THE EINSTEIN-KÄHLER METRIC ON $\{|\mathbf{z}|^2+|w|^{2p}<1\}$

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S. Y. Cheng and S. T. Yau showed in [2] that any C^2 bounded pseudoconvex domain in $\mathbf{C}^n$ has a complete Einstein-Kähler metric with negative Ricci curvature; their solution satisfied the Monge-Ampère equation $\operatorname{Det}[\partial^2 g/\partial z_i \partial z_{\bar{j}}]=e^{(n+1)g}$, $g=\infty$ on the boundary, where the metric is given by $(\partial^2 g/\partial z_i \partial z_{\bar{j}}) dz^i \otimes d\bar{z}^j$. N. Mok and S. T. Yau [4] have extended this result to arbitrary bounded pseudoconvex domains in $\mathbf{C}^n$. Explicit solutions, however, are only known in the very simplest cases. The purpose of this paper is to describe the Einstein-Kähler metric for the domain $\Omega_p=\{|\mathbf{z}|^2+|w|^{2p}<1\}$, $p>0$. These domains exhibit a wide range of boundary behavior. For $p>1$, the special boundary points $|\mathbf{z}|=1$ are C^2 weakly pseudoconvex, and the domains interpolate between B^n and $B^{n-1}\times B$. For $\frac{1}{2}<p<1$, the domains are C^1 strictly convex. For $p<\frac{1}{2}$, the boundary intersects certain real planes in cusps.

The main technique is to use the $(2n-1)$-dimensional noncompact automorphism group of Ω and the biholomorphic invariance of the Einstein-Kähler metric to reduce the Monge-Ampère equation for the metric to an ordinary differential equation in the auxiliary function $X=|w|^2/(1-|\mathbf{z}|^2)^{1/p}$. This differential equation can be easily solved to give an implicit function in X; however, all information of interest is obtained by indirect methods.

The function X contains geometric information about the domain. The leaves $X=$ constant define a real foliation of the domain, the leaves of which converge at the special boundary points $|\mathbf{z}|=1$, $w=0$. The automorphism group of the domain preserves this foliation, and acts transitively within each leaf. Thus, any biholomorphically invariant quantity can be reduced to a function of X, and it assumes its full range of values arbitrarily near the special boundary points $|\mathbf{z}|=1$; in particular, any nonconstant biholomorphically invariant quantity exhibits no limiting behavior near these boundary points.

The results of these calculations have some interesting consequences. When $p>1$, the special boundary points are C^2 weakly pseudoconvex, and the Riemannian sectional curvature for the domain is bounded between negative constants. In particular, a local Schwarz lemma can be used to obtain bounds on the metric for any domain locally approximating Ω on the inside (see Theorems 4 & 5). On the other hand, there are C^1 strictly convex domains for which the Einstein-Kähler metric has strictly positive holomorphic sectional curvature in certain directions at some points (see Theorem 4). In all cases where $p>0$, volume estimates on the Einstein-Kähler metric for locally approximating domains can be obtained (Theorem 5).

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Throughout the paper, the coordinates $(\mathbf{z}, w) = (z_1, \ldots, z_{n-1}, w)$ will be used on $\mathbf{C}^n$.

1. The metric.

THEOREM 1. *Consider the bounded pseudoconvex domain*

$$\Omega = \{|\mathbf{z}|^2 + |w|^{2p} < 1\},$$

where $p > 0$. *Define the auxiliary function* $X = |w|^2/(1-|\mathbf{z}|^2)^{1/p}$. *Then there exists some function of* X, *say* $Y(X)$, *such that the Einstein–Kähler metric for* Ω *is given by*

$$g_{i\bar{j}} = Y(\log X)_{i\bar{j}} + (Y'/X)X_i X_{\bar{j}}$$

and the volume for the metric is

$$|g_{i\bar{j}}| = (Y/p)^{n-1} Y'(1-|\mathbf{z}|^2)^{-n-1/p}.$$

Further, Y *satisfies the ordinary differential equation in* X:

$$XY^{n-1}Y' = Y^{n+1} - pY^n + \alpha,$$

where

$$\alpha = \frac{(p-1)(np+1)^n}{(n+1)^{n+1}}, \qquad Y(0) = \frac{np+1}{n+1},$$

and $Y(1-X)$, $Y'(1-X)^2$ *are uniformly bounded from above and below by positive constants.*

The function X has been introduced primarily for notational convenience and computational simplicity; however, the importance of this auxilliary function should not be underestimated. For instance, $0 \le X < 1$ on Ω, and $X-1$ is a local defining function for the smooth strongly pseudoconvex boundary points of Ω. Thus, $Y(1-X)$ and $Y'(1-X)^2$ being uniformly bounded from above and below by positive constants really reflects how the Einstein–Kähler metric behaves asymptotically at the smooth strongly pseudoconvex boundary points. Near the special boundary points $|\mathbf{z}| = 1$, X can assume any value less than 1 in any arbitrarily small region, and for $c \in (0,1)$, the regions $X < c$ will be used as approach regions to the special boundary points in describing the asymptotic behavior of the metric. Finally, it should again be noticed that the function X is preserved under the automorphism group of Ω, and that the automorphism group is transitive on any leaf $X = \text{constant}$.

The remainder of this section will provide the calculations necessary to establish Theorem 1. The basic idea is to use the automorphism group of Ω and the transformation properties of the Einstein–Kähler metric to reduce the differential equation for the metric to an ordinary differential equation.

A. THE TRANSFORMATION FORMULA. During the course of the calculations, it will prove convenient to introduce the function $-\phi(\mathbf{z}, w) = e^{-g(\mathbf{z}, w)}$. Then

$$g_{i\bar{j}}(\mathbf{z}, w) = \frac{\partial^2}{\partial z_i \partial z_{\bar{j}}}(-\log(-\phi(\mathbf{z}, w))),$$

and $(-\phi(\mathbf{z}, w))^{-(n+1)}$ is the volume for the Einstein–Kähler metric.

For any automorphism F of the domain, the invariance properties of the metric imply

$$\begin{aligned}(-\phi(\mathbf{z}, w))^{-(n+1)} &= \operatorname{Det}[g_{i\bar{j}}(\mathbf{z}, w)]\\ &= |dF(\mathbf{z}, w)|^2 \operatorname{Det}[g_{i\bar{j}}(F(\mathbf{z}, w))]\\ &= |dF(\mathbf{z}, w)|^2(-\phi(F(\mathbf{z}, w)))^{-(n+1)}\end{aligned}$$

$$\phi(\mathbf{z}, w) = |dF(\mathbf{z}, w)|^{-2/(n+1)}\phi(F(\mathbf{z}, w)), \tag{1.1}$$

where $|dF(\mathbf{z}, w)|$ denotes the absolute value of the determinant of the differential of the mapping F. Formula (1.1) will be referred to as the transformation property of ϕ.

B. AUTOMORPHISMS OF $\Omega = \{|\mathbf{z}|^2 + |w|^{2p} < 1\}$. The domain Ω is obviously invariant under unitary transformations in the $\mathbf{z}$ coordinates and the w rotations $(\mathbf{z}, w) \to (\mathbf{z}, we^{i\theta})$ for arbitrary $\theta \in \mathbf{R}$. The transformation property then implies that $\phi(\mathbf{z}, w) = \phi(|\mathbf{z}|, 0, \ldots, 0, |w|)$, where $|\mathbf{z}| = (\sum_{i=1}^{n-1} |z_i|^2)^{1/2}$ is the usual length of the vector $\mathbf{z}$. The determination of ϕ now depends upon the two parameters $|\mathbf{z}|, |w|$. Consider the additional automorphisms given by

$$F(\mathbf{z}, w) = \left[\frac{z_1 - \eta}{1 - \eta z_1}, \frac{(1-\eta^2)^{1/2}}{1-\eta z_1} z_2, \ldots, \frac{(1-\eta^2)^{1/2}}{1-\eta z_1} z_{n-1}, \frac{(1-\eta^2)^{1/2p}}{(1-\eta z_1)^{1/p}} w\right].$$

Then

$$|dF(\mathbf{z}, w)|^2 = \left|\frac{(1-\eta^2)^{(np+1)/2p}}{(1-\eta z_1)^{(np+1)/p}}\right|^2.$$

Applying the transformation property for ϕ,

$$\phi(\eta, 0, \ldots, 0, w) = \phi\left(0, \ldots, 0, \frac{(1-\eta^2)^{1/2p}}{(1-\eta^2)^{1/p}} w\right)(1-\eta^2)^{(np+1)/p(n+1)}$$

so

$$\phi(\mathbf{z}, w) = \phi(0, \ldots, 0, (1-|\mathbf{z}|^2)^{-1/2p}|w|)(1-|\mathbf{z}|^2)^{(np+1)/p(n+1)}.$$

Define $X(\mathbf{z}, w) = |w|^2/(1-|\mathbf{z}|^2)^{1/p}$. Then $X < 1$ on Ω and for some function $h(X) > 0$,

$$-\phi(\mathbf{z}, w) = (1-|\mathbf{z}|^2)^{(np+1)/p(n+1)} h(X). \tag{1.2}$$

C. GENERAL FORM OF THE METRIC.

$$\begin{aligned} g_{i\bar{j}}(\mathbf{z}, w) &= (-\log(-\phi(\mathbf{z}, w)))_{i\bar{j}}\\ &= \frac{np+1}{n+1}(\log X)_{i\bar{j}} - \left(\frac{h'}{h}\right)' X_i X_{\bar{j}} - \frac{h'}{h} X_{i\bar{j}}\\ &= \left(\frac{np+1}{n+1} - X\frac{h'}{h}\right)(\log X)_{i\bar{j}} - \left[\left(\frac{h'}{h}\right)' + \frac{1}{X}\frac{h'}{h}\right] X_i X_{\bar{j}}\end{aligned}$$

$$g_{i\bar{j}}(\mathbf{z}, w) = Y(\log X)_{i\bar{j}} + \frac{Y'}{X} X_i X_{\bar{j}}, \tag{1.3}$$

where

$$Y=\left(\frac{np+1}{n+1}-X\frac{h'}{h}\right).$$

Using the fact that $X_i X_{\bar{j}}$ is a matrix of rank 1, the volume of the metric is

$$\begin{aligned}\mathrm{Det}[g_{i\bar{j}}(\mathbf{z},w)]&=\mathrm{Det}[g_{i\bar{j}}(|\mathbf{z}|,0,\dots,0,w)]\\&=(Y/p)^{n-1}(1-|\mathbf{z}|^2)^{-n}XY'|w|^{-2}\end{aligned}$$

$$\mathrm{Det}[g_{i\bar{j}}(\mathbf{z},w)]=(Y/p)^{n-1}Y'(1-|\mathbf{z}|^2)^{-n-1/p}. \tag{1.4}$$

D. THE EINSTEIN CONDITION. Recall that for the Kähler metric $g_{i\bar{j}}$, the components of the Ricci tensor are given by

$$R_{i\bar{j}}=\frac{-\partial^2}{\partial z_i\partial z_{\bar{j}}}(\log\mathrm{Det}[g_{i\bar{j}}]).$$

The condition that the Ricci curvature is equal to the negative constant $-(n+1)$ then becomes

$$\frac{\partial^2}{\partial z_i\partial z_{\bar{j}}}(\log\mathrm{Det}[g_{i\bar{j}}(\mathbf{z},w)])=(n+1)g_{i\bar{j}}(\mathbf{z},w).$$

Using (1.4) to compute the left-hand side, this equation becomes

$$\begin{aligned}&\frac{\partial^2}{\partial z_i\partial z_{\bar{j}}}(\log(Y^{n-1}Y'(1-|\mathbf{z}|^2)^{-(n+1/p)}))\\&=(np+1)(\log X)_{i\bar{j}}+\frac{(Y^{n-1}Y')'}{Y^{n-1}Y'}X_{i\bar{j}}+\left[\frac{(Y^{n-1}Y')'}{Y^{n-1}Y'}\right]'X_iX_{\bar{j}}\\&=\left((np+1)+X\frac{(Y^{n-1}Y')'}{Y^{n-1}Y'}\right)(\log X)_{i\bar{j}}+\left[\frac{X(Y^{n-1}Y')'}{Y^{n-1}Y'}\right]'\frac{X_iX_{\bar{j}}}{X}\\&=(n+1)Y(\log X)_{i\bar{j}}+(n+1)Y'\frac{X_iX_{\bar{j}}}{X}.\end{aligned}$$

Setting the coefficients in the last two lines equal (note that the coefficients of $X_iX_{\bar{j}}/X$ are just the derivatives of the coefficients of $(\log X)_{i\bar{j}}$):

$$(np+1)+X\frac{(Y^{n-1}Y')'}{Y^{n-1}Y'}=(n+1)Y.$$

Solving:

$$\begin{aligned}(np+1)Y^{n-1}Y'+[X(Y^{n-1}Y')]'-Y^{n-1}Y'&=(n+1)Y^nY'\\ [X(Y^{n-1}Y')]'&=(Y^{n+1})'-p(Y^n)'\end{aligned}$$

$$XY^{n-1}Y'=Y^{n+1}-pY^n+\alpha \tag{1.5}$$

for some constant α.

E. ESTIMATES FOR $h(X)$, $Y(X)$. It is important to know how the metric behaves at the boundary of Ω and hence how h, Y behave at $X\to 1$, as well as how h, Y behave at $X=0$.

For the boundary behavior, consider the complex line given by $\mathbf{z}=0$. This intersects $\partial\Omega$ at strongly pseudoconvex boundary points, and locally the function $X-1$ acts as a defining function. Using (1.3) and the asymptotic behavior of the Einstein-Kähler metric at the smooth, strongly pseudoconvex boundary points [1], it is immediate that $Y(1-X)$, $Y'(1-X)^2$ are uniformly bounded from above and below by positive constants in a neighborhood of the boundary.

For the behavior at $X=0$, consider the function $h\circ X(\mathbf{z},w)$. Since the metric and its volume $[-\phi(\mathbf{z},w)]^{-(n+1)}$ are smooth on Ω, and $(1-|\mathbf{z}|^2)^{(np+1)/p(n+1)}$ is smooth on Ω, it follows from (1.2) that $h\circ X$ is a smooth strictly positive function on Ω. The symmetry of $h\circ X$ implies that $\nabla h\circ X(0,\dots,0)=0$, or

$$\frac{\partial}{\partial w}h\circ X(0,\dots,0)=h'(0)\frac{\bar{w}}{(1-|\mathbf{z}|^2)^{1/p}}\bigg|_{(0,\dots,0)}=0,$$

whence $Xh'(X)\,|_{X=0}=0$. Since $Y=(np+1)/(n+1)-X(h'/h)$ by definition, $Y(0)=(np+1)/(n+1)$. Using (1.4) and (1.5), it is clear that

$$XY^{n-1}Y'\,|_{X=0}=0 \quad\text{and}\quad Y^{n+1}-pY^n+\alpha\,|_{X=0}=0.$$

Solving:

$$\frac{\alpha}{Y^{n+1}(0)}=\left(\frac{p}{Y(0)}-1\right)=\left(\frac{p-1}{np+1}\right); \tag{1.6}$$

$$\alpha=\frac{(p-1)(np+1)^n}{(n+1)^{n+1}}. \tag{1.7}$$

Finally, it is clear from the positivity of the metric that $Y'(X)>0$. This, together with (1.6), implies a fact that will be needed later in the analysis of the curvature, namely

$$\frac{\alpha}{Y^{n+1}(X)}\in\begin{cases}\left(0,\dfrac{p-1}{np+1}\right] & \text{if } p\geq 1,\\[2ex] \left[\dfrac{p-1}{np+1},0\right) & \text{if } p<1.\end{cases} \tag{1.8}$$

2. The curvature tensor. The calculations presented in this section will establish the following result.

THEOREM 2. *The components of the curvature tensor for the metric in Theorem* 1 *are given by*

$$\begin{aligned}R_{i\bar{j}k\bar{l}}=&-\{g_{i\bar{j}}g_{k\bar{l}}+g_{i\bar{l}}g_{k\bar{j}}\}\left(1-\frac{n\alpha}{Y^{n+1}}\right)\\&-Y^2\{(\log X)_{i\bar{j}}(\log X)_{k\bar{l}}+(\log X)_{i\bar{l}}(\log X)_{k\bar{j}}\}\left(\frac{(n+1)\alpha}{Y^{n+1}}\right)\\&-\left(\frac{Y'}{X}\right)^2\{X_iX_{\bar{j}}X_kX_{\bar{l}}\}\left(\frac{n(n+1)\alpha}{Y^{n+1}}\right),\end{aligned}$$

where

$$\frac{\alpha}{Y^{n+1}} \in \begin{cases} \left(0, \dfrac{p-1}{np+1}\right] & \textit{if } p \geq 1, \\ \left[\dfrac{p-1}{np+1}, 0\right) & \textit{if } p < 1. \end{cases}$$

A. SOME NOTATION AND FORMULAE. Throughout this section, the coordinates $(\mathbf{z}, w) = (z_1, \dots, z_{n-1}, w)$ will be used on Ω. By z_i, it will be meant

$$\begin{cases} z_i & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases}$$

The computations will be performed at the point $(z_1, 0, \dots, 0, w)$, but the final formula for the curvature will be quite general. It can be routinely verified that the following formulae hold at the point $(z_1, 0, \dots, 0, w)$:

$$g_{i\bar{j}} = Y(\log X)_{i\bar{j}} + (Y'/X)X_i X_{\bar{j}},$$

$$g^{\bar{j}k} = \left[\begin{array}{c|c|c} (p/Y)(1-|\mathbf{z}|^2)^2 & 0 \ \cdot \ \cdot \ \cdot \ 0 & -\bar{z}_i w(1-|\mathbf{z}|^2)/Y \\ \hline 0 & & \\ \vdots & (p/Y)(1-|\mathbf{z}|^2)I & \\ 0 & & \\ \hline & -z_i \bar{w}(1-|\mathbf{z}|^2)/Y & |w|^2(|\mathbf{z}|^2/pY + 1/XY') \end{array}\right],$$

$$\frac{X_{\bar{j}}}{X} g^{\bar{j}k} = \left(0, \dots, \frac{w}{XY'}\right),$$

$$(\log X)_{i\bar{j}} g^{\bar{j}k} = \frac{1}{Y}\left[\begin{array}{c|c} I & -w\bar{z}_i/p(1-|\mathbf{z}|^2) \\ \hline 0 & 0 \end{array}\right],$$

$$(\log X)_{i\bar{j}k} = (\log X)_{i\bar{j}} \frac{\bar{z}_k}{1-|\mathbf{z}|^2} + (\log X)_{\bar{j}k} \frac{\bar{z}_i}{1-|\mathbf{z}|^2},$$

$$(\log X)_{i\bar{j}k\bar{l}} = p\left\{(\log X)_{i\bar{j}}(\log X)_{k\bar{l}} + (\log X)_{k\bar{j}}(\log X)_{i\bar{l}} + (\log X)_{i\bar{j}\bar{l}} \frac{\bar{z}_k}{1-|\mathbf{z}|^2} + (\log X)_{\bar{j}k\bar{l}} \frac{\bar{z}_i}{1-|\mathbf{z}|^2}\right\},$$

$$(\log X)_{ik\bar{l}} = \frac{X_{ik\bar{l}}}{X} - \frac{X_{ik}X_{\bar{l}}}{X^2} - \frac{X_k}{X}(\log X)_{i\bar{l}} - \frac{X_i}{X}(\log X)_{k\bar{l}},$$

$$\left[\frac{X_{ik}X_{\bar{j}}}{X}\right]_{\bar{l}} = X_{\bar{j}}(\log X)_{ik\bar{l}} + \frac{X_{\bar{j}}X_k}{X}(\log X)_{i\bar{l}} + \frac{X_i X_{\bar{j}}}{X}(\log X)_{k\bar{l}} + \frac{X_{ik}X_{\bar{j}\bar{l}}}{X}.$$

B. GENERAL FORM OF THE CURVATURE. Recall the formula for the components of the curvature tensor:

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}k\bar{l}} + g_{ik\bar{p}} g^{\bar{p}q} g_{q\bar{j}\bar{l}}.$$

Computing:

$$g_{ik\bar{p}} = Y(\log X)_{i\bar{p}k} + Y'\{(\log X)_{i\bar{p}}X_k + (\log X)_{k\bar{p}}X_i\} + (Y'X_{ik} + Y''X_iX_k)(X_{\bar{p}}/X),$$

$$\begin{aligned}
g_{i\bar{j}k\bar{l}} = {} & Y(\log X)_{i\bar{j}k\bar{l}} + Y'\{(\log X)_{i\bar{j}k}X_{\bar{l}} + (\log X)_{i\bar{j}\bar{l}}X_k + (\log X)_{\bar{j}k\bar{l}}X_i \\
& \qquad + (X_{ik}[X_{\bar{j}}/X])_{\bar{l}} + (\log X)_{i\bar{j}}X_{k\bar{l}} + (\log X)_{k\bar{j}}X_{i\bar{l}}\} \\
& + Y''\left\{(\log X)_{i\bar{j}}X_kX_{\bar{l}} + (\log X)_{k\bar{j}}X_iX_{\bar{l}} + (\log X)_{i\bar{l}}X_{\bar{j}}X_k \right. \\
& \qquad \left. + X_i\frac{X_{\bar{j}}X_{k\bar{l}}}{X} + \frac{X_{ik}X_{\bar{j}}X_{\bar{l}} + X_iX_kX_{\bar{j}\bar{l}}}{X}\right\} \\
& + Y'''\left\{\frac{X_iX_{\bar{j}}X_kX_{\bar{l}}}{X}\right\} \\
= {} & pY\{(\log X)_{i\bar{j}}(\log X)_{k\bar{l}} + (\log X)_{i\bar{l}}(\log X)_{\bar{j}k}\} \\
& + Y\left\{(\log X)_{i\bar{j}\bar{l}}\frac{\bar{z}_k}{1-|\mathbf{z}|^2} + (\log X)_{\bar{j}k\bar{l}}\frac{\bar{z}_i}{1-|\mathbf{z}|^2}\right\} \\
& + Y'\left\{X_i(\log X)_{\bar{j}k\bar{l}} + X_{\bar{j}}(\log X)_{ik\bar{l}} + X_k(\log X)_{i\bar{j}\bar{l}} + X_{\bar{l}}(\log X)_{i\bar{j}k} \right. \\
& \qquad + (\log X)_{i\bar{j}}X_{k\bar{l}} + (\log X)_{k\bar{j}}X_{i\bar{l}} + (\log X)_{i\bar{l}}\frac{X_{\bar{j}}X_k}{X} + (\log X)_{k\bar{l}}\frac{X_iX_{\bar{j}}}{X} \\
& \qquad \left. + \frac{X_{ik}X_{\bar{j}\bar{l}}}{X}\right\} \\
& + Y''\left\{(\log X)_{i\bar{j}}X_kX_{\bar{l}} + (\log X)_{k\bar{l}}X_iX_{\bar{j}} + (\log X)_{i\bar{l}}X_kX_{\bar{j}} + (\log X)_{k\bar{j}}X_iX_{\bar{l}} \right. \\
& \qquad \left. + \frac{X_{ik}X_{\bar{j}}X_{\bar{l}} + X_iX_kX_{\bar{j}\bar{l}}}{X} + \frac{X_iX_{\bar{j}}X_kX_{\bar{l}}}{X^2}\right\} \\
& + Y'''\left\{\frac{X_iX_{\bar{j}}X_kX_{\bar{l}}}{X}\right\}.
\end{aligned}$$

Writing $g_{ik\bar{p}}g^{\bar{p}q}$ as a row vector indexed by q:

$$\begin{aligned}
g_{ik\bar{p}}g^{\bar{p}q} = {} & \left[\left[\frac{Y\bar{z}_k}{1-|\mathbf{z}|^2} + Y'X_k\right](\log X)_{i\bar{p}} + \left[\frac{Y\bar{z}_i}{1-|\mathbf{z}|^2} + Y'X_i\right](\log X)_{k\bar{p}}\right]g^{\bar{p}q} \\
& + (Y'X_{ik} + Y''X_iX_k)\frac{X_{\bar{p}}}{X} \\
= {} & \left[\frac{Y\bar{z}_k}{1-|\mathbf{z}|^2} + Y'X_k\right]\frac{1}{Y}\left[\delta_i^q, -\frac{\bar{z}_iw}{p(1-|\mathbf{z}|^2)}\right] \\
& + \left[\frac{Y\bar{z}_i}{1-|\mathbf{z}|^2} + Y'X_i\right]\frac{1}{Y}\left[\delta_k^q, -\frac{\bar{z}_kw}{p(1-|\mathbf{z}|^2)}\right] \\
& + (Y'X_{ik} + Y''X_iX_k)\left(0, \ldots, 0, \frac{w}{XY'}\right)
\end{aligned}$$

where $(\delta_i^q, 0)$ denotes the vector with 1 in the ith spot if $i < n$, and the 0-vector if $i = n$.

$$g_{ik\bar p}g^{\bar pq}g_{q\bar j\bar l}=g_{ik\bar p}g^{\bar pq}\left[\left[\frac{Yz_j}{1-|\mathbf{z}|^2}+Y'X_{\bar j}\right](\log X)_{q\bar l}+\left[\frac{Yz_l}{1-|\mathbf{z}|^2}+Y'X_{\bar l}\right](\log X)_{q\bar j}\right.$$
$$\left.+(Y'X_{\bar j\bar l}+Y''X_{\bar j}X_{\bar l})\frac{X_q}{X}\right]$$
$$=\left[\frac{Y\bar z_k}{1-|\mathbf{z}|^2}+Y'X_k\right]\frac{1}{Y}\left[\frac{Yz_j}{1-|\mathbf{z}|^2}+Y'X_{\bar j}\right](\log X)_{i\bar l}$$
$$+\left[\frac{Y\bar z_i}{1-|\mathbf{z}|^2}+Y'X_i\right]\frac{1}{Y}\left[\frac{Yz_l}{1-|\mathbf{z}|^2}+Y'X_{\bar l}\right](\log X)_{\bar jk}$$
$$+\left[\frac{Y\bar z_i}{1-|\mathbf{z}|^2}+Y'X_i\right]\frac{1}{Y}\left[\frac{Yz_j}{1-|\mathbf{z}|^2}+Y'X_{\bar j}\right](\log X)_{k\bar l}$$
$$+\left[\frac{Y\bar z_k}{1-|\mathbf{z}|^2}+Y'X_k\right]\frac{1}{Y}\left[\frac{Yz_l}{1-|\mathbf{z}|^2}+Y'X_{\bar l}\right](\log X)_{i\bar j}$$
$$+(Y'X_{ik}+Y''X_iX_k)(Y'X_{\bar j\bar l}+Y''X_{\bar j}X_{\bar l})(1/XY').$$

Subtracting and cancelling terms:

$$R_{i\bar jk\bar l}=-g_{i\bar jk\bar l}+g_{ik\bar p}g^{\bar pq}g_{q\bar j\bar l}$$
$$=-pY\{(\log X)_{i\bar j}(\log X)_{k\bar l}+(\log X)_{i\bar l}(\log X)_{\bar jk}\}$$
$$-XY'\left\{(\log X)_{i\bar j}(\log X)_{k\bar l}+(\log X)_{i\bar l}(\log X)_{\bar jk}+(\log X)_{i\bar j}\frac{X_kX_{\bar l}}{X^2}\right.$$
$$\left.+(\log X)_{k\bar j}\frac{X_iX_{\bar l}}{X^2}+(\log X)_{i\bar l}\frac{X_kX_{\bar j}}{X^2}+(\log X)_{k\bar l}\frac{X_iX_{\bar j}}{X^2}\right\}$$
$$-Y''\left\{(\log X)_{i\bar j}X_kX_{\bar l}+(\log X)_{i\bar l}X_{\bar j}X_k+(\log X)_{\bar jk}X_iX_{\bar l}+(\log X)_{k\bar l}X_iX_{\bar j}\right.$$
$$\left.+\frac{X_iX_{\bar j}X_kX_{\bar l}}{X^2}\right\}$$
$$-Y'''\left\{\frac{X_iX_{\bar j}X_kX_{\bar l}}{X}\right\}$$
$$+\frac{Y'Y'}{Y}\left\{X_kX_{\bar j}(\log X)_{i\bar l}+X_iX_{\bar j}(\log X)_{k\bar l}+X_iX_{\bar l}(\log X)_{k\bar j}\right.$$
$$\left.+X_kX_{\bar l}(\log X)_{i\bar j}\right\}$$
$$+\frac{Y''Y''}{Y'}\left\{\frac{X_iX_{\bar j}X_kX_{\bar l}}{X}\right\};$$

$$R_{i\bar jk\bar l}=-\{(\log X)_{i\bar j}(\log X)_{k\bar l}+(\log X)_{i\bar l}(\log X)_{k\bar j}\}(pY+XY')$$
$$-\{(\log X)_{i\bar j}X_kX_{\bar l}+(\log X)_{i\bar l}X_{\bar j}X_k+(\log X)_{\bar jk}X_iX_{\bar l}+(\log X)_{k\bar l}X_iX_{\bar j}\}$$
$$\times\left(\frac{Y'}{X}+Y''-\frac{Y'Y'}{Y}\right) \tag{2.1}$$
$$-\{X_iX_{\bar j}X_kX_{\bar l}\}\left(\frac{Y''}{X^2}+\frac{Y'''}{X}-\frac{Y''Y''}{XY'}\right).$$

REMARK. The terms in { } are of the same form as the terms that arise in the expansion of $(g_{i\bar{j}}g_{k\bar{l}}+g_{i\bar{l}}g_{k\bar{j}})$.

C. INVOKING THE EINSTEIN CONDITION. The coefficients in (2.1) can be simplified by using (1.5) and its first derivative:

$$XY^{n-1}Y'=Y^{n+1}-pY^n+\alpha$$
$$XY^{n-1}Y''=Y'\{(n+1)Y^n-(np+1)Y^{n-1}-(n-1)XY^{n-2}Y'\}.$$

Simplifying the coefficients in (2.1):

$$(pY+XY')=Y^2+\frac{\alpha}{Y^{n-1}}=Y^2\left(1+\frac{\alpha}{Y^{n+1}}\right),$$

$$\left(Y''+\frac{Y'}{X}-\frac{Y'Y'}{Y}\right)=\left(\frac{XY'}{Y}\right)'\frac{Y}{X}$$
$$=(Y-p+\alpha Y^{-n})'\frac{Y}{X}$$
$$=\frac{YY'}{X}\left(1-\frac{n\alpha}{Y^{n+1}}\right),$$

$$\left(\frac{Y'''}{X}+\frac{Y''}{X^2}-\frac{Y''Y''}{XY'}\right)=\left(\frac{XY''}{Y'}\right)'\frac{Y'}{X^2}$$
$$=\left((n+1)Y-(np+1)-(n-1)X\frac{Y'}{Y}\right)'\frac{Y'}{X^2}$$
$$=\frac{Y'}{X^2}((n+1)Y'-(n-1)(Y-p+\alpha Y^{-n})')$$
$$=\left(\frac{Y'}{X}\right)^2\left(2+n(n-1)\frac{\alpha}{Y^{n+1}}\right).$$

The expression for the curvature of the Einstein-Kähler metric then becomes:

$$\begin{aligned}R_{i\bar{j}k\bar{l}}=&-Y^2\{(\log X)_{i\bar{j}}(\log X)_{k\bar{l}}+(\log X)_{i\bar{l}}(\log X)_{k\bar{j}}\}\left(1+\frac{\alpha}{Y^{n+1}}\right)\\&-Y\frac{Y'}{X}\{(\log X)_{i\bar{j}}X_kX_{\bar{l}}+(\log X)_{i\bar{l}}X_kX_{\bar{j}}+(\log X)_{k\bar{j}}X_iX_{\bar{l}}\\&\qquad+(\log X)_{k\bar{l}}X_iX_{\bar{j}}\}\left(1-\frac{n\alpha}{Y^{n+1}}\right)\\&-\left(\frac{Y'}{X}\right)^2\{X_iX_{\bar{j}}X_kX_{\bar{l}}\}\left(2+n(n-1)\frac{\alpha}{Y^{n+1}}\right)\end{aligned}\tag{2.2}$$

or

$$\begin{aligned}R_{i\bar{j}k\bar{l}}=&-\{g_{i\bar{j}}g_{k\bar{l}}+g_{i\bar{l}}g_{k\bar{j}}\}\left(1-\frac{n\alpha}{Y^{n+1}}\right)\\&-Y^2\{(\log X)_{i\bar{j}}(\log X)_{k\bar{l}}+(\log X)_{i\bar{l}}(\log X)_{k\bar{j}}\}\left(\frac{(n+1)\alpha}{Y^{n+1}}\right)\\&-\left(\frac{Y'}{X}\right)^2\{X_iX_{\bar{j}}X_kX_{\bar{l}}\}\left(\frac{n(n+1)\alpha}{Y^{n+1}}\right).\end{aligned}\tag{2.3}$$

3. Analysis of the results. The behavior of the Einstein–Kähler metric and its curvature tensor on smooth bounded strongly pseudoconvex domains in $\mathbf{C}^n$ is already known. The interest in the domains $\Omega_p = \{|\mathbf{z}|^2 + |w|^{2p} < 1\}$, $p > 0$, is that they provide a range of examples which contain domains with C^2 weakly pseudoconvex points ($p > 1$), C^1 strictly convex domains which are not C^2 ($\frac{1}{2} < p < 1$), and pseudoconvex domains with corners ($p < \frac{1}{2}$). In each case, the interesting boundary points are $\{(\mathbf{z}, w)\colon |\mathbf{z}| = 1,\ w = 0\}$. The behavior of the metric and its curvature tensor at these points is summarized.

THEOREM 3. *For the domain* $\Omega_p = \{|\mathbf{z}|^2 + |w|^{2p} < 1\}$, $p > 0$, *and for any fixed size* $c \in (0, 1)$ *of the approach region* $\{X < c\}$ *to the special boundary points* $|\mathbf{z}| = 1$, *the following quantities are uniformly bounded from above and below by positive constants* (*u represents a holomorphic vector of unit Euclidean length, and* $|u|_g^2$ *its length in the Einstein–Kähler geometry*).

(1) $(1 - |\mathbf{z}|^2)^2 |u|_g^2$ *if u is a multiple of* $(z_1, \ldots, z_{n-1}, 0)$.

(2) $(1 - |\mathbf{z}|^2)|u|_g^2$ *if u is Euclidean perpendicular to* $(z_1, \ldots, z_{n-1}, 0)$ *and* $(0, \ldots, 1)$.

(3) $(1 - |\mathbf{z}|^2)^{1/p} |u|_g^2$ *if* $u = (0, \ldots, 0, 1)$.

Also, for $p > \frac{1}{2}$,

(4) $|g_{i\bar{j}}|(\text{distance to } \partial\Omega)^{n+1}(1 - |\mathbf{z}|^2)^{(1-p)/p}$ *is uniformly bounded above and below by positive constants on* Ω. ($|g_{i\bar{j}}|$ *is the volume for the Einstein–Kähler metric, and distance to* $\partial\Omega$ *refers to Euclidean distance.*)

Proof. Parts (1), (2), and (3) follow directly from Theorem 1. For part (4), notice that Y, Y' are uniformly equivalent to $(1 - X)^{-1}, (1 - X)^{-2}$ respectively (see Theorem 1) and that the defining function $(1 - X^p)(1 - |\mathbf{z}|^2) = 1 - |\mathbf{z}|^2 - |w|^{2p}$ is uniformly equivalent to distance to the boundary for $p > \frac{1}{2}$. □

Notice that the approach region to the special boundary points is given in terms of X. For $p > \frac{1}{2}$, these approach regions are larger than the conical approach regions which are commonly used in such descriptions of the metric. However, for $p < \frac{1}{2}$, the domain itself has directions with cusps, and the approach regions reflect this by also having directions with cusps.

The theorem indicates that the Einstein–Kähler metric behaves as usual in the complex normal direction and in the smooth strongly pseudoconvex complex tangential directions. In the special directions, the metric grows large slower than usual for $p > 1$ (weakly pseudoconvex directions) and faster than usual for $p < 1$ (non-smooth "sharper" directions).

The fourth statement in the above theorem indicates the interplay between the smooth strongly pseudoconvex boundary points and the points $|\mathbf{z}| = 1$ in their effect upon the growth rate of the metric.

THEOREM 4. *Consider the Einstein–Kähler metric on the domain*

$$\Omega_p = \{|\mathbf{z}|^2 + |w|^{2p} < 1\}, \quad p > 0.$$

(a) *For* $p \geq 1$ (*the* C^2 *weakly pseudoconvex case*), *the Riemannian sectional curvature is bounded between negative constants on* Ω_p.

(b) *There exist* C^1 *strictly convex domains (and hence* C^2 *strongly convex domains) for which the holomorphic sectional curvature is strictly positive at some points in some directions.*

(c) *The noncompact automorphism group has elements which move any compact subdomain arbitrarily close to the boundary point* $z_1 = 1$. *Hence, the entire range of behavior that the curvature tensor exhibits on* Ω_p *is exhibited arbitrarily close to* $(1, 0, \ldots, 0)$. *In particular, the curvature tensor does not exhibit any limiting behavior at* $(1, 0, \ldots, 0)$.

Proof. (a) The curvature tensor (2.3) is expressed as the sum of three tensors, each of which represents the constant negative holomorphic sectional curvature tensor for a possibly degenerate metric. ($Y(\log X)_{i\bar{j}}$ and $Y'X_i X_{\bar{j}}/X$ represent degenerate metrics, each of which is dominated by the metric $g_{i\bar{j}}$.) By a well-known formula (see e.g. [3, p. 166]), the Riemannian curvature tensor can be explicitly written down for each of the three tensors. The first tensor (representing the nondegenerate metric $g_{i\bar{j}}$) will give a strictly negative contribution to the Riemannian sectional curvature tensor; the remaining two tensors will give non-positive contributions. The result, of course, is a tensor bounded between negative constants.

(b) Choose $p = 3/4$, $n = 10$. From (1.7), $\alpha Y^{-(n+1)} \in [-1/34, 0)$. Then

$$(2 + n(n-1)\alpha Y^{-(n+1)}) \in [2 - 90/34, 2)$$

certainly has some negative values within its possible range on $\Omega_{3/4}$. From (2.2), there are some points at which the holomorphic sectional curvature for the $\partial/\partial w$ plane is positive.

(c) Obvious. □

This range of examples of Einstein-Kähler metrics can now be used to give metric estimates for more general domains by using the localized comparison principles of [1]. Volume estimates use only the fact that the metric is Einstein-Kähler (i.e., bounds on the Ricci curvature), whereas the metric estimates will make use of the fact that for $p > 1$, the holomorphic bisectional curvature is bounded above by a negative constant.

THEOREM 5. *Consider a pseudoconvex domain* $\tilde{\Omega} \subset C^n$ *with complete Einstein-Kähler metric* $\tilde{g}_{i\bar{j}}$ *and with* $(1, 0, \ldots, 0) \in \partial\tilde{\Omega}$. *Consider also* $\Omega = \{|\mathbf{z}|^2 + |w|^{2p} < 1\}$ *with Einstein-Kähler metric* $g_{i\bar{j}}$. *Let D be a bounded neighborhood of* $(1, 0, \ldots, 0)$. *Define*

$$\beta_g(\mathbf{z}, w) = \exp[-\rho_g((\mathbf{z}, w), \partial(D \cap \Omega))], \qquad \beta_{\tilde{g}}(\mathbf{z}, w) = \exp[-\rho_{\tilde{g}}((\mathbf{z}, w), \partial(D \cap \tilde{\Omega}))],$$

where ρ_g *and* $\rho_{\tilde{g}}$ *denote distance in the metrics g and* $\tilde{g}$ *respectively. Assume that* $\beta_g(\mathbf{z}, w) \to 0$ *and* $\beta_{\tilde{g}}(\mathbf{z}, w) \to 0$ *as* $(\mathbf{z}, w) \to (1, 0, \ldots, 0)$ *(hence* $g, \tilde{g}$ *are "locally complete" at* $(1, 0, \ldots, 0)$*).*

(a) $p > 1$, $D \cap \tilde{\Omega} \subset D \cap \Omega$: *For any* $D_\epsilon = \{\beta_{\tilde{g}} \le 1 - \epsilon\}$, $\epsilon > 0$, *there is a constant* $c > 0$ *such that* $[g_{i\bar{j}}] \le c[\tilde{g}_{i\bar{j}}]$ *on* D_ϵ.

(b) $p > 0$, $D \cap \tilde{\Omega} \subset D \cap \Omega$: *For any* $D_\epsilon = \{\beta_{\tilde{g}} \le 1 - \epsilon\}$, $\epsilon > 0$, *there is a constant* $\delta > 0$ *such that* $|g_{i\bar{j}}| \le |\tilde{g}_{i\bar{j}}|(1 + O(|\beta|^\delta))$ *on* D_ϵ.

(c) $p > 0$, $D \cap \Omega \subset D \cap \tilde{\Omega}$: *For any* $D_\epsilon = \{\beta_g \le 1 - \epsilon\}$, $\epsilon > 0$, *there is a constant* $\delta > 0$ *such that* $|\tilde{g}_{i\bar{j}}| \le |g_{i\bar{j}}|(1 + O(|\beta|^\delta))$ *on* D_ϵ.

Proof. This is a direct application of Lemma III and Lemma IV in [1]. □

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