SPECTRAL THEORY OF SELF-ADJOINT HANKEL MATRICES

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The purpose of this paper is to determine, up to unitary equivalence, the absolutely continuous part of a bounded symmetric infinite Hankel matrix, in terms of the symbol of the operator. Since, according to Hartman's theorem, continuous symbols correspond to compact operators, the continuous spectrum must be connected somehow with the discontinuities of the symbol. For jump discontinuities, this is born out by the theory of the essential spectrum [6, Chapter 6] in which each discontinuity contributes a segment to the essential spectrum of length proportional to the jump.

To the author's knowledge, the only known result on multiplicity theory for the continuous spectrum is Rosenblum's work [8] on the Hilbert matrix, which he shows by explicit diagonalization to have uniform Lebesgue spectrum of multiplicity one on the interval $[0, \pi]$. Our main result gives a complete description of the absolutely continuous part in the case of symbols with a finite number of smooth jumps. We show that each discontinuity contributes a direct summand to the absolutely continuous part having uniform Lebesgue spectrum of multiplicity one, on a certain interval—the same interval that it contributes to the essential spectrum.

We shall obtain this result from a theorem first stated by Ismagilov in 1963 [3], and later proved in [2].

THEOREM (Ismagilov). *Let A and B be bounded self-adjoint operators and set $H = A + B$. If the product AB is of trace class, then the absolutely continuous part of H is unitarily equivalent to the direct sum of the absolutely continuous parts of A and B.*

This theorem is, in fact, a theorem of trace class scattering theory, generalizing the classic Kato–Rosenblum theorem on stability of absolutely continuous parts under trace class perturbations [7, p. 16]. It may be proved as a consequence of another generalization of the Kato–Rosenblum theorem due to Pearson [5, §7, p. 24], which we shall also use.

THEOREM (Pearson). *Let A be self-adjoint on $\mathcal{H}$, B self-adjoint on $\mathcal{H}'$, and J bounded from $\mathcal{H}$ to $\mathcal{H}'$. If $BJ - JA$ is trace class then the wave operator*

$$\Omega_+ = \operatorname*{s-lim}_{t\to\infty} e^{iBt} J e^{-iAt} P_a(A)$$

exists, where $P_a(A)$ is the projection onto the absolutely continuous subspace of A.

Received February 15, 1985.
The author was supported by NSF Grant MCS-82-02115-01.
Michigan Math. J. 33 (1986).

This result, with $A=B$, was also proved in an earlier paper of Carey and Pincus [9, Lemma 4.1, p. 57]. This reference, pointed out by the referee, does not seem to be known in the scattering theory literature.

For general information about Hankel operators, and references to the literature, we refer the reader to the excellent recent monograph of Power [6].

1. Hankel operators and symbols. Let l_2 be the Hilbert space of complex square-summable *one-sided* sequences $(x_0, x_1, x_2, \dots)$. The space l_2 will be freely identified with the Hardy space $H^2(\Delta)$ by the Fourier correspondence

$$(c_n) \to \sum_{n=0}^{\infty} c_n e^{in\theta}$$

which is unitary if the norm on $H^2(\Delta)$ is taken to be

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta\right)^{1/2}.$$

Here Δ denotes the unit disc.

An operator H of the form

$$(Hx)_n = \sum_{k=0}^{\infty} c_{n+k} x_k$$

is called a *Hankel operator*. A *symbol* for H is a function $f(e^{i\theta})$ such that

$$c_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{-i(n+1)\theta} f(e^{i\theta})\, d\theta \tag{1.1}$$

for $n \geq 0$. Since the negative Fourier coefficients of f are arbitrary, the symbol is far from unique. We denote the Hankel operator with symbol f by $H(f)$. Note that its adjoint is $H(f)^* = H(\tilde{f})$, where $\tilde{f}(e^{i\theta}) = \bar{f}(e^{-i\theta})$. If f is bounded, Nehari's theorem [6, Chapter 1] asserts that $H(f)$ is bounded. *We shall work only with bounded symbols f throughout this paper.*

1.1. LEMMA. *If f and g are bounded and have disjoint supports, then $H(f)^*H(g)$ is trace class.*

Proof. Let $u(e^{i\theta})$ and $v(e^{i\theta})$ be H^2 functions with Fourier coefficients u_k and v_k. Put $A = H(f)^*H(g)$. Then

$$\begin{aligned}\langle Au, v\rangle &= \lim_{r\uparrow 1} \sum_{nlk} r^k \bar{c}_{n+k} a_{k+l} u_l \bar{v}_n \\ &= \frac{1}{4\pi^2}\lim_{r\uparrow 1}\sum_{nlk}\int_0^{2\pi}\int_0^{2\pi} r^k e^{i(n+k+1)\theta} e^{-i(k+l+1)\phi} \bar{f}(e^{i\phi}) g(e^{i\theta}) u_l \bar{v}_n\, d\theta\, d\phi.\end{aligned}$$

The sums over l and n give $u(e^{-i\theta})$ and $\bar{v}(e^{-i\theta})$, while the sum over k gives $(1-re^{i(\theta-\phi)})^{-1}$. Changing variables to $\theta' = -\theta$, $\phi' = -\phi$, and letting r tend to 1 therefore gives

$$\langle Au, v\rangle = \int_0^{2\pi}\int_0^{2\pi} a(\theta,\phi) u(e^{i\phi}) \bar{v}(e^{i\theta})\, d\theta\, d\phi,$$

where

$$a(\theta,\phi)=(4\pi^2)^{-1}(1-e^{i(\theta-\phi)})^{-1}e^{i\theta}g(e^{i\theta})e^{-i\phi}\bar{f}(e^{i\phi}). \tag{1.2}$$

We must show that this kernel, which is bounded because $g\bar{f}$ vanishes when $\theta-\phi$ is near zero, defines a trace class operator. Write (1.2) as

$$a(\theta,\phi)=b(\theta,\phi)h(\theta)\bar{k}(\phi),$$

where $b(\theta,\phi)$ is a C^∞ function on the torus which is equal to $(1-e^{i(\theta-\phi)})^{-1}$ on the support of $g(e^{i\theta})\bar{f}(e^{i\phi})$. Expanding $b(\theta,\phi)$ in a Fourier series gives

$$a(\theta,\phi)=\sum_{n,j=0}^{\infty} b_{nj}e^{ih\theta}h(\theta)e^{ij\phi}\bar{k}(\phi)$$

$$=\sum_{n,j=0}^{\infty} b_{nj}h_n(\theta)\bar{k}_j(\phi),$$

where $h_n(\theta)=e^{in\theta}h(\theta)$ and $k_j(\phi)=e^{-ij\phi}k(\phi)$, and where b_{nj} is rapidly decreasing. Hence

$$A=\sum_{n,j=0}^{\infty} b_{nj}\langle\cdot,k_j\rangle h_n$$

is an absolutely convergent sum of rank one operators, and so is trace class. □

2. An abstract theorem. If T is self-adjoint on the *separable* Hilbert space $\mathcal{H}$, we shall write $P_a(T)$ for the absolutely continuous projection of T, $\mathcal{H}_a(T)=P_a(T)\mathcal{H}$ for the absolutely continuous subspace, and T_a for the absolutely continuous part of T, i.e. the restriction of T to $\mathcal{H}_a(T)$.

Let $\mathcal{H}^2=\mathcal{H}\oplus\mathcal{H}$, and define an operator $J\colon\mathcal{H}^2\to\mathcal{H}$ by

$$Jf=J\begin{pmatrix}f_1\\ f_2\end{pmatrix}=f_1+f_2.$$

The adjoint of J is

$$J^*x=\begin{pmatrix}x\\ x\end{pmatrix}.$$

Let $L\colon\mathcal{H}^2\to\mathcal{H}^2$ be defined by

$$L=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},$$

and note that

$$|Jf|^2=|f|^2+(Lf,f). \tag{2.1}$$

Let A be bounded on $\mathcal{H}$ and define the self-adjoint operators $T=A+A^*$ on $\mathcal{H}$, and

$$S=\begin{pmatrix}0 & A\\ A^* & 0\end{pmatrix}$$

on $\mathcal{H}^2$. Note that S is unitarily equivalent to its negative; in fact, $WSW^*=-S$, where W is the unitary involution

$$W=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

on $\mathcal{H}^2$.

2.1. THEOREM. *Let A be a bounded operator on a separable Hilbert space $\mathcal{H}$. If A^2 is trace class, then the absolutely continuous parts of T and S are unitarily equivalent.*

Since S_a is unitarily equivalent to $-S_a$, we have:

2.2. COROLLARY. *T_a is unitarily equivalent to $-T_a$.*

Proof of Theorem. We claim that the two-space wave operator

$$\Omega_+ f = \operatorname*{s-lim}_{t\to\infty} e^{iTt} J e^{-iSt} P_a(S) f$$

exists. It suffices to prove this for f in the dense subspace $S\mathcal{H}_a^2(S)$ of $\mathcal{H}_a^2(S)$; that is, with f replaced by Sf. This is equivalent to proving the existence of

$$\operatorname*{s-lim}_{t\to\infty} e^{iTt} JSe^{-iSt} P_a(S). \tag{2.2}$$

Existence of (2.2) follows from Pearson's theorem [5, §7, p. 24] because

$$T(JS)-(JS)S=(A^{*2}, A^2)$$

is trace class.

Moreover, Ω_+ is isometric. For this, it suffices to prove that

$$\lim_{t\to\infty} |Je^{-iSt} f|^2 = |f|^2 \tag{2.3}$$

for $f \in \mathcal{H}_a^2(S)$. Replacing f by Sf, as above, and using (2.1) yields

$$|Je^{-iSt}Sf|^2-|Sf|^2=(SLSe^{-iSt}f, e^{-iSt}f).$$

Now by the Riemann–Lebesgue lemma, $e^{-iSt}f \to 0$ weakly for absolutely continuous vectors, and the operator

$$SLS=\begin{pmatrix} 0 & A^2 \\ A^{*2} & 0 \end{pmatrix}$$

is compact. Hence, $SLSe^{-iSt}f \to 0$ strongly, which gives (2.3).

It follows [4] that $\Omega_+\mathcal{H}^2$ is a reducing subspace of T and that the part of T in $\Omega_+\mathcal{H}^2$ is unitarily equivalent to S_a.

Now consider the square of T:

$$T^2=(A+A^*)^2 = AA^*+A^*A+A^2+A^{*2}.$$

The last two terms are trace class, so T_a^2 is equivalent to the absolutely continuous part of $D=AA^*+A^*A$. But the product $(AA^*)(A^*A)=AA^{*2}A$ is also trace class, so by Ismagilov's theorem [2], D_a is unitarily equivalent to

$$\begin{pmatrix} (AA^*)_a & 0 \\ 0 & (A^*A)_a \end{pmatrix},$$

which is exactly the absolutely continuous part of

$$S^2 = \begin{pmatrix} AA^* & 0 \\ 0 & A^*A \end{pmatrix}.$$

Hence, T_a^2 and S_a^2 are unitarily equivalent.

We have now established three facts: (1) T_a contains S_a, (2) S_a is unitarily equivalent to $-S_a$, and (3) T_a^2 is unitarily equivalent to S_a^2. Let $t(\lambda)$ and $s(\lambda)$ be the multiplicity functions of T_a and S_a. Our three facts translate into the relations:

(2.4) $$t(\lambda) \geq s(\lambda)$$

and

(2.5) $$s(-\lambda) = s(\lambda)$$

for a.e. λ, and

(2.6) $$t(\lambda^{1/2}) + t(-\lambda^{1/2}) = s(\lambda^{1/2}) + s(-\lambda^{1/2})$$

for a.e. $\lambda > 0$. Replacing $\lambda^{1/2}$ by $\lambda > 0$ gives

(2.7) $$t(\lambda) + t(-\lambda) = s(\lambda) + s(-\lambda) = 2s(\lambda)$$

for $\lambda > 0$. Using (2.4), (2.7), (2.4) and (2.5) successively, we obtain $s(\lambda) \leq t(\lambda) = 2s(\lambda) - t(-\lambda) \leq 2s(\lambda) - s(-\lambda) = s(\lambda)$. Hence $t(\lambda) = s(\lambda)$ a.e., so that T_a and S_a are unitarily equivalent. □

REMARK. We have *not* established completeness, that is, that $\Omega_+ \mathcal{H}^2 = \mathcal{H}_a(T)$, although this must hold if T_a has finite multiplicity. It would be interesting to know if this is true.

2.3. COROLLARY. *If, in addition, A is congruent to a self-adjoint operator B, then T_a is unitarily equivalent to $B_a \oplus -B_a$.*

Proof. If $A = UBU$ where U is unitary, then $AA^* = UB^2U^*$ and $A^*A = U^*B^2U$. Hence, T_a^2 is unitarily equivalent to $B_a^2 \oplus B_a^2$. In terms of multiplicities, this says that

$$t(\lambda) + t(-\lambda) = 2(b(\lambda) + b(-\lambda))$$

for a.e. $\lambda > 0$. By Corollary 2.1, $t(\lambda) = t(-\lambda)$ a.e., so we obtain

$$t(\lambda) = b(\lambda) + b(-\lambda),$$

which is equivalent to the result. □

2.4. REMARK. If $A = UBU$ where B is only *normal*, we obtain $AA^* = U|B|^2U^*$ and $A^*A = U^*|B|^2U$, and so, by the same proof, T_a is unitarily equivalent to $-|B|_a \oplus |B|_a$.

3. Symmetry of absolutely continuous parts. In this section, we digress to prove a result on symmetry of the spectrum of $H(f)$ in the origin. For the corresponding result for essential spectra, see [6, p. 59 and p. 61, problem 6]. The result is very general; there is no condition (except boundedness) on f except near ± 1. *We assume that $H(f)$ is self-adjoint.*

3.1. THEOREM. *If $f(e^{i\theta})$ is C^2 on a neighborhood of $+1$ and -1, then the absolutely continuous parts of $H(f)$ and $-H(f)$ are unitarily equivalent.*

Proof. If $g=\frac{1}{2}(f+\tilde{f})$, then g is also C^2 near ± 1 and satisfies $g=\tilde{g}$, and $H(g)=H(f)$. Let $\chi(e^{i\theta})$ be a non-negative C^∞ function which vanishes near ± 1, but $\chi\equiv 1$ off a neighborhood of ± 1 where g is C^2, with $\chi=\tilde{\chi}$. Then the absolutely continuous part of $H(g)=H(\chi g)+H(g(1-\chi))$ is unitarily equivalent to that of $H(\chi g)$, since $g(1-\chi)$ is C^2 and hence $H(g(1-\chi))$ is trace class. (See Lemma 5.2 below.)

We may therefore assume that f vanishes near ± 1, and $f=\tilde{f}$. Write $f=f_+ + f_-$, where

$$f_+(\xi)=\begin{cases} f(\xi) & \operatorname{Im}\xi\geq 0,\\ 0 & \operatorname{Im}\xi<0.\end{cases}$$

Then $\tilde{f}_+ = f_-$. Thus

$$H(f)=H(f_+)+H(f_-)=H(f_+)+H(f_+)^*,$$

where $H(f_+)^2=H(f_-)^*H(f_+)$ is trace class by Lemma 1.1. The result follows from Corollary 2.2 with $A=H(f_+)$.

4. A special case. In this section, we shall determine the absolutely continuous spectrum for symbols with jumps only at a point ξ and its complex conjugate $\bar{\xi}$. The general case is reduced to this one in the next section.

For $|\xi|=1$, the diagonal unitary operator

$$D(\xi): e^{in\theta}\to e^{in\theta}\xi^n$$

is translation on $H^2(\Delta)$:

$$D(\xi): f(e^{i\theta})\to f(\xi e^{i\theta}),$$

and so has the group property $D(\xi\xi')=D(\xi)D(\xi')$. One computes that

$$H(D(\xi)f)=D(\xi)H(f)D(\xi). \tag{4.1}$$

This corresponds to the new Hankel matrix $(\xi^{n+m}c_{n+m})$. Note that (4.1) expresses a *congruence*, and is a unitary equivalence only for $\xi=\pm 1$.

Let H_1 be the Hilbert matrix, with $c_n=(n+1)^{-1}$. H_1 has the symbol

$$f_1(e^{i\theta})=\theta,\quad 0<\theta<2\pi,$$

which has a jump of magnitude 2π at $\xi=e^{i\theta}=1$. According to Rosenblum [8], H_1 has only absolutely continuous spectrum, with uniform multiplicity one on $[0,\pi]$.

Let $f_\xi=D_\xi f_1$, and $H_\xi=D(\xi)H_1D(\xi)$.

For $\xi=-1$, H_{-1} is unitarily equivalent to H_1. It has the sequence $c_n=(-1)^n(n+1)^{-1}$ and the symbol

$$f_{-1}(e^{i\theta})=\theta+\pi,\quad -\pi<\theta<\pi,$$

which jumps by 2π at $\xi=-1$.

Let α be a complex number, and define the self-adjoint operator

$$T(\xi,\alpha)=\alpha H_\xi+\bar{\alpha}H_{\bar{\xi}}.$$

The symbol $\alpha f_\xi+\bar{\alpha}f_{\bar{\xi}}$ of $T(\xi,\alpha)$ has a jump $2\pi\alpha$ at ξ, and a jump $2\pi\bar{\alpha}$ at $\bar{\xi}$.

4.1. THEOREM. (a) *The operators* H_1 *and* H_{-1} *have Lebesgue spectrum with uniform multiplicity* 1 *on* $[0,\pi]$.

(b) *The absolutely continuous part of* $T(\xi,\alpha)$ *has uniform multiplicity* 1 *on the interval* $[-|\alpha|\pi, |\alpha|\pi]$.

REMARK. It would be interesting to find an explicit diagonalization of $T(\xi,\alpha)$ similar to that of H_1 in [8].

Proof of Theorem 4.1. Part (a) has already been proved. For part (b), take $A=\alpha D_\xi H_1 D_\xi$. Then $T=A+A^*$, and

$$A^2=\alpha^2 H(f_\xi)H(f_\xi)=\alpha^2 H(\tilde{f}_\xi)^*H(f_\xi)$$

is trace class by Lemma 1.1. If α were real, the result would follow immediately from Corollary 2.3 with $B=\alpha H_1$. In general, it follows from Remark 2.4. □

REMARK. If $\alpha=\rho e^{i\delta}$ and $\xi=e^{i\theta}$, the *sequence* corresponding to the Hankel matrix $T(\xi,\alpha)$ is $c_n=2\rho(n+1)^{-1}\cos(n\theta+\delta)$.

5. The main theorem. For a function $f(\xi)$ on the circle, define

$$f(\xi\pm)=\lim_{h\to 0\pm} f(\xi e^{ih})$$

and define the *jump* at ξ to be

$$j(\xi)=f(\xi-)-f(\xi+).$$

We shall call $f(\xi)$ *piecewise* C^2 if f is C^2 on the complement of a finite number of points $\xi_1,\dots,\xi_n$ at which $f(\xi\pm)$ and $f'(\xi\pm)$ exist. If ξ is a jump of such a function, define the interval

$$I(\xi)=[-\tfrac{1}{2}|j(\xi)|,\tfrac{1}{2}|j(\xi)|]$$

whenever $\xi^2\neq 1$. If $\xi=\pm 1$, take $I(\xi)=[0,\frac{1}{2}j(\xi)]$ (or $[\frac{1}{2}j(\xi),0]$ if $j(\xi)$ is negative). Let $M(\xi)$ be the operator of multiplication by λ on $L_2(I(\xi),d\lambda)$.

5.1. THEOREM. *If* f *is piecewise* C^2 *and* $H(f)$ *is self-adjoint then the absolutely continuous part of* $H(f)$ *is unitarily equivalent to the direct sum*

$$\bigoplus \sum_\xi M(\xi),$$

where ξ *runs over all jumps of* f *with* $\operatorname{Im}\xi\geq 0$.

IMPORTANT REMARK. Note that the symbol f is defined by (1.1), which is slightly non-standard.

We shall require:

5.2. LEMMA. *If* f *is continuous and piecewise* C^2, *then* $H(f)$ *is trace class.*

Proof. Integration by parts shows that the Fourier coefficients c_n of f are $O(n^{-2})$. The result follows from [1, Corollary 1.4]. □

Note that the derivative of f may *jump* at ξ.

Proof of Theorem 5.1. If $H(f)$ is self-adjoint, then $f=\tilde{f}+\bar{h}$ where $h\in H^\infty$. Since $\bar{h}=f-\tilde{f}$, h can have only jump discontinuities, and is therefore continuous by a theorem of Lindelöf [6, p. 60]. It follows that f and $\tilde{f}$ have the same points of discontinuity, so that if ξ is a jump of f then so is $\bar{\xi}$ and $j(\bar{\xi})=\overline{j(\xi)}$. Moreover, the function $g=\frac{1}{2}(f+\tilde{f})$ has the same jumps as f, and of the same magnitude, and satisfies $g=\tilde{g}$. Replacing f by g, *we may assume that* $f=\tilde{f}$.

Let $\xi_1,\dots,\xi_n$ be the jumps of f with $\operatorname{Im}\xi_i\geq 0$. Choose a C^∞ partition of unity $\psi_0,\psi_1,\dots,\psi_n$ such that (i) $\psi_i=\tilde{\psi}_i$, (ii) $\psi_1,\dots,\psi_n$ have disjoint supports, and (iii) ψ_i is identically one on a neighborhood of ξ_i $(i=1,\dots,n)$. Then $\psi_0=1-(\psi_1+\cdots+\psi_n)$ vanishes in a neighborhood of all jumps. Write

$$H(f)=H(\psi_1 f)+\cdots+H(\psi_n f)+H(\psi_0 f).$$

The last term is trace class, and so makes no contribution to the absolutely continuous part. The remaining terms have pairwise trace class products; for example,

$$H(\psi_1 f)H(\psi_2 f)=H(\tilde{\psi}_1\tilde{f})^*H(\psi_2 f)=H(\psi_1 f)^*H(\psi_2 f)$$

is trace class by Lemma 1.1. By Ismagilov's theorem, $H(f)_a$ is unitarily equivalent to the direct sum

$$\bigoplus\sum_{i=1}^{n} H(\psi_i f)_a.$$

It remains to show that $H(\psi_i f)_a$ is unitarily equivalent to $T(\xi_i,\alpha_i)_a$ with $\alpha_i=j(\xi_i)$. The difference of these two operators is Hankel with symbol

$$g=f\psi_i-(\alpha_i f_1+\bar{\alpha}_i\tilde{f}_1).$$

But g, which is obviously piecewise C^2, is also *continuous* since the jumps at ξ_i and $\bar{\xi}_i$ cancel. By Lemma 5.1, $H(g)$ is trace class. □

6. Concluding remarks. The proof has made use of the fact that certain Hankel operators are trace class. We have relied for this on some rather crude sufficient conditions. Much better conditions, even characterizations, for nuclearity of Hankel operators are known [6, Chapter 3]. Essentially, something a little better than differentiability is required, where we have assumed C^2. Thus our theorems (3.1 and 5.1) admit improvement in this direction.

We have not pushed the smoothness hypotheses for two reasons: (1) We desired to present simple, easily stated hypotheses. Both characterizations of nuclearity in [6], for example, involve the *analytic* symbol, while our symbols are *never* analytic, since H^∞ functions do not jump. (2) In general, trace class methods in scattering theory rarely yield the sharpest results, so that we would hope to obtain the best results by some other means.

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