H}: \tilde{X}_2\times[0,1]\to\tilde{X}_1$ of H. Note that $\tilde{H}(x,0)=x$ and $\tilde{H}(\tilde{X}_2\times[0,1])\cap A=\varnothing$, since $\tilde{H}(\tilde{X}_2\times[\epsilon,1])\subseteq \mathrm{Cl}_{\tilde{X}_1} H(X\times[\epsilon,1])\subseteq\tilde{X}_1-A$. Moreover, $\tilde{H}(\tilde{X}_2\times(0,1])$ misses a G_δ-set $\hat{A}\subseteq A$. Similarly, $\tilde{H}$ is closed over A (from the density of X in $\tilde{X}_2$). By [11] the set $C(\tilde{H})$ of points in $\tilde{X}_1$ over which $\tilde{H}$ is closed is of type G_δ in $\tilde{X}_1$, and it contains A. Let $\tilde{X}_3=\tilde{X}_2-(\mathrm{Cl}_{\tilde{X}_2} C(\tilde{H})-C(\tilde{H}))-(\mathrm{Cl}_{\tilde{X}_2}\hat{A}-\hat{A})$, and observe that $\tilde{X}_3\supseteq X$ and $\tilde{X}_3$ is of type G_δ in $\tilde{X}_2$. Inductively choose a sequence $\{\tilde{X}_n\}_{n=4}^\infty$ such that $X\subseteq\tilde{X}_n\subseteq\tilde{X}_{n-1}$, $\tilde{X}_n$ is a G_δ-subset of $\tilde{X}_{n-1}$, and $\tilde{H}(\tilde{X}_n\times[0,1])\subseteq\tilde{X}_{n-1}$. Then let $\tilde{X}=\bigcap_{n=1}^\infty\tilde{X}_n$, and note that $\tilde{H}\mid\tilde{X}\times[0,1]: \tilde{X}\times[0,1]\to\tilde{X}$ is a map closed over $\tilde{A}=\mathrm{Cl}_{\tilde{X}} A$ with $\tilde{H}(x,0)=x$ and $\tilde{H}(\tilde{X}\times(0,1])\cap\tilde{A}=\varnothing$. Hence, by Corollary 1.2, $\tilde{A}$ is a strong Z-set in $\tilde{X}$, which is a G_δ-set containing X.

2. Strong universality. Let $\mathcal{C}$ be a class of (separable metric) spaces. We say that $\mathcal{C}$ is a *topological class* if for every $C\in\mathcal{C}$ and every homeomorphism $h: C\to D$ it follows that $D\in\mathcal{C}$. A topological class $\mathcal{C}$ is *hereditary with respect to closed (open) subsets* if every closed (open) subset of any $C\in\mathcal{C}$ belongs to $\mathcal{C}$.

A (separable, metric) space $X\in$ ANR is $\mathcal{C}$-*universal* if for every map $f: C\to X$ of a space $C\in\mathcal{C}$ and for every $\mathcal{U}\in\mathrm{cov}(X)$ there exists a Z-embedding $h: C\to X$ such that $(f,h)<\mathcal{U}$ (an embedding $h: C\to X$ is a *Z-embedding* if $h(C)$ is a Z-set in X).

A space X is *strongly* $\mathcal{C}$-*universal* if for every map $f: C\to X$ from a space $C\in\mathcal{C}$, for every closed subset $D\subseteq C$ such that $f\mid D: D\to X$ is a Z-embedding, and for every $\mathcal{U}\in\mathrm{cov}(X)$, there exists a Z-embedding $h: C\to X$ such that $h\mid D= f\mid D$ and $(f,h)<\mathcal{U}$.

PROPOSITION 2.1. *Let $\mathcal{C}$ be a topological class hereditary with respect to closed subsets, and let $X\in$ ANR be strongly $\mathcal{C}$-universal. Then every open subset of X is strongly $\mathcal{C}$-universal.*

Proof. Let d be a metric on X, U an open subset of X, and $f: C\to U$ a map of a space $C\in\mathcal{C}$ into U such that $f\mid D: D\to U$ is a Z-embedding for some closed set $D\subseteq C$. We have $U=\bigcup_{n=1}^\infty U_n$, where $U_n=\{x\in X: d(x,X-U)\geq 2^{-n}\}$. Let $A_n= f^{-1}(U_n)$ and $B_n=f^{-1}(U-\mathrm{int}\,U_{n+1})$. Then $C=\bigcup_{n=1}^\infty A_n$, $A_n\subseteq\mathrm{int}\,A_{n+1}$, and A_n, B_n are disjoint closed subsets of C. For a given map $\epsilon: U\to(0,1)$ we shall construct a sequence $\{f_n: C\to U\}$ satisfying the following conditions:

(i) $f_n\mid B_n\cup D=f\mid B_n\cup D$,
(ii) $f_n\mid A_n\cup D: A_n\cup D\to U$ is a Z-embedding.
(iii) $f_n\mid A_{n-1}\cup B_n\cup D=f_{n-1}\mid A_{n-1}\cup B_n\cup D$,
(iv) f_n is ϵ_n-close to f_{n-1}, where $\epsilon_n: X\to(0,1)$ is a map such that $\epsilon_n(x)= 2^{-n}\min\{\epsilon(x), d(x,X-U)\}$ for $x\in U_{n+1}$, and
(v) $f_n(A_{n+2})\subseteq U_{n+2}$.

Without loss of generality, $A_0=\varnothing$ and $B_0=C$, so we can set $f_0=f$. Let us assume that f_{n-1} has been constructed. Since X is a strongly $\mathcal{C}$-universal ANR, there is an ϵ_n-homotopy $g_t: C\to X$ such that $g_0=f_{n-1}$, $g_1: C\to X$ is a Z-embedding, and $g_1\mid A_{n-1}\cup(D-\mathrm{int}\,B_n)=f_{n-1}\mid A_{n-1}\cup(D-\mathrm{int}\,B_n)$. We can also assume that $g_t(c)=f(c)$ for each $c\in D-\operatorname{int}B_n$. Define $f_n\colon C\to X$ by $f_n(c)=g_{\lambda(c)}(c)$, where $\lambda\colon C\to[0,1]$ is a Urysohn function satisfying $\lambda(B_n)=\{0\}$, $\lambda(A_n)=\{1\}$. To check (v) let $c\in A_{n+2}$. If $c\in A_{n+1}$, then $f_{n-1}(c)\in U_{n+1}$ and hence, by (iv), $\frac{1}{2}d(f_{n-1}(c),X-U)\le d(f_n(c),X-U)$, which implies $d(f_n(c),X-U)\ge\frac{1}{2}2^{-(n+1)}$ and thus $f_n(c)\in U_{n+2}$. If $c\in A_{n+2}-A_{n+1}\subseteq B_n$, then $f_n(c)=f(c)\in U_{n+2}$. Consequently, (v) holds and it implies $f_n(C)\subseteq U$.

Define a map $h\colon C\to U$ by $h=\lim_{n\to\infty}f_n$. It is clear that h is ϵ-close to f. Also, (ii), (iii) and (iv) imply that h is an embedding. Note that

$$h(C)=\bigcup_{n=0}^{\infty}h(A_{n+1}-\operatorname{int}A_n)=\bigcup_{n=0}^{\infty}f_{n+1}(A_{n+1}-\operatorname{int}A_n)$$

is a locally finite union of Z-sets in U, and hence $h(C)$ is a Z-set in U. □

The following proposition helps detecting strong universality.

PROPOSITION 2.2. *Let $\mathcal{C}$ be a topological class that is hereditary with respect to both closed and open subsets. If each open subset of $X\in$ ANR is $\mathcal{C}$-universal, and if every Z-set in X is a strong Z-set, then X is strongly $\mathcal{C}$-universal.*

Proof. Let $f\colon C\to X$ be a given map from $C\in\mathcal{C}$ and the $D\subseteq C$ be a closed subset such that $f\mid D\colon D\to X$ is a (strong) Z-embedding. By Lemma 1.1 we can approximate f by a map $g\colon C\to X$ such that $g\mid D=f\mid D$, $g(C-D)\cap f(D)=\varnothing$, and so that g is closed over $g(D)=f(D)$. Apply the hypotheses to the map $g\mid C-D\colon C-D\to X-g(D)$ to produce a Z-embedding $g'\colon C-D\to X-g(D)$. If g' is sufficiently close to $g\mid C-D$, then the map $\hat{g}\colon C\to X$ defined by $\hat{g}\mid D=g\mid D$, $\hat{g}\mid C-D=g'$ is a Z-embedding close to f with $\hat{g}\mid D=f\mid D$. □

A topological class $\mathcal{C}$ is *additive* if $C\in\mathcal{C}$ whenever C can be expressed as the union of two of its closed subsets that belong to $\mathcal{C}$.

For a topological class $\mathcal{C}$ we can form the class $\mathcal{C}_\sigma$ that consists of all spaces C that can be written as $C=\bigcup_{n=1}^{\infty}C_n$, where C_n is a closed subset of C with $C_n\in\mathcal{C}$, $n=1,2,\ldots$. Clearly, if $\mathcal{C}$ is hereditary with respect to closed subsets, then $\mathcal{C}_\sigma$ is hereditary with respect to both closed and open subsets.

PROPOSITION 2.3. *Let $\mathcal{C}$ be an additive topological class hereditary with respect to closed subsets. Suppose $X\in$ ANR can be written as $X=\bigcup_{i=1}^{\infty}X_i$, where each X_i is a strong Z-set in X. If X is strongly $\mathcal{C}$-universal, then X is strongly $\mathcal{C}_\sigma$-universal.*

Proof. By Proposition 2.2 (see also Lemma 1.9 and Proposition 1.7) it suffices to show that each open subset $U\subseteq X$ is $\mathcal{C}_\sigma$-universal. Without loss of generality (see Lemma 1.3 and Proposition 2.1) we can assume that $U=X$.

Let $f\colon C\to X$ be a map of $C\in\mathcal{C}_\sigma$. We assume first that C is an open subset of some $C'\in\mathcal{C}$. Write $C=\bigcup_{i=1}^{\infty}C_i$, where $\operatorname{int}C_i\supseteq C_{i-1}$, C_i is a closed subset of C, and $C_i\in\mathcal{C}$. For a given $\mathcal{U}\in\operatorname{cov}(X)$, choose a sequence $\{\mathcal{U}_i\in\operatorname{cov}(X)\}_{i=1}^{\infty}$ such that $\operatorname{St}\mathcal{U}_i<\mathcal{U}_{i-1}$ and $\operatorname{St}\mathcal{U}_1<\mathcal{U}$. Without loss of generality, $\varnothing=X_1\subseteq X_2\subseteq X_3\subseteq\cdots$. We shall construct a sequence $\{f_i\colon C\to X\}_{i=1}^{\infty}$ such that

(1) $f_i \mid C_i : C_i \to X$ is a Z-embedding.
(2) $f_i \mid C_{i-1} = f_{i-1} \mid C_{i-1}$;
(3) $\mathrm{Cl}_X f_i(C - \mathrm{int}\, C_{i+1}) \cap X_{i+1} = \varnothing$; and
(4) f_i and f_{i-1} are close with respect to $\mathcal{U}_i$; the function $\epsilon_i : X \to (0,1]$, where

$$\epsilon_0 = 1 \quad \text{and} \quad \epsilon_i = \min\{\tfrac{1}{2}\epsilon_{i-1}, \delta_k^j : 1 \le j \le i-1,\ j-1 \le k \le i-1\};$$

and $\delta_k^j : X \to (0,1]$ is a map with

$$\delta_k^j(x) = \tfrac{1}{4} d(x, X_j) \quad \text{for } x \in \mathrm{Cl}_X f_k(C - \mathrm{int}\, C_j).$$

We set $f_0 = f$. Assuming that f_{i-1} has been constructed, choose a Z-embedding $v_i : C_i \to X$ such that $v_i \mid C_{i-1} = f_{i-1} \mid C_{i-1}$, and v_i is so close to $f_{i-1} \mid C_i$ that there is an extension $\tilde{v}_i : C \to X$ of v_i, which is close to f_{i-1}. Let $h_t : X \to X$ be a small homotopy such that $\mathrm{Cl}_X h_1(X) \cap X_{i+1} = \varnothing$, and define $f_i : C \to X$ by $f_i(c) = h_{\lambda(c)}(\tilde{v}_i(c))$, where $\lambda : X \to [0,1]$ is a Urysohn function such that $\lambda(C_i) = \{0\}$ and $\lambda(C - \mathrm{int}\, C_{i+1}) = \{1\}$.

The reader can verify that $f' = \lim_{i\to\infty} f_i$ is a Z-embedding $\mathcal{U}$-close to f.

Now consider the general case of $C \in \mathcal{C}_\sigma$. Using the fact that for each map $g : C \to X$ the restriction $g \mid C_i - C_{i-1}$ can be approximated by embeddings, we construct a sequence $\{f_i : C \to X\}$ satisfying

(5) $f_i \mid C_i$ is a Z-embedding,
(6) $f_i \mid C_{i-1} = f_{i-1} \mid C_{i-1}$,
(7) f_i is a closed map over $f_i(C_i)$ and $f_i(C - C_i) \cap f_i(C_i) = \varnothing$,
(8) $d(f_i(c), f_{i-1}(c)) \le \frac{1}{2} d(f_{i-1}(c), f_{i-1}(C_{i-1}))$ for $c \in C - C_{i-1}$,
(9) $\mathrm{Cl}_X f_i(C) \cap (X_i - f_{i-1}(C_{i-1})) = \varnothing$,
(10) $d(f_i(c), f_{i-1}(c)) < \frac{1}{4} d(f_{i-1}(c), X_{i-1})$ for $c \in C - C_{i-1}$, and
(11) $(f_i, f_{i-1}) < \mathcal{U}_i$ for a sequence $\{\mathcal{U}_i \in \mathrm{cov}(X)\}$, with mesh $\mathcal{U}_i < 2^{-i}$ and $\mathrm{St}\, \mathcal{U}_i < \mathcal{U}_{i-1}$.

As usual, $f_0 = f$. Suppose f_{i-1} has been constructed, and consider

$$f_{i-1} \mid C - C_{i-1} : C - C_{i-1} \to X - f_{i-1}(C_{i-1}).$$

By the case already discussed, we can find a Z-embedding

$$h : C_i - C_{i-1} \to X - f_{i-1}(C_{i-1})$$

so close to $f_{i-1} \mid C_i - C_{i-1}$ that h extends to $\tilde{h} : C - C_{i-1} \to X - f_{i-1}(C_{i-1})$ and $\tilde{h}$ is close to $f_{i-1} \mid C - C_{i-1}$. Note that by Lemma 1.4 each Z-set in $X - f_{i-1}(C_{i-1})$ is a strong Z-set, and hence by Lemma 1.1 there is a map $\tilde{h}' : C - C_{i-1} \to X - f_{i-1}(C_{i-1})$ close to $\tilde{h}$ with $\tilde{h}' \mid C_i - C_{i-1} = \tilde{h} \mid C_i - C_{i-1}$, $\tilde{h}'(C - C_i) \cap \tilde{h}(C_i - C_{i-1}) = \varnothing$, and $\tilde{h}'$ is closed over $\tilde{h}'(C_i - C_{i-1})$.

Define $f_i : C \to X$ by $f_i \mid C_{i-1} = f_{i-1} \mid C_{i-1}$, $f_i \mid C - C_{i-1} = \tilde{h}'$. Then

$$f' = \lim_{i\to\infty} f_i : C \to X$$

is a closed embedding $\mathrm{St}\, \mathcal{U}_1$-close to f. Note that f' may not be a Z-embedding (f' can even be a homeomorphism!).

To get a Z-embedding, fix a countable set $A_0 = \{\alpha_0^1, \alpha_0^2, \dots\}$ dense in the space of maps $Q \to X$. Along with the sequence $\{f_i: C \to X\}$ we construct countable dense sets $A_i = \{\alpha_i^1, \alpha_i^2, \dots\} \subseteq C(Q, X)$ so that

(12) $\alpha_i^k(Q) \cap f_i(C_i) = \varnothing$ for $k \le i$,

(13) $\alpha_i^k = \alpha_{i-1}^k$ for $k \le i-1$, and

(14) $d(\alpha_i^k, \alpha_{i-1}^k) < 2^{-k}$ for $k \ge i$.

This construction is possible because compact subsets of X (in particular, $\bigcup_{k \le i} \alpha_i^k(Q)$) are (strong) Z-sets in X (see Corollary 1.6). Then $\{\alpha_1^1, \alpha_2^2, \dots\} \subseteq C(Q, X)$ is a dense set, and $f'(C) \cap (\bigcup_{k=1}^\infty \alpha_k^k(Q)) = \varnothing$. It follows that $f'(C)$ is a Z-set in X. □

The following corollary summarizes the results.

COROLLARY 2.4. *Let $\mathcal{C}$ be an additive topological class hereditary with respect to closed subsets. Suppose $X \in$ ANR can be written as $X = \bigcup_{i=1}^\infty X_i$, where each X_i is a strong Z-set in X. Then the following statements are equivalent:*

(i) X is strongly $\mathcal{C}$-universal,

(ii) X is strongly $\mathcal{C}_\sigma$-universal,

(iii) each open subset of X is $\mathcal{C}_\sigma$-universal,

(iv) each open subset of X is strongly $\mathcal{C}_\sigma$-universal.

We close this section by three propositions detecting strong universality of certain spaces.

For a space X and a basepoint $* \in X$, we define the *weak product*

$$W(X, *) = \{(x_1, x_2, \dots) \in X^\infty : x_n = * \text{ for almost all } n\}$$

(with the subspace topology).

PROPOSITION 2.5. *Let $X \in$ ANR, $X \ne \{\text{point}\}$. Then the space X^∞ ($W(X, *)$ for a basepoint $* \in X$) is strongly universal for the class $\mathcal{C}$ of spaces homeomorphic to a closed subset of X^∞ (respectively $W(X, *)$).*

Proof. Note that $X^\infty \cong (X^\infty)^\infty$ (and $W(X, *) \cong W(W(X, *), *)$) and hence each point in X^∞ ($W(X, *)$) can be represented as $(x_1, x_2, \dots)$, where $x_i \in X^\infty$ ($W(X, *)$). We assume that the metric d on X^∞ ($W(X, *)$) is chosen so that $d(x, x') \le 2^{-k-2}$ if x and x' agree on the first k coordinates.

Let $g: C \to X^\infty$ ($g: C \to W(X, *)$) be a closed embedding such that $g(C) \not\ni * = (*, *, \dots)$; if necessary replace $g = (g_1, g_2, \dots)$ by $g' = (*', g_1, g_2, \dots)$ for some $*' \ne *$; and let $f: C \to X^\infty$ (or $W(X, *)$) be a given map. We also assume that $f \mid D: D \to X^\infty$ ($W(X, *)$) is a Z-embedding, and that $\epsilon: X^\infty \to (0, 1)$ (resp. $\epsilon: W(X, *) \to (0, 1)$) is a Lipschitz map.

Note that X can be embedded into a complete AR $\tilde{X}$ so that $\tilde{X} - X$ is locally homotopy negligible in $\tilde{X}$ [22], and hence X^∞ embeds into $\tilde{X}^\infty \cong s = (-1, 1)^\infty$ or $Q = [-1, 1]^\infty$ [23], so that $s - X^\infty$ (or $Q - X^\infty$) is locally homotopy negligible. (The same is true for $W(X, *) \subseteq W(\tilde{X}, *) \subseteq \tilde{X}^\infty \cong s$ or Q.) Consequently, every Z-set in X^∞ ($W(X, *)$) is a strong Z-set. Hence by Lemma 1.1 we can assume that $f(C - D) \cap f(D) = \varnothing$, and that f is closed over $f(D)$.

Define $\delta: X^\infty \to [0,1)$ ($\delta: W(X,*) \to [0,1)$) by

$$\delta(x) = \min\{\epsilon(x),\ d(x, f(D))\}.$$

For $2^{-k-1} \le \delta(f(c)) \le 2^{-k}$, $k = 0, 1, 2, \dots$, define

$$f'(c) = \left(f_1(c), f_2(c), \dots, f_k(c), H_{k+1}\left(c, \frac{1}{\delta(f(c))} - k\right),\right.$$

$$\left. g(c), g(c), G\left(c, \frac{1}{\delta(f(c))} - k\right), *, *, *, \dots\right),$$

where $H_{k+1}: C \times [0,1] \to X^\infty$ ($W(X,*)$) is a homotopy between g and f_{k+1}, and $G: C \times [0,1] \to X^\infty$ ($W(X,*)$) is a homotopy between $*$ and g (f_i is the ith coordinate of f).

For $c \in D$, let $f'(c) = f(c)$.

Note that $d(f(c), f'(c)) \le \frac{1}{2}\delta(f(c))$, and $\delta(f(c)) \le 2\delta(f'(c))$. These inequalities imply that if $f'(c_n) \to x \in f(D)$ then $c_n \to f^{-1}(x)$, and hence f' is closed over $f'(D)$. It is left to the reader to show that f' is a Z-embedding. □

PROPOSITION 2.6. *Let $\mathcal{C}$ be a topological class, and suppose $X \in$ ANR is strongly $\mathcal{C}$-universal. If $Y \in$ ANR, then $X \times Y$ is strongly $\mathcal{C}$-universal provided every Z-set in $X \times Y$ is a strong Z-set, and provided $C \in \mathcal{C}$ implies $C \times [0,1] \in \mathcal{C}$.*

Proof. Let $\epsilon: X \times Y \to (0, \frac{1}{2})$ be a given Lipschitz map, and $f: C \to X \times Y$ a map from $C \in \mathcal{C}$, with $f \mid D: D \to X \times Y$ being a (strong) Z-embedding for a closed set $D \subseteq C$. By Lemma 1.1 we can assume that $f(C - D) \cap f(D) = \varnothing$ and that f is closed over D. Let $\delta: X \times Y \to [0, \frac{1}{2})$ be defined by $\delta(x) = \min\{\epsilon(x), d(x, f(D))\}$. For $n = 1, 2, \dots$, choose a Z-embedding $g_n: C \to X$ such that g_n is 2^{-n-4} homotopic to $p_X f: C \to X$, where $p_X: X \times Y \to X$ is the projection. Without loss of generality, we assume that $g_n(C) \cap g_m(C) = \varnothing$ for $n \ne m$. Note that g_n and g_{n+1} are 2^{-n-3} homotopic, and let $H_n: C \times [0,1] \to X$ be a Z-embedding such that $H_n(c, 0) = g_n(c)$, $H_n(c, 1) = g_{n+1}(c)$, and $\operatorname{diam} H_n(\{c\} \times [0,1]) < 2^{-n-3}$ for $c \in C$. Also, without loss of generality, $H_n(C \times [0,1]) \cap H_m(C \times [0,1]) = \varnothing$ for $|n - m| > 1$, and $H_n(C \times [0,1]) \cap H_{n+1}(C \times [0,1]) = g_{n+1}(C)$ (by strong universality of X). For $2^{-k-1} \le \delta(f(c)) \le 2^{-k}$, $k = 1, 2, \dots$, define

$$f'(c) = \left(H_k\left(c, \frac{1}{\delta(f(c))} - k\right), p_Y f(c)\right),$$

where $p_Y: X \times Y \to Y$ is the projection. For $\delta(f(c)) = 0$ (i.e., for $c \in D$) let $f'(c) = f(c)$. Then $f': C \to X \times Y$ is a Z-embedding δ-close to f. □

The next proposition detects strong $\mathcal{C}$-universality of manifolds modeled on strong $\mathcal{C}$-universal spaces.

PROPOSITION 2.7. *Suppose $X \in$ ANR has an open cover $\mathcal{U}$ such that every $U \in \mathcal{U}$ is strongly $\mathcal{C}$-universal for a topological class $\mathcal{C}$ hereditary with respect to closed subsets. Then X is strongly $\mathcal{C}$-universal.*

Proof. By Proposition 2.1 we can assume that $\mathcal{U}=\{U_1, U_2, \dots\}$ is countable and locally finite. Find $\mathcal{V}=\{V_1, V_2, \dots\}\in\operatorname{cov}(X)$ so that $\mathrm{Cl}_X V_i\subseteq U_i$, and pick $\mathcal{W}_0\in\operatorname{cov}(X)$ so that $\mathrm{St}(\mathrm{Cl}_X V_i, \mathcal{W}_0)\subseteq U_i$, $i=1,2,3,\dots$. Let $\{\mathcal{W}_i\in\operatorname{cov}(X)\}$ be a sequence with $\mathrm{St}\,\mathcal{W}_i<\mathcal{W}_{i-1}$, $i=1,2,\dots$.

For $C\in\mathcal{C}$, a closed subset $D\subseteq C$, and a map $f\colon C\to X$ such that $f\mid D\colon D\to X$ is a Z-embedding, define $C_i=f^{-1}(\mathrm{Cl}_X V_i)$. We construct a sequence $\{f_i\colon C\to X\}$ with the following properties:

(1) $f_i\mid D\cup C_1\cup\cdots\cup C_{i-1}=f_{i-1}\mid D\cup C_1\cup\cdots\cup C_{i-1}$,

(2) $f_i\mid D\cup C_1\cup\cdots\cup C_i\colon D\cup C_1\cup\cdots\cup C_i\to X$ is a Z-embedding, and

(3) $(f_i, f_{i-1})<\mathcal{W}_i$.

Setting $f_0=f$ we proceed inductively, assuming that f_{i-1} has been constructed. Note that by construction $\mathrm{Cl}_X f_{i-1}(C_i)\subseteq U_i$, and that $f_{i-1}\colon C_i\to U_i$ restricted to $C_i\cap(D\cup C_1\cup\cdots\cup C_{i-1})$ is a Z-embedding. By the strong $\mathcal{C}$-universality of U_i, there is a Z-embedding $g\colon C_i\to U_i$ such that

$$g\mid C_i\cap(D\cup C_1\cup\cdots\cup C_{i-1})=f_{i-1}\mid C_i\cap(D\cup C_1\cup\cdots\cup C_{i-1}).$$

We can also assume that g is so close to $f_{i-1}\mid C_i$ that g extends to $f_i\colon C\to X$, so that properties (1)–(3) hold.

Define $f'\colon C\to X$ by $f'=\lim_{i\to\infty} f_i$. Then f' is a Z-embedding $\mathcal{W}_0$-close to f, and $f'\mid D=f\mid D$.

3. $\mathcal{C}$-absorbing sets in s-manifolds. A natural generalization of the notion of an (f.d.) cap (finite dimensional compact absorption property) set [2] is the notion of a $\mathcal{C}$-absorbing set. *In what follows, $\mathcal{C}$ will be an additive topological class hereditary with respect to closed subsets,* for example, the class of (finite dimensional) compact metric spaces. As usual, s denotes the pseudo-interior $(-1,1)^\infty$ of the Hilbert cube $Q=[-1,1]^\infty$.

We say that a subset X of an s-manifold M is a *$\mathcal{C}$-absorbing set* (*in M*) if $M-X$ is locally homotopy negligible in M, $X=\bigcup_{n=1}^\infty X_n$ where each X_n is a Z-set in X and $X_n\in\mathcal{C}$, and X is strongly $\mathcal{C}$-universal. It follows that X is an ANR [22] and that every Z-set in X is a strong Z-set in X.

The following result is a slight modification of the well-known theorems about homeomorphisms between cap sets, Z-skeletons, absorbing sets, and pseudoboundaries (cf. [2], [5], [24], [27], [14]).

THEOREM 3.1. *Let X and Y be two $\mathcal{C}$-absorbing sets in an s-manifold M. Then for every $\mathcal{U}\in\operatorname{cov}(M)$ there exists a homeomorphism $h\colon X\to Y$ that is $\mathcal{U}$-close to the inclusion $X\subseteq M$.*

Proof (cf. [5], [24], [27]). Write $X=\bigcup_{n=1}^\infty X_n$, $Y=\bigcup_{n=1}^\infty Y_n$ as in the definition. Let $\{\mathcal{U}_n\}$ be a sequence of open covers of M such that $\mathrm{St}\,\mathcal{U}_{n+1}<\mathcal{U}_n$ and $\operatorname{mesh}\mathcal{U}_n<2^{-n}$. To find a homeomorphism $h\colon X\to Y$ it suffices to construct sequences of homeomorphisms $\{f_n\colon K_n\to L_n\}$ and $\{g_n\colon L'_n\to K'_n\}$, where K_n, K'_n, L_n, L'_n are G_δ-subsets of M with $K_n\cap K'_n\supseteq X$ and $L_n\cap L'_n\supseteq Y$, such that the following conditions are satisfied:

(a)$_n$ $f_n \mid X$ is $\mathcal{U}_n$-close to $f_{n-1} \mid X$ and $g_n \mid Y$ is $\mathcal{U}_n$-close to $g_{n-1} \mid Y$,
(b)$_n$ $f_n \mid X_{n-1} = f_{n-1} \mid X_{n-1}$ and $g_n \mid Y_{n-1} = g_{n-1} \mid Y_{n-1}$,
(c)$_n$ $f_n(X_n)$ is a Z-set in Y and $g_n(Y_n)$ is a Z-set in X,
(d)$_n$ $g_n f_n \mid X_n = \mathrm{id}_{X_n}$ and $f_n g_n \mid Y_n = \mathrm{id}_{Y_n}$.
Then the maps $f = \lim_{n\to\infty} f_n$ and $g = \lim_{n\to\infty} g_n$ are well-defined and continuous, and $fg = \mathrm{id}_Y$, $gf = \mathrm{id}_X$.

Letting $K_0 = K_0' = L_0 = L_0' = M$ and $f_0 = g_0 = \mathrm{id}_M$ we proceed inductively. Assume that f_i, g_i (satisfying (a)$_i$–(d)$_i$ for $i = 1, \ldots, n-1$) have been constructed. We will construct a map f_n satisfying (a)$_n$–(c)$_n$. Since Y is strongly $\mathcal{C}$-universal, there is a Z-embedding $h: X_n \cup g_{n-1}(Y_{n-1}) \to Y$ $\mathcal{U}_{n+6}$-homotopic to $f_{n-1} \mid X_n \cup g_{n-1}(Y_{n-1})$ such that $h \mid X_{n-1} \cup g_{n-1}(Y_{n-1}) = f_{n-1} \mid X_{n-1} \cup g_{n-1}(Y_{n-1})$. By Lavrentiev's theorem [10] there is a homeomorphism $\tilde{h}: A \to B$ between two G_δ-subsets A and B of K_{n-1} and L_{n-1} respectively, satisfying

$$X_n \cup g_{n-1}(Y_{n-1}) \subseteq A \subseteq \mathrm{Cl}_{K_{n-1}}(X_n \cup g_{n-1}(Y_{n-1})),$$

$$h(X_n \cup g_{n-1}(Y_{n-1})) \subseteq B \subseteq \mathrm{Cl}_{L_{n-1}} h(X_n \cup g_{n-1}(Y_{n-1})),$$

and

$$\tilde{h} \mid X_n \cup g_{n-1}(Y_{n-1}) = h.$$

We can assume that $\tilde{h}$ is $\mathcal{U}_{n+5}$-homotopic to $f_{n-1} \mid A$. Define

$$K_n = K_{n-1} - (Cl_{K_{n-1}}(X_n \cup g_{n-1}(Y_{n-1})) - A),$$

$$L_n = L_{n-1} - (\mathrm{Cl}_{L_{n-1}} h(X_n \cup g_{n-1}(Y_{n-1})) - B).$$

Then K_n and L_n, being G_δ-subsets of M such that $M - K_n$, $M - L_n$ are locally homotopy negligible (and hence σ-Z-sets), are s-manifolds, and $X \subseteq K_n \subseteq K_{n-1}$, $Y \subseteq L_n \subseteq L_{n-1}$. Let $\alpha: K_{n-1} \to K_n$ be a homeomorphism so close to the identity $\mathrm{id}_{K_{n-1}}$ that $f_{n-1}\alpha^{-1} \mid X$ is $\mathcal{U}_{n+6}$-homotopic to $f_{n-1} \mid X$. Let $\beta: L_{n-1} \to L_n$ be a homeomorphism $\mathcal{U}_{n+6}$-homotopic to $\mathrm{id}_{L_{n-1}}$. Then $\beta f_{n-1}\alpha^{-1} \mid X$ is $\mathcal{U}_{n+4}$-homotopic to $f_{n-1} \mid X$ and $\beta f_{n-1}\alpha^{-1} \mid A$ is $\mathcal{U}_{n+2}$-homotopic to $\tilde{h}$. Let $\gamma: L_n \to L_n$ be a homeomorphism of L_n $\mathcal{U}_{n+2}$-homotopic to id_{L_n} such that $\gamma\beta f_{n-1}\alpha^{-1} \mid A = \tilde{h}$ (the Z-set Unknotting Theorem for s-manifolds). Then the homeomorphism $f_n = \gamma\beta f_{n-1}\alpha^{-1}: K_n \to L_n$ satisfies (a)$_n$–(c)$_n$.

The construction of a homeomorphism $g_n: L_n' \to K_n'$ satisfying (a)$_n$–(d)$_n$ is similar, and is left to the reader. □

The powerful Z-set Unknotting Theorem for s-manifolds carries over to $\mathcal{C}$-absorbing sets.

THEOREM 3.2. *Let X be a $\mathcal{C}$-absorbing set in an s-manifold M, let $\mathcal{U} \in \mathrm{cov}(X)$, and suppose that $h: A \to B$ is a homeomorphism between Z-sets A and B in X. If h is $\mathcal{U}$-homotopic to the inclusion $A \subseteq X$, and if $\mathcal{V} \in \mathrm{cov}(X)$, then there is a homeomorphism $H: X \to X$ such that $H \mid A = h$ and $(H, \mathrm{id}_X) < \mathrm{St}(\mathcal{U}, \mathcal{V})$.*

Proof. Find a G_δ-subset $\tilde{X}$ of M containing X, open covers $\tilde{\mathcal{U}}$, $\tilde{\mathcal{V}}$ of $\tilde{X}$, and a homeomorphism $\tilde{h}: \tilde{A} \to \tilde{B}$ between two Z-sets in $\tilde{X}$ such that $\tilde{\mathcal{U}} \mid X = \mathcal{U}$, $\tilde{\mathcal{V}} \mid X = \mathcal{V}$, $\tilde{A} \cap X = A$, $\tilde{B} \cap X = B$, $\tilde{h} \mid A = h$, and $\tilde{h}$ is $\mathrm{St}(\tilde{\mathcal{U}}, \tilde{\mathcal{W}})$-homotopic to the inclusion $\tilde{A}\subseteq\tilde{X}$, where $\tilde{\mathcal{W}}\in\operatorname{cov}(\tilde{X})$ is chosen so that $\operatorname{St}^3\tilde{\mathcal{W}}<\tilde{\mathcal{V}}$. As before, $\tilde{X}$ is an s-manifold, and X is a $\mathcal{C}$-absorbing set in $\tilde{X}$. Applying [3] we obtain a homeomorphism $f:\tilde{X}\to\tilde{X}$ which is $\operatorname{St}(\tilde{\mathcal{U}},\operatorname{St}\tilde{\mathcal{W}})$-homotopic to $\mathrm{id}_{\tilde{X}}$. Note that $f(X)\cong X$ is a $\mathcal{C}$-absorbing set in $\tilde{X}$. By Proposition 2.1, $f(X)-\tilde{B}$ and $X-\tilde{B}$ are $\mathcal{C}$-absorbing sets in $\tilde{X}-\tilde{B}$. Applying Theorem 3.1, we find a homeomorphism $g: f(X)-\tilde{B}\to X-\tilde{B}$ so close to identity that the map $\tilde{g}:(f(X)-\tilde{B})\cup B\to X$ is well-defined by

$$\tilde{g}(x)=\begin{cases} g(x), & x\in f(X)-\tilde{B}\\ x, & x\in B\end{cases}$$

and so that $\tilde{g}$ is a homeomorphism $\tilde{\mathcal{W}}$-close to id. Finally, $H=\tilde{g}f\mid X$ is a homeomorphism of X onto itself such that $H\mid A=h$, and H is $\operatorname{St}(\mathcal{U},\mathcal{V})$-close to id_X. □

Next, we identify the set of near homeomorphisms between two $\mathcal{C}$-absorbing sets as being precisely equal to the set of fine homotopy equivalences between them.

THEOREM 3.3. *Let X and Y be $\mathcal{C}$-absorbing sets in an s-manifold M, and let $f: X\to Y$ be a fine homotopy equivalence. Then f is a near-homeomorphism.*

Proof. First note that if $f_1: X_1\to Y_1$ is an extension of f onto G_δ-subsets $X_1\supseteq X$ and $Y_1\supseteq Y$, then f_1 is a fine homotopy equivalence. Indeed, it is easy to see that f_1 is a UV^∞-map, since if U is an open set in Y_1 we get a commutative diagram

$$\begin{array}{ccc} f^{-1}(U\cap Y)=f_1^{-1}(U)\cap X & \hookrightarrow & f_1^{-1}(U)\\ f_1\downarrow & & f_1\downarrow\\ U\cap Y & \hookrightarrow & U\end{array}$$

in which the maps on the top, bottom and on the left are homotopy equivalences. Hence $f_1: f_1^{-1}(U)\to U$ is a homotopy equivalence.

Let $\mathcal{U}\in\operatorname{cov}(Y)$. Find an open cover $\tilde{\mathcal{U}}$ of an open subset Y_2 of Y_1 that contains Y with $\tilde{\mathcal{U}}\mid Y=\mathcal{U}$. If we set $X_2=f_1^{-1}(Y_2)$ and $f_2=f_1\mid X_2: X_2\to Y_2$, then f_2 is a fine homotopy equivalence between two s-manifolds. It follows [13] that f_2 is a near-homeomorphism. Choose a homeomorphism $g: X_2\to Y_2$ $\mathcal{U}$-close to f_2. Since $g(X)$ and Y are $\mathcal{C}$-absorbing sets in Y_2, there is a homeomorphism $h: g(X)\to Y$ $\tilde{\mathcal{U}}$-close to inclusion. Thus $hg: X\to Y$ is a homeomorphism $\operatorname{St}\mathcal{U}$-close to f. □

REMARK 3.4. The same proof shows that, for every $\alpha\in\operatorname{cov}(Y)$, $\mathcal{V}\in\operatorname{cov}(Y)$, and $\tilde{\alpha}=\operatorname{St}(\alpha,\mathcal{V})$, any α-equivalence [13] $f: X\to Y$ from a $\mathcal{C}$-absorbing set X is $\operatorname{St}^2\tilde{\alpha}$-close to a homeomorphism.

4. Resolving incomplete ANR's. Our ultimate goal is to give a topological characterization of $\mathcal{C}$-absorbing sets. We are willing to include the strong $\mathcal{C}$-universality into the hypotheses, but we would like to replace the assumption that the space embeds nicely into an s-manifold by intrinsic statements about the space. Whatever our assumptions are, they have to imply that a fine homotopy equivalence (a *resolving map*) $f: X\to Y$ from a $\mathcal{C}$-absorbing set X to the space Y that satisfies our assumptions is a near-homeomorphism. In this section we construct the resolving map f.

H. Torunczyk [25] showed that if Y is a complete ANR, then there is a fine homotopy equivalence $f: M \to X$ from an s-manifold M. An alternate proof, based on Miller's techniques, is given in [8].

The following lemma states that a resolving map $f: M \to Y$ from an s-manifold to a *complete* ANR Y can be improved over a Z-set.

LEMMA 4.1. *Let $f: M \to Y$ be a fine homotopy equivalence, where M is an s-manifold and Y is a complete* ANR. *Assume that Z is a Z-set in M. Then, for every $\mathcal{U} \in \operatorname{cov}(Y)$ and for every map $\beta: Z \to Y$ that is $\mathcal{U}$-homotopic to $f \mid Z$, there exists a fine homotopy equivalence $\varphi: M \to Y$ such that $\varphi \mid Z = \beta$ and $(\varphi, f) < \operatorname{St}^2 \mathcal{U}$.*

Proof. Let $\{\mathcal{V}_n \in \operatorname{cov}(Y)\}$ be a sequence such that $\operatorname{St} \mathcal{V}_1 < \mathcal{U}$, $\operatorname{St} \mathcal{V}_n < \mathcal{V}_{n-1}$, and $\operatorname{mesh} \mathcal{V}_n < 2^{-n}$. Let $\gamma_n: Z \to M$ be a Z-embedding such that $f\gamma_n$ and β are $\mathcal{V}_{n+1}$-homotopic. Then $f\gamma_n$ and $f\gamma_{n+1}$ are $\operatorname{St}(\mathcal{V}_{n+1}, \mathcal{V}_{n+2})$-homotopic, and hence γ_n and γ_{n+1} are $f^{-1}(\mathcal{V}_n)$-homotopic. Let $h_n: M \to M$ be a homeomorphism $f^{-1}(\mathcal{V}_n)$-close to id_M with $h_n \gamma_n = \gamma_{n+1}$. Similarly, let $h: M \to M$ be a homeomorphism $f^{-1}(\operatorname{St} \mathcal{U})$-close to id_M such that $h \mid Z = \gamma_1$. Define $\varphi: M \to Y$ by

$$\varphi = \lim_{n \to \infty} f h_n h_{n-1} \cdots h_1 h. \qquad \square$$

THEOREM 4.2. *Let Y be an* ANR. *Then there exists an s-manifold M such that for every $\mathcal{C}$-absorbing set $X \subseteq M$ there exists a fine equivalence $\varphi: X \to Y$.*

Proof. Let $\tilde{Y}$ be a complete ANR that contains Y such that $\tilde{Y} - Y$ is locally homotopy negligible in $\tilde{Y}$ [22]. Let $f: M \to \tilde{Y}$ be a fine homotopy equivalence from an s-manifold M. For a $\mathcal{C}$-absorbing set $X \subseteq M$, write $X = X_1 \cup X_2 \cup \cdots$ so that $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ with each X_i a Z-set in X and $X_i \in \mathcal{C}$. By Lemma 4.1 there is a sequence $\{f_n: M \to \tilde{Y}\}$ of fine homotopy equivalences such that $f_n(\operatorname{Cl}_M X_n) \subseteq Y$, $f_n \mid \operatorname{Cl}_M X_{n-1} = f_{n-1} \mid \operatorname{Cl}_M X_{n-1}$, and $(f_n, f_{n-1}) < 2^{-n}$. Consequently, $\tilde{\varphi} = \lim_{n \to \infty} f_n: M \to \tilde{Y}$ is a fine homotopy equivalence with $\tilde{\varphi}(X) \subseteq Y$. Thus $\varphi = \tilde{\varphi} \mid X: X \to Y$ is a fine homotopy equivalence. $\square$

COROLLARY 4.3. *Suppose $f: X \to Y$ is a fine homotopy equivalence from a $\mathcal{C}$-absorbing set X in an s-manifold M to an* ANR *Y, $\mathcal{U} \in \operatorname{cov}(Y)$, and $Z \subseteq X$ is a Z-set. Then for every map $\beta: Z \to Y$ that is $\mathcal{U}$-homotopic to $f \mid Z: Z \to Y$ there is a fine homotopy equivalence $\varphi: X \to Y$ such that $(\varphi, f) < \operatorname{St}^4 \mathcal{U}$ and $\varphi \mid Z = \beta$.*

Proof. Completing Y to an ANR $\tilde{Y}$ so that $\tilde{Y} - Y$ is locally homotopy negligible, extending $\mathcal{U}, f, Z, \beta$; and then trimming back, we can assume that f is the restriction of a fine homotopy equivalence $\tilde{f}: M \to \tilde{Y}$, that $\mathcal{U}$ is the restriction of $\tilde{\mathcal{U}} \in \operatorname{cov}(\tilde{Y})$, that β is the restriction of $\tilde{\beta}: \tilde{Z} \to \tilde{Y}$ (where $\tilde{Z} = \operatorname{Cl}_M Z$), and that $\tilde{\beta}$ is $\operatorname{St} \tilde{\mathcal{U}}$-homotopic to $\tilde{f} \mid \tilde{Z}$. By Lemma 4.1 there is a fine homotopy equivalence $\tilde{\varphi}: M \to \tilde{Y}$ such that $(\tilde{f}, \tilde{\varphi}) < \operatorname{St}^3 \tilde{\mathcal{U}}$ and $\tilde{\varphi} \mid \tilde{Z} = \tilde{\beta}$. Proceed as in the proof of Theorem 4.2 to obtain a fine homotopy equivalence $\tilde{\varphi}: M \to \tilde{Y}$ such that $(\tilde{\varphi}', \tilde{\varphi}) < \tilde{\mathcal{U}}$, $\tilde{\varphi}'(X) \subseteq Y$, and $\tilde{\varphi}' \mid \tilde{Z} = \tilde{\varphi} \mid \tilde{Z}$. Then $\varphi = \tilde{\varphi}' \mid X: X \to Y$ is the desired fine homotopy equivalence.

5. Intrinsic characterization of $\mathcal{C}$-absorbing sets.

THEOREM 5.1. *Let $\mathcal{C}$ be an additive topological class hereditary with respect to closed subsets, and let Ω be a $\mathcal{C}$-absorbing set in an s-manifold M. If X is a strongly $\mathcal{C}$-universal* ANR *that can be written as $X=\bigcup_{i=1}^{\infty} X_i$ where each X_i is a strong Z-set in X and $X_i \in \mathcal{C}$, then each fine homotopy equivalence $f:\Omega \to X$ is a near-homeomorphism.*

The proof of 5.1 is based on the following special case:

LEMMA 5.2 (The Strong Z-set Shrinking Theorem). *Suppose that X is an* ANR *and $X=\Omega \cup Z$, where $Z \in \mathcal{C}$, Z is a strong Z-set in X, $\Omega \cap Z=\varnothing$, and Ω is a $\mathcal{C}$-absorbing set in an s-manifold M. Then the inclusion $i:\Omega \to X$ is a near-homeomorphism.*

Proof. Let $\tilde{X} \supset X$ be a complete ANR such that $\tilde{X}-X$ is locally homotopy negligible in $\tilde{X}$. By Lemma 1.11 we can assume that $\tilde{X}=\tilde{\Omega}\cup\tilde{Z}$, where $\tilde{Z}=\mathrm{Cl}_{\tilde{X}} Z$, $\tilde{Z}$ is a strong Z-set in $\tilde{X}$, and (by trimming back and Lavrentiev's theorem) $\tilde{\Omega}$ is an s-manifold. By [4] the inclusion $\tilde{i}:\tilde{\Omega}\to\tilde{X}$ is a near-homeomorphism, and $\tilde{X}$ is an s-manifold. Since both Ω and X are $\mathcal{C}$-absorbing sets in $\tilde{X}$, the conclusion follows from Theorem 3.3.

Proof of Theorem 5.1. Choose a sequence $\{\mathcal{U}_i \in \mathrm{cov}(X)\}$ such that $\mathrm{St}\,\mathcal{U}_i < \mathcal{U}_{i-1}$, and build a sequence $\{f_i:\Omega\to X\}$ of fine homotopy equivalences such that:

(1) $(f_i, f_{i-1})<\mathcal{U}_i$,

(2) $f_i=f_{i-1}$ on $\bigcup_{j=1}^{i-1}\Omega_j \cup f_{i-1}^{-1}(\bigcup_{j=1}^{i-1} X_j)$,

(3) f_i is a homeomorphism over $\bigcup_{j=1}^{i}(f_i(\Omega_j)\cup X_j)$ and f_i is a closed map over this set,

(4) $d(f_i(\omega), f_{i-1}(\omega))\le 2^{-i}\min\{1, d(f_{i-1}(\omega), \bigcup_{j=1}^{i-1}(f_{i-1}(\Omega_j)\cap X_j))\}$.

We let $\Omega_0=\varnothing$, $X_0=\varnothing$, and $f_0=f$ ($\Omega=\bigcup_{i=1}^{\infty}\Omega_i$ is the representation of Ω as in the definition of a $\mathcal{C}$-absorbing set). Assume that f_{i-1} satisfying (1)–(4) has been constructed. We set $Z=\bigcup_{j=1}^{i-1}(X_j\cup f_{i-1}(\Omega_j))$. Observe that $f_{i-1}(\Omega-f_{i-1}^{-1}(Z))\subseteq X-Z$. By Propositions 2.1 and 2.3, the space $X-Z$ is $\mathcal{C}_\sigma$-universal. Thus there is a Z-embedding $v:\Omega_i-f_{i-1}^{-1}(Z)\to X-Z$ such that

$$d(v(\omega), f_{i-1}(\omega))<2^{-i+1}\min\{1, d(f_{i-1}(\omega), Z)\}$$

and v is $\mathcal{U}_{i+5}$-close to $f_{i-1}\,|\,\Omega_i-f_{i-1}^{-1}(Z)$. Let $g:\Omega-f_{i-1}^{-1}(Z)\to X-Z$ be a fine homotopy equivalence such that g is $\mathcal{U}_{i+1}$-close to $f_{i-1}\,|\,\Omega-f_{i-1}^{-1}(Z)$,

$$g\,|\,\Omega_i-f_{i-1}^{-1}(Z)=v, \quad \text{and} \quad d(g(\omega), f_{i-1}(\omega))<2^{-(i+1)}\min(1, d(f_{i-1}(\omega), Z))$$

(see Corollary 4.3). Let $C=g(\Omega_i-f_{i-1}^{-1}(Z))\cap(X_i-Z)$. Consider the adjunction space $(\Omega-f_{i-1}^{-1}(Z))\cup_g C$ together with the corresponding maps $p:\Omega-f_{i-1}^{-1}(Z)\to(\Omega-f_{i-1}^{-1}(Z))\cup_g C$ and $q:(\Omega-f^{-1}(Z))\cup_g C\to X-Z$ such that $g=qp$. Observe that

$$(\Omega-f_{i-1}^{-1}(Z))\cup_g C=((\Omega-f_{i-1}^{-1}(Z))-g^{-1}(C))\cup C,$$

where $(\Omega-f_{i-1}^{-1}(Z))-g^{-1}(C)$, being an open subset of Ω, is a $\mathcal{C}$-absorbing (and hence a $\mathcal{C}_\sigma$-absorbing) set in *some* s-manifold, and C is a strong Z-set with $C\in\mathcal{C}_\sigma$ (see Lemma 1.10.) By Lemma 5.2 the adjunction space $(\Omega-f_{i-1}^{-1}(Z))\cup_g C$ is $\mathcal{C}_\sigma$-

absorbing in some s-manifold. Thus p, being a fine homotopy equivalence, is a near-homeomorphism (by Theorem 3.3).

Let $h:\Omega - f_{i-1}^{-1}(Z) \to (\Omega - f_{i-1}^{-1}(Z)) \cup_g C$ be a homeomorphism so close to p that qh is $\mathcal{U}_{i+1}$-close to g and $d(qh(\omega), g(\omega)) < 2^{-(i+1)} \min\{1, d(g(\omega), Z)\}$. Also, by Theorem 3.2, we can assume that $h \mid \Omega_i - f_{i-1}^{-1}(Z) = p \mid \Omega_i - f_{i-1}^{-1}(Z)$. Define $f_i: \Omega \to X$ by setting $f_i = f_{i-1}$ on $f_{i-1}^{-1}(Z)$ and $f_i = qh$ on $\Omega - f_{i-1}^{-1}(Z)$.

The reader can check that $f': \Omega \to X$ defined as $f' = \lim_{n\to\infty} f_n$ is a homeomorphism St $\mathcal{U}_1$-close to f.

A direct consequence of Theorems 5.1 and 4.2 is the following characterization theorem.

THEOREM 5.3. *Assume that for an additive topological class* $\mathcal{C}$ *hereditary with respect to closed subsets there exists a* $\mathcal{C}$*-absorbing set* Ω *in* s. *Then* $X \in \mathrm{AR}$ *is homeomorphic to* Ω if and only if $X \in \mathcal{C}_\sigma$, *X is strongly* $\mathcal{C}$*-universal, and* $X = \bigcup_{i=1}^{\infty} X_i$, *where each* X_i *is a strong Z-set in X.*

COROLLARY 5.4. *If* $\Omega \subseteq s$ *is a* $\mathcal{C}$*-absorbing set for an additive topological class* $\mathcal{C}$ *containing* $[0,1]$ *and hereditary with respect to closed subsets, with the property that* $C_1, C_2 \in \mathcal{C}$ *imply* $C_1 \times C_2 \in \mathcal{C}$, *then a necessary and sufficient condition that* $\Omega \times X$ *be homeomorphic to* Ω *is that X be a retract of* Ω.

Proof. Necessity being obvious, note that every Z-set in $\Omega \times X$ is a strong Z-set, since $\Omega \times X$ embeds into $s \times \tilde{X} \cong s$ for a complete $\tilde{X} \in \mathrm{AR}$ that contains X with $\tilde{X} - X$ locally homotopy negligible in $\tilde{X}$, and hence $\Omega \times X$ embeds into s so that the complement is locally homotopy negligible. Thus Proposition 2.6 implies that $\Omega \times X$ is strongly $\mathcal{C}$-universal, and the assumption about $\mathcal{C}$ guarantees that $\Omega \times X \in \mathcal{C}_\sigma$. If $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where each Ω_i is a (strong) Z-set in Ω, then $\Omega \times X = \bigcup_{i=1}^{\infty} (\Omega_i \times X)$, where each $\Omega_i \times X$ is a (strong) Z-set in $\Omega \times X$. Hence by Theorem 5.3 (or 3.1), $\Omega \times X \cong \Omega$. □

COROLLARY 5.5. (i) *The topological type of* $W(X, *)$ *does not depend on the choice of the basepoint* $* \in X$, *for* $X \in \mathrm{AR}$.

(ii) *For* $X, Y \in \mathrm{AR}$ *we have* $W(X, *) \cong W(Y, *)$ *if and only if X embeds as a closed subset into* $W(Y, *)$ *and Y embeds as a closed subset into* $W(X, *)$.

Proof. Let $\mathcal{C}_X = \{$spaces homeomorphic to a closed subset of $W(X, *)\}$. It is clear that $\mathcal{C}_X$ is hereditary with respect to closed subsets. To show that it is additive, fix a space $C = A \cup B$, where $A, B \in \mathcal{C}_X$ are closed in C. Let $e_A: A \to W(X, *)$, $e_B: B \to W(X, *)$ be closed embeddings. Using the fact that

$$W(X, *) \cong W(W(X, *), *)$$

we can assume that e_A, e_B are Z-embeddings. Note that $W(X, *) - e_A(A - B) \in \mathrm{AR}$ by [22], since $e_A(A-B)$ is locally homotopy negligible in $W(X, *)$. Let $g: B \to W(X, *) - e_A(A-B)$ be a map extending $e_A \mid A \cap B: A \cap B \to W(X, *) - e_A(A-B)$. Similarly, let $f: A \to W(X, *) - e_B(B-A)$ be a map extending $e_B \mid A \cap B: A \cap B \to W(X, *) - e_B(B-A)$. Finally, define a closed embedding $h: C \to W(X, *)^2 \cong W(X, *)$ by

$$h(x)=\begin{cases}(e_A(x),e_B(x)), & x\in A\cap B,\\ (e_A(x),f(x)), & x\in A-B,\\ (g(x),e_B(x)), & x\in B-A.\end{cases}$$

To finish the proof of (ii), note that by Proposition 2.5 and Theorem 5.3 $W(X, *)$ is characterized among spaces $W(T, *)$, $T\in$ AR, as being strongly $\mathcal{C}_X$-universal, and that under assumptions of (ii) $\mathcal{C}_X=\mathcal{C}_Y=\{$spaces homeomorphic to a closed subset of $W(Y, *)\}$, since $W(Y, *)$ embeds as a closed subset of $W(X, *)$ and vice versa. Indeed, the homogeneity of $W(X, *)$ (Theorem 3.2) says that we can assume that the given closed embedding $Y\to W(X, *)$ preserves basepoints, and hence $W(Y, *)$ embeds as a closed subset of $W(W(X, *), *)\cong W(X, *)$. Finally, (ii) implies (i). □

COROLLARY 5.6 (The Triangulation Theorem). *Suppose that a topological class $\mathcal{C}$ is additive, hereditary with respect to closed subsets, and has the property that if $C\in\mathcal{C}$ and if $n\geq 0$, then $[-1,1]^n\times C\in\mathcal{C}$. Also assume that $\Omega\subseteq s$ is a $\mathcal{C}$-absorbing set. Then*

(i) *any s-manifold M contains a $\mathcal{C}$-absorbing set, and*

(ii) *a space X is an Ω-manifold (i.e., admits an open cover by sets homeomorphic to open subsets of Ω) if and only if there is a locally finite countable simplicial complex K such that $X\cong|K|\times\Omega$.*

Proof. (i) By the triangulation theorem for s-manifolds, $M\cong|K|\times s$ for some locally finite countable simplicial complex K. Then $|K|\times\Omega\subseteq|K|\times s\cong M$ is a $\mathcal{C}$-absorbing set in M.

(ii) Suppose that X is an Ω-manifold, and let K be a locally finite countable simplicial complex such that $|K|$ and X have the same homotopy type. By Proposition 2.7, X is strongly $\mathcal{C}$-universal. By Theorem 4.2 there is a fine homotopy equivalence $f\colon Y\to X$ from a $\mathcal{C}$-absorbing set Y in an s-manifold M. Since M and $|K|$ have the same homotopy type, it follows that $M\cong|K|\times s$, and hence $|K|\times\Omega$ (being a $\mathcal{C}$-absorbing set in $|K|\times s$) is homeomorphic to Y. Finally, f is a near-homeomorphism by Theorem 5.1, and therefore $X\cong Y\cong|K|\times\Omega$. □

COROLLARY 5.7 (The Open Embedding Theorem). *Let $\mathcal{C}$ and Ω be as in Corollary 5.6. Then any Ω-manifold X embeds as an open subset of Ω.*

Proof. By Corollary 5.6, X is a $\mathcal{C}$-absorbing set in an s-manifold M. By the Open Embedding Theorem for s-manifolds [17], M embeds as an open subset of s. Then both X and $M\cap\Omega$ are $\mathcal{C}$-absorbing sets in M, and hence $X\cong M\cap\Omega$, the latter being open in Ω. □

6. Absorbing sets for classes of absolute Borel sets. In this section we derive from Theorem 5.3 characterization theorems for certain incomplete spaces.

It is well known that $\sigma=\{(t_i)\in s\colon t_i=0$ for almost all $i\}$ is strongly $\mathcal{C}_{\mathrm{fdc}}$-universal for the class $\mathcal{C}_{\mathrm{fdc}}$ of all finite-dimensional compacta. This fact is also a consequence of Proposition 2.5, since obviously $\sigma=W((-1,1),0)$. Thus we obtain a characterization theorem for σ, due to the second author.

COROLLARY 6.1 (see [20]). *A space* $X \in \mathrm{AR}$ ($X \in \mathrm{ANR}$) *is homeomorphic to* σ (*to a* σ*-manifold*) *if and only if:*

(i) *X is a countable union of finite-dimensional compacta,*

(ii) *X is strongly* $\mathfrak{C}_{\mathrm{fdc}}$*-universal, and*

(iii) $X = \bigcup_{i=1}^{\infty} X_i$, *where each* X_i *is a strong Z-set in X.*

Condition (iii) *can be replaced by the equivalent condition* (see Corollary 1.8 and Lemma 1.9.)

(iii′) *X satisfies the Strong Discrete Approximation Property.*

Similarly, $\Sigma = W(Q, *)$ is strongly $\mathfrak{C}_c$-universal for the class $\mathfrak{C}_c$ of all compacta.

COROLLARY 6.2 ([20]). *A space* $X \in \mathrm{AR}$ ($X \in \mathrm{ANR}$) *is homeomorphic to* Σ (*to a* Σ*-manifold*) *if and only if*

(i) *X is a countable union of compacta,*

(ii) *X is strongly* $\mathfrak{C}_c$*-universal, and*

(iii) $X = \bigcup_{i=1}^{\infty} X_i$, *where each* X_i *is a strong Z-set in X.*

Again, (iii) *can be replaced by*

(iii′) *X satisfies the Strong Discrete Approximation Property.*

Note that both $W(s, *)$ and $\Sigma \times s$ are $\mathfrak{M}_1$-absorbing sets for the class $\mathfrak{M}_1$ of all topologically complete spaces (by Propositions 2.5 and 2.6). So we have $W(s, *) \cong \Sigma \times S$, and the following.

COROLLARY 6.3. *A space* $X \in \mathrm{AR}$ ($X \in \mathrm{ANR}$) *is homeomorphic to* $\Sigma \times s$ ($\Sigma \times s$*-manifold*) *if and only if*

(i) $X = \bigcup_{i=1}^{\infty} X_i$, *where each* X_i *is a strong Z-set in X and each* X_i *is completely metrizable, and*

(ii) *X is strongly* $\mathfrak{M}_1$*-universal.*

The classes $\mathfrak{C}_c$ and $\mathfrak{M}_1$ are only the beginning of the hierarchy of *Borel classes.* Recall [6] that for each space X and for each countable ordinal α we can define *the additive Borelian class* α, $\mathfrak{a}_\alpha(X)$, and the *multiplicative Borelian class* α, $\mathfrak{M}_\alpha(X)$, of subsets of X as follows: $\mathfrak{a}_0(X)$ is the collection of all open subsets of X, and $\mathfrak{M}_0(X)$ is the collection of all closed subsets of X. Suppose that for $\zeta < \alpha$ the collections $\mathfrak{a}_\zeta(X)$ and $\mathfrak{M}_\zeta(X)$ have been defined. Then $\mathfrak{a}_\alpha(X)$ is the collection of all subsets of X that can be represented as $X_1 \cup X_2 \cup X_3 \cup \cdots$ with each $X_i \in \bigcup_{\zeta<\alpha} \mathfrak{M}_\zeta(X)$, and $\mathfrak{M}_\alpha(X)$ is the collection of all subsets of X that can be represented as $X_1 \cap X_2 \cap X_3 \cap \cdots$ with each $X_i \in \bigcup_{\zeta<\alpha} \mathfrak{a}_\zeta(X)$.

For a countable ordinal α we define *the absolute Borelian classes* $\mathfrak{a}_\alpha$ and $\mathfrak{M}_\alpha$. A space X belongs to $\mathfrak{a}_\alpha$ ($\mathfrak{M}_\alpha$) if and only if for any embedding $e: X \to Y$ we have $e(X) \in \mathfrak{a}_\alpha(Y)$ ($e(X) \in \mathfrak{M}_\alpha(Y)$). By a result of Lavrentiev [19], $X \in \mathfrak{a}_\alpha$ ($\alpha \geq 2$) if and only if $X \in \mathfrak{a}_\alpha(E)$ for some complete space E, and $X \in \mathfrak{M}_\alpha$ ($\alpha \geq 1$) if and only if $X \in \mathfrak{M}_\alpha(E)$ for some complete space E.

Note that $\mathfrak{a}_0 = \varnothing$; $\mathfrak{M}_0$ consists of all compacta, $\mathfrak{a}_1$ of all σ-compacta; $\mathfrak{M}_1$ is the collection of all completely metrizable spaces, and so forth. We construct $\mathfrak{M}_\alpha$-absorbing ($\mathfrak{a}_\alpha$-absorbing) set Ω_α (resp., Λ_α) in s. As we observed above, $\Omega_1 =$

$W(s, *) \cong \Sigma \times s$ and $\Lambda_1 = \Sigma = W(Q, *)$ will do. Suppose that the sets $\Omega_\zeta \subseteq s$ and $\Lambda_\zeta \subseteq s$ have been defined for all countable ordinals $\zeta < \alpha$. If $\alpha = \beta + 1$, let $\Omega_\alpha = \Lambda_\beta^\infty$; if α is a limit ordinal, define $\Omega_\alpha = \prod_{\zeta<\alpha} \Lambda_\zeta^\infty$. Since $s^\infty \cong s$, we can regard Ω_α as a subset of s, and set $\Lambda_\alpha = W(s - \Omega_\alpha, *)$. An easy argument establishes that $\Omega_\alpha \in \mathfrak{M}_\alpha$ for $\alpha \geq 2$ and that $\Lambda_\alpha \in \mathfrak{a}_\alpha$ for $\alpha \geq 1$.

The first step in proving the strong universality of these spaces consists of showing that there is at least *one* closed embedding of each space in the corresponding class. The proof is based on an idea of R. Sikorski (cf. [21], [12]).

LEMMA 6.3. *Suppose $X \in \mathfrak{M}_\alpha$ ($X \in \mathfrak{a}_\alpha$) for $\alpha \geq 2$ is embedded into the Hilbert cube Q. Then there is an embedding $\varphi_\alpha: Q \to s$ such that $\varphi_\alpha^{-1}(\Omega_\alpha) = X$ ($\varphi_\alpha^{-1}(\Lambda_\alpha) = X$).*

Proof. First consider the case $\alpha = 2$, and $X \in \mathfrak{M}_2$. Let $v: Q \to s$ be an embedding such that $v(Q) \cap \Lambda_1 = \varnothing$ (it exists since $\Lambda_1 = \Sigma$ is a σ-Z-set in s). We have $X = \bigcap_{i=1}^\infty A_i$, where each $A_i \subseteq Q$ is σ-compact. There are homeomorphisms $h_i: s \to s$ such that $h_i^{-1}(\Lambda_1) \cap v(Q) = v(A_i)$. Define an embedding $\varphi_2: Q \to s^\infty \cong s$ by $\varphi_2(q) = \{h_i(v(q))\}$; then $\varphi_2^{-1}(\Omega_2) = X$.

If $X \in \mathfrak{a}_2$, then $Q - X \in \mathfrak{M}_2$; hence the above applied to $Q - X$ implies that there is a map $\psi: Q \to s$ such that $\psi^{-1}(\Omega_2) = Q - X$, that is, $\psi^{-1}(s - \Omega_2) = X$. If we define $\varphi_2: Q \to s^\infty$ by $\varphi_2(q) = (\psi(q), *, *, \ldots)$, then $\varphi_2^{-1}(\Lambda_2) = X$.

Now assume that $X \in \mathfrak{M}_\alpha$, $\alpha > 2$. For simplicity assume that $\alpha = \beta + 1$ (the case when α is a limit ordinal is analogous). Then X can be written as $X = \bigcap_{i=1}^\infty A_i$ with each $A_i \in \mathfrak{a}_\beta$. By induction, there is an embedding $\psi_i: Q \to s$ such that $\psi_i^{-1}(\Lambda_\beta) = A_i$. Define $\varphi_\alpha: Q \to s^\infty \cong s$ by $\varphi_\alpha(q) = (\psi_1(q), \psi_2(q), \ldots)$. Then

$$\varphi_\alpha^{-1}(\Omega_\alpha) = \bigcap_{i=1}^\infty A_i = X.$$

The case $X \in \mathfrak{a}_\alpha$ ($\alpha > 2$) is completely analogous to the case $\alpha = 2$. □

PROPOSITION 6.4. *For a countable ordinal $\alpha \geq 1$, the space Ω_α is $\mathfrak{M}_\alpha$-absorbing, and Λ_α is $\mathfrak{a}_\alpha$-absorbing.*

Proof. As shown above, the statement is true for $\alpha = 1$. It is clear that each Ω_α (and Λ_α) is a countable union of Z-sets, and that both Ω_α and Λ_α are embedded into s with locally homotopy negligible complements. Lemma 6.3 coupled with Proposition 2.5 implies that Ω_α (Λ_α) is strongly $\mathfrak{M}_\alpha$-universal (strongly $\mathfrak{a}_\alpha$-universal). □

THEOREM 6.5. *A space $X \in \mathrm{AR}$ is homeomorphic to Ω_α (or Λ_α) for $\alpha \geq 1$ if and only if*

(i) *$X = \bigcup_{i=1}^\infty X_i$, where each $X_i \in \mathfrak{M}_\alpha$ ($X_i \in \mathfrak{a}_\alpha$) and X_i is a strong Z-set in X, and*

(ii) *X is strongly $\mathfrak{M}_\alpha$-universal (strongly $\mathfrak{a}_\alpha$-universal).*

Moreover, if X is a retract of Ω_α (or Λ_α) then $X \times \Omega_\alpha \cong \Omega_\alpha$ ($X \times \Lambda_\alpha \cong \Lambda_\alpha$). Also, the Triangulation Theorem 5.6 and the Open Embedding Theorem 5.7 hold for manifolds modeled on Ω_α (or Λ_α), $\alpha \geq 1$.

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