GROUPS OF AUTOMORPHISMS OF HYPERELLIPTIC KLEIN SURFACES OF GENUS THREE

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Dedicated to Prof. E. Linés on the occasion of his academic jubilee

1. Introduction. The problem of determining the group of automorphisms of a surface is a classical one that goes back to Hurwitz [15]. The study of the groups of automorphisms of Klein surfaces has grown in the last ten years. Our goal in this paper is to determine the group of automorphisms of each hyperelliptic Klein surface of genus 3. The same question was solved for genus 1 in [1] and for genus 2 in [9].

In Riemann surfaces these questions were first studied by Wiman [31]. He solves the problem for genus 2. More recently some results about hyperelliptic Riemann surfaces of genus 3 have been obtained by A. and I. Kuribayashi [16, 17, 18].

The techniques used in this paper, involving NEC groups, are different from those of the Riemann case.

We now describe the contents of our paper. In §2 we introduce the terminology about Klein surfaces and NEC groups, and establish a technical result.

Section 3 is devoted to the study of some properties of the automorphisms of Klein surfaces of genus three, and in §4 we introduce the method for checking when a group of automorphisms of an arbitrary bordered compact Klein surface is the full group.

In §5 we obtain the main result. All the groups that are the full group of automorphisms of each hyperelliptic Klein surface of genus 3 are classified according to the topological type of the surface.

From the well-known functorial equivalence established by Alling and Greenleaf [2] between the category of real irreducible algebraic curves and the one of bordered Klein surfaces, and our results [8] about the relation between hyperelliptic Klein surfaces and hyperelliptic real algebraic curves, we will translate in §6 the results obtained for surfaces in §5 to the language of curves.

2. NEC groups and Klein surfaces. Klein surfaces, introduced from a modern point of view by Alling and Greenleaf [2], are studied by means of NEC groups since the results of Preston and May.

If X is a Klein surface of algebraic genus $p \geq 2$, it may be expressed as D/Γ, where $D = \{z \in \mathbf{C} \mid \operatorname{Im}(z) \geq 0\}$ and Γ is an NEC group (see Preston [25]).

NEC groups, introduced by Wilkie [30], are discrete subgroups of the group G of isometries of the hyperbolic plane, with compact quotient space.

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An NEC group may include orientation reversing isometries. NEC groups are classified according to their signature [19], that has the form

$$(g, \pm, [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

and determines a presentation of the group given by generators

(i) $x_1, \dots, x_r$
(ii) $e_1, \dots, e_k$
(iii) $c_{10}, \dots, c_{1s_1}, \dots, c_{k0}, \dots, c_{ks_k}$
(iv) (if sign '+') $a_1, b_1, \dots, a_g, b_g$
(if sign '−') $d_1, \dots, d_g$

and relations

(i) $x_i^{m_i} = 1,\ i = 1, \dots, r$
(ii) $c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1,\ i = 1, \dots, k,\ j = 1, \dots, s_i$
(iii) $e_i^{-1}c_{i0}e_i c_{is_i} = 1,\ i = 1, \dots, k$
(iv) (if sign '+') $x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1$
(if sign '−') $x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_g^2 = 1.$

From now on the letters x, a, b, c, d, e, will be used just for these canonical generators of the group. We call proper periods the numbers m_i, period-cycles the brackets $(n_{i1}, \dots, n_{is_i})$ and period of the period-cycles the numbers n_{ij}.

Γ having the above signature, the surface D/Γ has topological genus g and k boundary components, and is orientable if and only if the sign is '+'. In fact, in the above result of Preston, the group Γ has neither proper periods nor periods in the period-cycles.

May [23] proved that if H is a group of automorphisms of the surface D/Γ, it may be expressed as Γ'/Γ, Γ' being another NEC group. The full group of automorphisms of D/Γ is $N_G(\Gamma)/\Gamma$, $N_G(\Gamma)$ being the normalizer of Γ in G.

A Klein surface is called hyperelliptic if its canonical double covering is hyperelliptic (see [8]).

The following result, that appears in [10], will be used along the paper:

THEOREM 2.1 [10]. *Let Γ be an NEC group with a non-empty period-cycle $(n_1, \dots, n_s)$, and $c_0, \dots, c_s$ the corresponding reflections. Let Γ_0 be a normal subgroup of Γ, with even index N. If $c_i, c_{i+1}, \dots, c_j \in \Gamma_0$, $c_{i-1}, c_{j+1} \notin \Gamma_0$, and n is the index of $c_{i-1}c_{j+1}$ mod Γ_0, then*

(a) *necessarily n_i, n_{j+1} are even, and*
(b) *among the period-cycles of the signature of Γ_0 there are at least $N/2n$ between those that have the forms*

$$\left(\frac{n_i}{2}, n_{i+1}, \dots, n_j, \frac{n_{j+1}}{2}, n_j, \dots, n_{i+1}, \overset{(n)}{\dots}, \frac{n_i}{2}, n_{i+1}, \dots, n_j, \frac{n_{j+1}}{2}, n_j, \dots, n_{i+1}\right)$$

and

$$\left(\frac{n_{j+1}}{2}, n_j, \dots, n_{i+1}, \frac{n_i}{2}, n_{i+1}, \dots, n_j, \overset{(n)}{\dots}, \frac{n_{j+1}}{2}, n_j, \dots, n_{i+1}, \frac{n_i}{2}, n_{i+1}, \dots, n_j\right).$$

3. Some results on Klein surfaces of genus three. First of all, we obtain the possible orders of the automorphisms of a compact bordered Klein surface $X = D/\Gamma$ of genus 3, according to the signature of Γ.

If X is a compact Klein surface of topological genus g and $k \neq 0$ boundary components, its algebraic genus p is $2g+k-1$ when X is orientable, and $g+k-1$ when X is non-orientable.

Thus, if X has algebraic genus 3, Γ has one of the following signatures:

$$(0, +, [\text{—}], \{(\text{—}), (\text{—}), (\text{—}), (\text{—})\}),$$
$$(1, +, [\text{—}], \{(\text{—}), (\text{—})\}),$$
$$(3, -, [\text{—}], \{(\text{—})\}),$$
$$(2, -, [\text{—}], \{(\text{—}), (\text{—})\}),$$
$$(1, -, [\text{—}], \{(\text{—}), (\text{—}), (\text{—})\}).$$

THEOREM 3.1. *If X is a compact non-orientable Klein surface with boundary, of algebraic genus* 3, *then:*

(i) *If X has one boundary component, the order of each automorphism is* 2, 3, *or* 4.

(ii) *If X has two boundary components, the order of each automorphism is* 2 *or* 4.

(iii) *If X has three boundary components, the order of each automorphism is* 2, 3, 4, *or* 6.

In all cases for each value there is a Klein surface with an automorphism of this order.

Proof. (i) X being D/Γ, from [6] the maximum order of an automorphism in this case is 4, and the value 4 is attained. To show that the order 3 is attainable, we consider an NEC group Γ' with signature $(1, -, [3], \{(\text{—})\})$ and the epimorphism θ from Γ' onto $Z/3$ given by $\theta(x_1)=x$, $\theta(e_1)=x$, $\theta(c_1)=1$, $\theta(d_1)=x^2$. By [3], $\ker\theta \simeq \Gamma$.

(ii) From [12] the order of an automorphism is a power of 2. As it is at most 6 [22], it suffices to meet an automorphism of order 4. If $X=D/\Gamma$, we take an NEC group Γ' with signature $(0, +, [\text{—}], \{(2,2)(\text{—})\})$ and the epimorphism θ from Γ' onto $Z/4$ defined by $\theta(e_1)=x$, $\theta(e_2)=x^3$, $\theta(c_{10})=x^2$, $\theta(c_{11})=1$, $\theta(c_{12})=x^2$, $\theta(c_{20})=x^2$. Furthermore, as $e_1^2c_{20}$ belongs to $\ker\theta$, by [14] the signature of $\ker\theta$ has the sign '$-$'. Using the fundamental region of Γ' we obtain a region of $\ker\theta$ having two holes without non-empty period-cycles, and so $\ker\theta$ is Γ.

(iii) By [12] each order is $2^\alpha 3^\beta$ and by [22] the highest possible value is 6. It suffices now to construct an automorphism of order 6, and another one of order 4.

For the order 6, we take Γ' with signature $(0, +, [6], \{(2,2)\})$ and the epimorphism θ from Γ' onto $Z/6$ given by $\theta(x_1)=x$, $\theta(e_1)=x^5$, $\theta(c_1)=\theta(c_3)=x^3$, $\theta(c_2)=1$.

On the other hand, taking an NEC group Γ' with signature

$$(0, +, [\text{—}], \{(2,2)(\text{—})\}),$$

the epimorphism θ from Γ' onto $Z/4$ is defined by $\theta(e_1)=x$, $\theta(e_2)=x^3$, $\theta(c_{10})=x^2$, $\theta(c_{11})=1$, $\theta(c_{12})=x^2$, $\theta(c_{20})=1$.

In both cases it is easy to check, using the same arguments in the cases above, that $\ker\theta \simeq \Gamma$, X being D/Γ. □

THEOREM 3.2. *If X is a compact orientable Klein surface with boundary, of algebraic genus* 3, *then:*

(i) *If X has four boundary components, the order of each automorphism is* 2, 3, *or* 4.

(ii) *If X has two boundary components, the order of each automorphism is* 2, 3, 4, *or* 6.

In both cases for each value there is a Klein surface with an automorphism of this order.

Proof. (i) It has been already proved in [7].

(ii) If X has two boundary components, from [12] we know that the order of each automorphism is $2^\alpha 3^\beta$. From [22], the order is at most $2g=6$. So it is enough to find an automorphism having order 6, and another one of order 4. To do that, writing $X=D/\Gamma$, we need to find an NEC group Γ' and an epimorphism θ from Γ' onto Z/m, $m=4$ or 6, whose kernels were isomorphic to Γ.

First, let $m=6$. If Γ' is an NEC group with signature $(0, +, [2,6], \{(—)\})$, we choose the epimorphism θ from Γ' onto $Z/6$ given by $\theta(x_1)=x$, $\theta(x_2)=x^3$, $\theta(e_1)=x^2$, $\theta(c_1)=1$.

When $m=4$, Γ' has signature $(0, +, [4,4], \{(—)\})$, and now we consider θ from Γ' onto $Z/4$ defined by $\theta(x_1)=x$, $\theta(x_2)=x$, $\theta(e_1)=x^2$, $\theta(c_1)=1$.

In both cases, it is easy to check that $\ker\theta \simeq \Gamma$. □

REMARK. We will see later that not all possible orders of automorphisms of each type of Klein surfaces of genus 3 are realized for hyperelliptic surfaces.

As we saw in §2, if G is a group of automorphisms of the Klein surface D/Γ, there exists an NEC group Γ' such that $\Gamma'/\Gamma = G$. We are going now to deduce properties on the signature of Γ' if G is to be a group of automorphisms of a Klein surface of genus 3.

THEOREM 3.3. *Let $X=D/\Gamma$ be a Klein surface of genus* 3 *and $G=\Gamma'/\Gamma$ a group of automorphisms of X. If the signature of Γ' has no empty period-cycle, it has at least two consecutive periods equal to two in a period-cycle.*

Proof. As Γ is a subgroup of Γ' having period-cycles, all of them empty, Γ' must have period-cycles. Since Γ' has no empty period-cycle, every period-cycle in its signature has at least one period. In the first place we consider that every period-cycle has a unique period: let $(m_1), (m_2), \dots, (m_p)$ be the period-cycles. Then we have c_{i1}, c_{i2} such that $c_{i1}^2 = c_{i2}^2 = (c_{i1}c_{i2})^{m_i} = 1$ for each $i=1,\dots,p$. From the relation $e_i^{-1}c_{i1}e_ic_{i2}=1$ we conclude that for every i, c_{i1} and c_{i2} belong to Γ, or none of them belong to Γ. If both c_{i1}, c_{i2} are in Γ then their product is in Γ because Γ is a subgroup, and m_i appears as a period in a period-cycle. If, on the other hand, no c_{ij} is in Γ, then, since Γ is normal, it contains no reflections and so no period-cycles.

Thus there must be period-cycles with at least two periods. As Γ has period-cycles, there exist $c_1, \dots, c_p \in \Gamma'$ such that $c_1, \dots, c_p \in \Gamma$. Let $i \leq p$. Without loss of generality we may suppose that there exist $c, c' \in \Gamma'$ such that $c^2 = c'^2 = c_i^2 = (cc_i)^{m_1} = (c_i c')^{m_2} = 1$. Neither c nor c' belong to Γ; otherwise, among the periods of the period-cycles of Γ there would appear m_1 or m_2, which is impossible. Let k be the least integer such that $(cc')^k \in \Gamma$. Using Theorem 2.1, m_1 and m_2 are even numbers and $m_1/2$ and $m_2/2$ appear as periods of a period-cycle of Γ, excepting if $m_1 = m_2 = 2$.

Consequently we have found a couple of consecutive periods equal to two in a period-cycle. □

THEOREM 3.4. *Let $X = D/\Gamma$ be a Klein surface, and $G = \Gamma'/\Gamma$ a group of automorphisms of X. Then each proper period and each period of the period-cycles of Γ' equals the order of some element of G.*

Proof. If there were proper periods in Γ' not satisfying the condition, by [3 and 4] there would appear proper periods in the signature of Γ. When G has odd order, all the period-cycles of Γ are empty [3]. Finally, G having even order, if there is a period m in a period-cycle, m different to all the orders of the elements of G, we have two reflections c_1, c_2 such that $(c_1 c_2)^m = 1$. If m is odd, either $c_1, c_2 \in \Gamma$, and then m is a period of a period-cycle of Γ, or $c_1, c_2 \notin \Gamma$; in this case $(c_1 c_2)^q \in \Gamma$, $q \mid m$, $q \neq m$, and then m/q is a proper period of Γ [4]. Lastly, if m is even, then $c_1, c_2 \in \Gamma$ or $c_1, c_2 \notin \Gamma$, and these cases are solved as above, or $c_1 \in \Gamma$, $c_2 \notin \Gamma$, and then $m/2$ is a period of a period-cycle of Γ. □

Let $|\Delta|$ be the area of a fundamental region of the NEC group Δ (see [27]). If Δ has signature

$$(g, \pm, [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}),$$

we have

$$|\Delta| = 2\pi\left(\alpha g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

α being 2 for sign '+' and 1 for sign '−'.

If $G = \Gamma'/\Gamma$, then $\text{order}(G) = |\Gamma|/|\Gamma'|$. In our case, $X = D/\Gamma$ being a Klein surface of genus 3, $|\Gamma| = 2$.

On the other hand, May [21] proved that the order of a group of automorphisms of a Klein surface of genus 3 is at most 24; and if it is less than 24, it is at most 16.

From Theorems 3.1 and 3.2 we know, furthermore, that the possible orders of G are 2, 3, 4, 6, 8, 9, 12, 16 or 24.

Keeping in mind that $|\Gamma'| = 2/\text{order}(G)$ and Theorems 3.3 and 3.4, by means of a simple but long calculation we obtain the possible signatures of Γ' in each case. This result is summarized in Table 1.

REMARK. Note that the order 9 does not appear in Table 1, because there is no NEC group Γ' satisfying the above conditions.

Order of G	Signatures of Γ'	
24	$(0, +, [-], \{(2,2,2,3)\})$	
16	$(0, +, [-], \{(2,2,2,4)\})$	
12	$(0, +, [2,3], \{(-)\})$	$(0, +, [-], \{(2,2,3,3)\})$
	$(0, +, [3], \{(2,2)\})$	$(0, +, [-], \{(2,2,2,6)\})$
8	$(0, +, [-], \{(2,2,2,2,2)\})$	$(0, +, [4], \{(2,2)\})$
	$(0, +, [2], \{(2,2,2)\})$	$(0, +, [-], \{(2)(-)\})$
	$(0, +, [-], \{(2,2,4,4)\})$	$(0, +, [2,4], \{(-)\})$
6	$(0, +, [2,6], \{(-)\})$	$(0, +, [-], \{(2,2,2,2,3)\})$
	$(0, +, [6], \{(2,2)\})$	$(0, +, [3,3], \{(-)\})$
	$(0, +, [2], \{(2,2,3)\})$	$(0, +, [-], \{(2,2,6,6)\})$
4	$(0, +, [-], \{(2,2,2,2,2,2)\})$	$(0, +, [-], \{(2,2)(-)\})$
	$(0, +, [2], \{(2,2,2,2)\})$	$(1, -, [-], \{(2,2)\})$
	$(0, +, [-], \{(2,2,2,4,4)\})$	$(0, +, [2,2,2], \{(-)\})$
	$(0, +, [-], \{(2,2,4,2,4)\})$	$(0, +, [4,4], \{(-)\})$
	$(0, +, [2,2], \{(2,2)\})$	$(1, -, [2], \{(-)\})$
	$(0, +, [4], \{(2,2,2)\})$	$(0, +, [2], \{(-)(-)\})$
3	$(0, +, [3], \{(-)(-)\})$	$(1, -, [3], \{(-)\})$
2	$(0, +, [-], \{(-)(-)(-)\})$	$(0, +, [2], \{(2,2,2,2,2,2)\})$
	$(0, +, [2,2], \{(-)(-)\})$	$(0, +, [-], \{(2,2,2,2,2,2,2,2)\})$
	$(0, +, [2,2,2,2], \{(-)\})$	$(1, -, [-], \{(-)(-)\})$
	$(0, +, [2], \{(2,2)(-)\})$	$(1, -, [2,2], \{(-)\})$
	$(0, +, [-], \{(2,2,2,2)(-)\})$	$(1, -, [2], \{(2,2)\})$
	$(0, +, [-], \{(2,2)(2,2)\})$	$(1, -, [-], \{(2,2,2,2)\})$
	$(0, +, [2,2,2], \{(2,2)\})$	$(1, +, [-], \{(-)\})$
	$(0, +, [2,2], \{(2,2,2,2)\})$	$(2, -, [-], \{(-)\})$

Table 1.

4. On the full group of automorphisms of a Klein surface. The purpose of this section is to obtain some results that allow us to decide when a group of automorphisms of a Klein surface is the full group of its automorphisms.

Given an NEC group Γ, we denote by $R(\Gamma, G)$ the set of isomorphisms $r: \Gamma \to G$ such that $r(\Gamma)$ is discrete and $D/r(\Gamma)$ is compact. Two elements $r_1, r_2 \in R(\Gamma, G)$ are said to be equivalent if for each $\gamma \in \Gamma$, there exists $g \in G$ verifying $r_1(\gamma) = g r_2(\gamma) g^{-1}$. The quotient space $T(\Gamma, G)$, the Teichmüller space of Γ, is homeomorphic to a cell of dimension $d(\Gamma)$. When Γ is a Fuchsian group with signature $(g, +, [m_1, \dots, m_r])$, it is known that $d(\Gamma) = 6(g-1) + 2r$, but Singerman proves in [28] that, if Γ is a proper NEC group, then $d(\Gamma) = \frac{1}{2} d(\Gamma^+)$, where Γ^+ is the canonical Fuchsian group associated to Γ.

The Teichmüller modular group $M(\Gamma)$ of Γ [20] is the quotient $\mathrm{Aut}(\Gamma)/I(\Gamma)$, where $\mathrm{Aut}(\Gamma)$ is the full group of automorphisms of Γ, and $I(\Gamma)$ denotes the inner automorphisms. $M(\Gamma)$ acts as a group of isometries, in the Teichmüller metric, on $T(\Gamma, G)$, and $T(\Gamma, G)/M(\Gamma) = M(\Gamma, G)$ is the modulus space of Γ.

Let σ, σ' be the signatures of two NEC groups. We say $\sigma \leq \sigma'$ (respectively $\sigma \lhd \sigma'$) if there exist NEC groups Γ and Γ' whose signatures are σ and σ', $d(\Gamma) = d(\Gamma')$, and Γ a subgroup of Γ' (respectively Γ a normal subgroup of Γ').

Denoting by σ^+ the signature of the canonical Fuchsian group Γ^+ associated to Γ, it is easy to deduce that two given signatures σ and σ' verify $\sigma \leq \sigma'$ (respectively $\sigma \lhd \sigma'$) if and only if there exist NEC groups Γ and Γ' with signatures σ and σ' verifying $\Gamma \leq \Gamma'$ (respectively $\Gamma \lhd \Gamma'$) and $\sigma^+ \leq \sigma'^+$ (respectively $\sigma^+ \lhd \sigma'^+$).

We will use afterwards the full list of pairs (σ, σ') with $\sigma \lhd \sigma'$ of [5] and the one obtained by Singerman [26] about pairs (σ^+, σ'^+) with $\sigma^+ \leq \sigma'^+$.

EXAMPLE 1. We are going to use the former criteria to show that the connected sum of two projective planes with two boundary components X and the torus with two boundary components Y, with Klein surface structure, both admit $Z/2$ as full group of automorphisms. It will be seen in §5 that these surfaces are not hyperelliptic.

Let Γ be an NEC group with signature $(0, +, [—], \{(2, 2)(2, 2)\})$, that we call σ, and consider the following epimorphisms θ and θ' from Γ onto $Z/2$:

$$\theta(e_1) = \theta(e_2) = \theta(c_{11}) = \theta(c_{13}) = \theta(c_{21}) = \theta(c_{23}) = x, \qquad \theta(c_{12}) = \theta(c_{22}) = 1;$$

$$\theta'(c_{11}) = \theta'(c_{13}) = \theta'(c_{21}) = \theta'(c_{23}) = x, \qquad \theta'(c_{12}) = \theta'(c_{22}) = \theta'(e_1) = \theta'(e_2) = 1.$$

The kernels of θ and θ' are easily seen to be two NEC groups with signature $(2, -, [—], \{(—)(—)\})$ and $(1, +, [—], \{(—)(—)\})$, respectively. This proves that in both cases $Z/2$ is a group of automorphisms of X and Y. As σ^+ is $(1, +, [2, 2, 2, 2])$, from [26] we know that there is no σ'^+, $\sigma^+ \leq \sigma'^+$; thus there is no σ' with $\sigma \leq \sigma'$ and consequently we can choose Γ to be maximal. Therefore $Z/2$ is in both cases the full group of automorphisms.

Now we prove a technical result that will be useful to delete possible groups as full group of automorphisms.

THEOREM 4.1. *Let Γ and Γ' be two* NEC *groups with signatures σ and σ', Γ being a normal subgroup of Γ', and let us call G' the quotient Γ'/Γ. Let Γ'' be an* NEC *group with signature σ'', with $\Gamma \subset \Gamma'' \subset \Gamma'$, and $\sigma'' \lhd \sigma'$. Let us assume that the canonical projection $\Gamma'' \to \Gamma''/\Gamma = G''$ is, up to isomorphisms of Γ'' and G'', the unique epimorphism between those groups having kernel Γ. Then for each pair of* NEC *groups (Γ^*, Γ''^*) with signatures σ and σ'' there exists an* NEC *group Γ'^* whose signature is σ' and $\Gamma^* \lhd \Gamma'^*$.*

Proof. Let $T_{\sigma'}(\Gamma) = \{\tau \in T(\Gamma, G) \mid$ there exists Γ_0 having signature σ' verifying that $\Gamma_0/\tau(\Gamma) = G'$ is a group of automorphisms of $D/\tau(\Gamma)\}$. In a similar way, we define $T_{\sigma''}(\Gamma) = \{\tau \in T(\Gamma, G) \mid$ there exists Γ_0 having signature σ'' verifying that $\Gamma_0/\tau(\Gamma) = G''$ is a group of automorphisms of $D/\tau(\Gamma)\}$.

It is enough to check that $T_{\sigma'}(\Gamma) \supset T_{\sigma''}(\Gamma)$.

From [13] and [20], we know that

$$T_{\sigma'}(\Gamma) = \bigcup_{\bar{\alpha} \in M(\Gamma)} \bar{\alpha}\Bigg(\sum_{i_\phi \in \Phi(\Gamma, \Gamma', \Gamma'/\Gamma)} i_\phi^*(T(\Gamma', G)) \Bigg),$$

and

$$T_{\sigma''}(\Gamma) = \bigcup_{\bar{\alpha} \in M(\Gamma)} \bar{\alpha}\left(\sum_{i_\phi \in \Phi(\Gamma, \Gamma'', \Gamma''/\Gamma)} i_\phi^*(T(\Gamma'', G))\right),$$

where $\Phi(\Gamma, \Delta, \Delta/\Gamma)$ is the family of all equivalence classes of surjections $\phi: \Delta \to \Delta/\Gamma$ with $\ker \phi \simeq \Gamma$ (modulo the actions of $\mathrm{Aut}(\Delta)$ and $\mathrm{Aut}(\Delta/\Gamma)$), and i_ϕ^* is the isometry induced by the inclusion $i_\phi: \ker \phi \to \Delta$. By assumptions, in our case $\Phi(\Gamma, \Gamma'', \Gamma''/\Gamma)$ has a unique element, which is the restriction of an element of $\Phi(\Gamma, \Gamma', \Gamma'/\Gamma)$. So, we need only to observe that $T(\Gamma'', G) \subset T(\Gamma', G)$, and this is clear since $\sigma'' \lhd \sigma'$ and consequently both spaces have the same dimension. □

EXAMPLE 2. We use the theorem above to prove that $Z/3$ is not the automorphism group of any sphere with four boundary components considered as Klein surface. When $Z/3$ is a group of automorphisms of such a surface X, the full group is D_3. It will be seen later that this surface is not hyperelliptic. (In fact, D_3 is not the group of automorphisms of any hyperelliptic Klein surface of genus 3.)

Let us suppose that $Z/3$ is a group of automorphisms of X. Then there would exist a group Γ'' and an epimorphism θ'' from Γ'' onto $Z/3$ having kernel $(0, +, [—], \{(—)(—)(—)(—)\})$. Using Table 1 the signature must be $(0, +, [3], \{(—)(—)\})$ or $(1, -, [3], \{(—)\})$. The second one may be deleted using [3]. Moreover, with the first signature (that we call σ''), up to automorphisms of Γ'' and $Z/3$, the unique epimorphism θ'' from Γ'' onto $Z/3$ having kernel $(0, +, [—], \{(—)(—)(—)(—)\})$ is the following:

$$\theta''(x_1) = x, \qquad \theta''(e_1) = x^2, \qquad \theta''(e_2) = \theta''(c_1) = \theta''(c_2) = 1.$$

Let now σ' be the signature $(0, +, [—], \{(2, 2, 2, 2, 3)\})$, and Γ' an NEC group having signature σ'. We consider the epimorphism θ' from Γ' onto $D_3 = \langle x, y \mid x^2 = y^2 = (xy)^3 = 1\rangle$ given by $\theta'(c_1) = x$, $\theta'(c_2) = 1$, $\theta'(c_3) = y$, $\theta'(c_4) = 1$, $\theta'(c_5) = y$, $\theta'(c_6) = x$. The kernel of θ' is again a group with signature

$$(0, +, [—], \{(—)(—)(—)(—)\}).$$

The preimage of the subgroup of D_3 generated by xy has signature

$$(0, +, [3], \{(—)(—)\}),$$

using the uniqueness of the signature, because its quotient by $\ker \theta'$ is $Z/3$.

So, as $\sigma'' \lhd \sigma'$ [5], we can apply Theorem 4.1 and obtain that whenever $Z/3$ is a group of automorphisms of X, so is D_3. In order to prove that in these cases D_3 is the full group of automorphisms it is enough, repeating the arguments of Example 1, to check that $\sigma'^+ = (0, +, [2, 2, 2, 2, 3])$ does not appear in the list of [26].

5. Groups of automorphisms of hyperelliptic Klein surfaces. Our goal in this section is to determine which groups are the full group of automorphisms of hyperelliptic Klein surfaces of genus 3.

We have introduced hyperelliptic Klein surfaces in §2 and we must recall the following characterization obtained in [8]:

Let Γ be an NEC group with signature $(g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\})$. Then D/Γ is hyperelliptic if and only if there exists a unique NEC group Γ_1 with $|\Gamma_1 : \Gamma| = 2$, and whose signature is:

$$(*)\quad \begin{cases} \text{(i)}\ (0, +, [-], \{(2, \overset{2k}{\ldots}, 2)\}) & \text{if } g = 0, \\ \text{(ii)}\ (0, +, [2, \overset{2g+k}{\ldots}, 2], \{(-)\}) & \text{if } g \neq 0 \text{ and } \Gamma \text{ has sign '+'}. \\ \text{(iii)}\ (0, +, [2, \overset{g}{\ldots}, 2], \{(2, \overset{2k}{\ldots}, 2)\}) & \text{if } \Gamma \text{ has sign '}-\text{'}. \end{cases}$$

Writing $\Gamma_1/\Gamma = \{\mathrm{id}, \rho\}$, it was proved in [8] that ρ, the automorphism of the hyperellipticity, is a central element of the group of automorphisms of D/Γ. So this group has even order.

If we call G the group of automorphisms of D/Γ, there exists an NEC group Γ' such that $G = \Gamma'/\Gamma$. So the strategy to follow is, first of all, look for NEC groups Γ' from which there is an epimorphism θ onto G, having kernel Γ, and such that $\Gamma \triangleleft \Gamma_1 \triangleleft \Gamma'$. For that we seek NEC groups Γ' and epimorphisms θ from Γ' onto G/ρ having kernel Γ_1, and extendable to an epimorphism $\tilde{\theta}$ from Γ' onto G with kernel Γ.

We list now the groups of order up to 24, satisfying the conditions that we have seen to be necessary. The notation for non-abelian groups is the one of [11]:

$$Z/2,\ Z/2\oplus Z/2,\ Z/4,\ Z/6,\ Z/2\oplus Z/2\oplus Z/2,\ Z/2\oplus Z/4,\ D_4,\ Q,$$

$$Z/2\oplus Z/6,\ D_6,\ \langle 2,2,3\rangle,\ Z/2\oplus Z/2\oplus Z/4,\ Z/2\oplus Z/2\oplus Z/2\oplus Z/2,$$

$$Z/4\oplus Z/4,\ Z/2\oplus D_4,\ Z/2\oplus Q,\ \langle 2,2,2\rangle_2,\ (4,4\,|\,2,2),\ \langle 2,2\,|\,4;2\rangle,$$

$$Z/2\oplus Z/2\oplus Z/6,\ Z/2\oplus D_6,\ Z/2\oplus A_4,\ Z/2\oplus\langle 2,2,3\rangle,\ \langle 2,3,3\rangle,\ \langle 4,6\,|\,2,2\rangle.$$

To manage these groups, we shall employ [29]. We shall also use the following presentations for some of these groups:

$$Z/n = \langle x \mid x^n = 1\rangle,$$

$$D_n = \langle x, y \mid x^2 = y^2 = (xy)^n = 1\rangle,$$

$$D_n \oplus Z/2 = \langle x, y, z \mid x^2 = y^2 = (xy)^n = z^2 = (zx)^2 = (zy)^2 = 1\rangle,$$

$$Z/n \oplus Z/2 = \langle x, y \mid x^n = y^2 = 1,\ xy = yx\rangle,$$

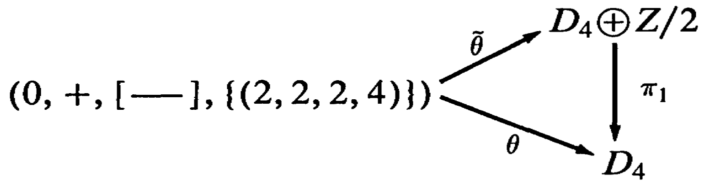
$$Z/2\oplus Z/2\oplus Z/2 = \langle x, y, z \mid x^2 = y^2 = z^2 = 1,\ xy = yx,\ xz = zx,\ yz = zy\rangle.$$

THEOREM 5.1. *Let D/Γ be a hyperelliptic Klein surface Γ having signature* $(0, +, [-], \{(-)(-)(-)(-)\})$. *The group of automorphisms of D/Γ is one of the following: $Z/2$, $Z/2\oplus Z/2$, $Z/2\oplus Z/2\oplus Z/2$,* or *$Z/2\oplus D_4$. Each of these groups is realized as the group of automorphisms of such a surface.*

Proof. From Theorem 3.2 we know that each automorphism of D/Γ has order 2, 3, or 4.

We are going to study the groups that may be the full group of automorphisms of D/Γ. The study is begun with the groups of order 24, and follows the decreasing sequence of orders. Let G be a group of order 24, which is the automorphisms group of D/Γ. Then if ρ belongs to the center of G, ρ having order two, G/ρ is a group of order 12 whose elements have orders 2, 3, or 4. The only group satisfying these conditions is A_4. Then if ρ is the automorphism of the hyperellipticity, there must be an epimorphism θ from the group $(0, +, [\text{—}], \{(2, 2, 2, 3)\})$ onto A_4 having kernel with signature $(0, +, [\text{—}], \{(2, 2, 2, 2, 2, 2, 2, 2)\})$. This is impossible using Theorem 2.1. So the group of automorphisms of D/Γ may not have order 24.

When G has order 16, G/ρ has order 8, and its elements have order 2 or 4. If ρ is the automorphism of the hyperellipticity, there must be an epimorphism θ from $(0, +, [\text{—}], \{(2, 2, 2, 4)\})$ onto G/ρ with kernel with signature $(0, +, [\text{—}], \{(2, 2, 2, 2, 2, 2, 2, 2)\})$. So G/ρ must be generated by elements of order 2, and G/ρ is neither $Z/2 \oplus Z/4$ nor Q. Let G/ρ be D_4. Then the epimorphism θ is defined by $\theta(c_1) = x$, $\theta(c_2) = \theta(c_3) = 1$, $\theta(c_4) = y$, $\theta(c_5) = x$. Thus, G is a group such that G/ρ is D_4; that is, $D_4 \oplus Z/2$, or $(4, 4 \mid 2, 2)$, or $\langle 2, 2 \mid 4; 2\rangle$. The two latter groups are not generated by elements of order 2, and so there is no epimorphism from $(0, +, [\text{—}], \{(2, 2, 2, 4)\})$ onto them. Now we are going to construct an epimorphism $\tilde{\theta}$ from $(0, +, [\text{—}], \{(2, 2, 2, 4)\})$ onto $D_4 \oplus Z/2$ with kernel $(0, +, [\text{—}], \{(\text{—})(\text{—})(\text{—})(\text{—})\})$ making commutative the following diagram:



(From now on, we will say that $\tilde{\theta}$ is compatible with θ when this diagram commutes.) The epimorphism $\tilde{\theta}$ is the following: $\tilde{\theta}(c_1) = x$, $\tilde{\theta}(c_2) = 1$, $\tilde{\theta}(c_3) = z$, $\tilde{\theta}(c_4) = y$, $\tilde{\theta}(c_5) = x$.

Finally G/ρ may not be $Z/2 \oplus Z/2 \oplus Z/2$ because, if there would exist an epimorphism from $(0, +, [\text{—}], \{(2, 2, 2, 4)\})$ onto $Z/2 \oplus Z/2 \oplus Z/2$, the three generators of this group must be the images of three reflections, and so by Theorem 2.1 the kernel is not the desired one.

There is no group of order 12 without elements of order 6 and non-trivial center.

When G has order 8, G/ρ may be $Z/4$ or $Z/2 \oplus Z/2$. The possible signatures are $(0, +, [\text{—}], \{(2, 2, 2, 2, 2)\})$, $(0, +, [2], \{(2, 2, 2)\})$, $(0, +, [\text{—}], \{(2, 2, 4, 4)\})$, $(0, +, [4], \{(2, 2)\})$, $(0, +, [\text{—}], \{(2)(\text{—})\})$ and $(0, +, [2, 4], \{(\text{—})\})$. We are looking for epimorphisms from these groups onto G/ρ having kernel

$$(0, +, [\text{—}], \{(2, 2, 2, 2, 2, 2, 2, 2)\}).$$

This is impossible in the case of the last signature. We begin by the study of the case $Z/4$. The three first signatures are deleted because $Z/4$ is not generated by elements of order two; and the fifth one is impossible because there would be at most four periods in the period-cycle of the kernel. The remaining signature is $(0, +, [4], \{(2, 2)\})$ and the desired epimorphism is the following: $\theta(x_1) = x$, $\theta(e_1) = x^3$, $\theta(c_1) = \theta(c_2) = \theta(c_3) = 1$. Then G must be $Z/2 \oplus Z/4$ and the epimorphism $\tilde{\theta}$ compatible with θ is: $\tilde{\theta}(x_1) = x$, $\tilde{\theta}(e_1) = x^3$, $\tilde{\theta}(c_1) = y$, $\tilde{\theta}(c_2) = 1$, $\tilde{\theta}(c_3) = y$.

If G/ρ is $Z/2 \oplus Z/2$, there is no epimorphism from groups having the five last signatures of the former paragraph onto G/ρ with kernel

$$(0,+,[\text{---}],\{(2,2,2,2,2,2,2,2)\}).$$

On the other hand, we have the following unique epimorphism θ from

$$(0,+,[\text{---}],\{(2,2,2,2,2)\})$$

onto $Z/2 \oplus Z/2$ with the above kernel: $\theta(c_1)=x$, $\theta(c_2)=\theta(c_3)=\theta(c_4)=1$, $\theta(c_5)=y$, $\theta(c_6)=x$. Thus G may be $Z/2 \oplus Z/4$, $Z/2 \oplus Z/2 \oplus Z/2$, D_4 or Q. As Q and $Z/2 \oplus Z/4$ are not generated by elements of order 2, there is not an epimorphism $\tilde{\theta}$ from $(0, +, [\text{---}], \{(2, 2, 2, 2, 2)\})$ onto these groups. There is not an epimorphism $\tilde{\theta}$ from $(0, +, [\text{---}], \{(2, 2, 2, 2, 2)\})$ onto D_4 compatible with θ, and so, by the uniqueness of θ, the group D_4 may be deleted. Finally, we have the epimorphism $\tilde{\theta}$ from $(0, +, [\text{---}], \{(2, 2, 2, 2, 2)\})$ onto $Z/2 \oplus Z/2 \oplus Z/2$, given by $\tilde{\theta}(c_1)=x$, $\tilde{\theta}(c_2)=1$, $\tilde{\theta}(c_3)=z$, $\tilde{\theta}(c_4)=1$, $\tilde{\theta}(c_5)=y$, $\tilde{\theta}(c_6)=x$, that is compatible with θ.

G may not have order 6, because $Z/6$ has elements with order 6.

When G has order 4, G/ρ is $Z/2$. The possible signatures giving periods in the period-cycles of the kernel are

$$(0,+,[\text{---}],\{(2,2,2,2,2,2)\}),\ (0,+,[2],\{(2,2,2,2)\}),$$
$$(0,+,[\text{---}],\{(2,2,2,4,4)\}),\ (0,+,[\text{---}],\{(2,2,4,2,4)\}),$$
$$(0,+,[2,2],\{(2,2)\}),\ (0,+,[4],\{(2,2,2)\}),$$
$$(0,+,[\text{---}],\{(2,2)(\text{---})\}),\quad \text{or} \quad (1,-,[\text{---}],\{(2,2)\}).$$

Groups with the last four signatures do not provide eight periods in the period-cycle of the kernel. As $Z/4$ is not generated by elements of order 2, there is no epimorphism from a group having one of the first four signatures onto $Z/4$. Finally, we have the following epimorphism θ from $(0, +, [2], \{(2, 2, 2, 2)\})$ onto $Z/2$: $\theta(x_1)=\theta(e_1)=x$, $\theta(c_1)=\theta(c_2)=\theta(c_3)=\theta(c_4)=\theta(c_5)=1$; and the epimorphism $\tilde{\theta}$ from $(0, +, [2], \{(2, 2, 2, 2)\})$ onto $Z/2 \oplus Z/2$, compatible with θ, is given by $\tilde{\theta}(x_1)=\tilde{\theta}(e_1)=x$, $\tilde{\theta}(c_1)=y$, $\tilde{\theta}(c_2)=1$, $\tilde{\theta}(c_3)=y$, $\tilde{\theta}(c_4)=1$, $\tilde{\theta}(c_5)=y$.

The group $Z/2$ is a group of automorphisms of D/Γ because it is a hyperelliptic Klein surface, and the corresponding signature is

$$(0,+,[\text{---}],\{(2,2,2,2,2,2,2,2)\}).$$

Now we see which of the possible groups that we have obtained are the full group of automorphisms.

Using the arguments of §4, as the signatures of the canonical Fuchsian groups associated to the NEC groups corresponding with $Z/2$, $Z/2 \oplus Z/2$, and $Z/2 \oplus Z/2 \oplus Z/2$ do not appear in the list of [26], the groups $Z/2$, $Z/2 \oplus Z/2$, and $Z/2 \oplus Z/2 \oplus Z/2$ are the full group of automorphisms of the surface.

Now we consider $Z/2 \oplus Z/4$. Since $\tilde{\theta}$ is unique modulo $\mathrm{Aut}(Z/2 \oplus Z/4)$ and automorphisms of the group having signature $(0, +, [4], \{(2, 2)\})$, and

$$(0,+,[4],\{(2,2)\}) \lhd (0,+,[\text{---}],\{(2,2,2,4)\}),$$

[5], when $Z/2 \oplus Z/4$ is a group of automorphisms, so is $Z/2 \oplus D_4$.

Finally, as $Z/2 \oplus D_4$ has order 16, there is no group of automorphisms strictly containing it. □

THEOREM 5.2. *Let D/Γ be a hyperelliptic Klein surface, Γ having signature $(1, +, [—], \{(—)(—)\})$. The group of automorphisms of D/Γ is one of the following: $Z/2$, $Z/2\oplus Z/2$, $Z/2\oplus Z/2\oplus Z/2$, D_6, or $Z/2\oplus D_4$. Each of these groups is realized as the group of automorphisms of such a surface.*

Proof. From Theorem 3.2 we have that each automorphism of D/Γ has order 2, 3, 4, or 6.

We begin studying the groups of order 24. Let G be a group of order 24, which is the automorphism group of D/Γ. Thus G/ρ is a group of order 12, whose elements have orders 2, 3, 4, or 6, and there is an epimorphism θ from $(0, +, [—], \{(2, 2, 2, 3)\})$ onto G/ρ having kernel $(0, +, [2, 2, 2, 2], \{(—)\})$. It would be $\theta(c_1)=x$, $\theta(c_2)=1$, $\theta(c_3)=y$, $\theta(c_4)=y$, $\theta(c_5)=x$, verifying $\mathrm{order}(xy)=6$, and so we would have six proper periods in the kernel. So the group of automorphisms of D/Γ may not have order 24.

When G has order 16, G/ρ has order 8. The unique possible group of order 8 G/ρ such that there exists an epimorphism θ from $(0, +, [—], \{(2, 2, 2, 4)\})$ onto G/ρ with kernel $(0, +, [2, 2, 2, 2], \{(—)\})$ is D_4. Thus G is $D_4\oplus Z/2$, or $(4, 4 \mid 2, 2)$, or $\langle 2, 2 \mid 4; 2\rangle$. The two latter groups are not generated by elements of order 2, and so there is no epimorphism from $(0, +, [—], \{(2, 2, 2, 4)\})$ onto them. The epimorphism θ from $(0, +, [—], \{(2, 2, 2, 4)\})$ onto D_4 is given by $\theta(c_1)=\theta(c_2)=x$, $\theta(c_3)=1$, $\theta(c_4)=y$, $\theta(c_5)=x$. The epimorphism $\tilde{\theta}$ from $(0, +, [—], \{(2, 2, 2, 4)\})$ onto $D_4\oplus Z/2$, compatible with θ, is $\tilde{\theta}(c_1)=xz$, $\tilde{\theta}(c_2)=x$, $\tilde{\theta}(c_3)=1$, $\tilde{\theta}(c_4)=y$, $\tilde{\theta}(c_5)=xz$.

If G has order 12, G/ρ has order 6; thus G/ρ is $Z/6$ or D_3. The possible signatures would be

$$(0, +, [2,3], \{(—)\}),\quad (0, +, [3], \{(2,2)\}),$$
$$(0, +, [—], \{(2,2,3,3)\}),\quad \text{and}\quad (0, +, [—], \{(2,2,2,6)\}).$$

No group with these signatures can be mapped onto $Z/6$ with kernel

$$(0, +, [2,2,2,2], \{(—)\}).$$

Only in the last case is there an epimorphism onto D_3 having kernel

$$(0, +, [2,2,2,2], \{(—)\}):$$

$\theta(c_1)=y$, $\theta(c_2)=1$, $\theta(c_3)=\theta(c_4)=x$, $\theta(c_5)=y$. Then G must be D_6, and the epimorphism $\tilde{\theta}$ which sends c_1 to y, c_2 to 1, c_3 to $x(xy)^3$, c_4 to x, c_5 to y, is compatible with θ.

When G has order 8, G/ρ is $Z/4$ or $Z/2\oplus Z/2$. The possible signatures are $(0, +, [—], \{(2, 2, 2, 2, 2)\})$, $(0, +, [2], \{(2, 2, 2)\})$, $(0, +, [—], \{(2, 2, 4, 4)\})$, $(0, +, [4], \{(2, 2)\})$, $(0, +, [—], \{(2)(—)\})$, and $(0, +, [2, 4], \{(—)\})$. If G/ρ is $Z/4$, the three first signatures give groups from which there is no epimorphism onto G/ρ, because this is not generated by elements of order 2. The two following signatures do not provide the desired kernel. So we have only the group

$(0, +, [2, 4], \{(\text{—})\})$. Then we have the epimorphism θ onto $Z/4$, given by $\theta(x_1)=1$, $\theta(x_2)=x$, $\theta(e_1)=x^3$, $\theta(c_1)=1$. If G/ρ is $Z/4$, then G is $Z/2\oplus Z/4$, and we have the epimorphism $\tilde{\theta}$, compatible with θ, given by $\tilde{\theta}(x_1)=y$, $\tilde{\theta}(x_2)=x$, $\tilde{\theta}(e_1)=x^3y$, $\tilde{\theta}(c_1)=1$.

If G/ρ is $Z/2\oplus Z/2$, the four last signatures do not provide the desired kernel. Considering $(0, +, [\text{—}], \{(2, 2, 2, 2, 2)\})$, we have the epimorphism θ onto $Z/2\oplus Z/2$, defined by $\theta(c_1)=x$, $\theta(c_2)=1$, $\theta(c_3)=\theta(c_4)=\theta(c_5)=y$, $\theta(c_6)=x$; and if we take $(0, +, [2], \{(2, 2, 2)\})$, we have θ' onto $Z/2\oplus Z/2$, given by $\theta'(x_1)=1$, $\theta'(e_1)=1$, $\theta'(c_1)=x$, $\theta'(c_2)=1$, $\theta'(c_3)=y$, $\theta'(c_4)=x$. Now we are going to seek epimorphisms onto G compatible with θ and θ'. The groups Q and $Z/2\oplus Z/4$ are not generated by elements of order 2; so there are no such epimorphisms onto these groups. If G were D_4, the possible epimorphisms $\tilde{\theta}$ onto D_4, compatible with θ, have a kernel with a unique period-cycle, and there is no $\tilde{\theta}'$ compatible with θ'. Finally, when G is $Z/2\oplus Z/2\oplus Z/2$, we have an epimorphism $\tilde{\theta}$ from $(0, +, [\text{—}], \{(2, 2, 2, 2, 2)\})$ onto $Z/2\oplus Z/2\oplus Z/2$ given by $\tilde{\theta}(c_1)=x$, $\tilde{\theta}(c_2)=1$, $\tilde{\theta}(c_3)=yz$, $\tilde{\theta}(c_4)=y$, $\tilde{\theta}(c_5)=yz$, $\tilde{\theta}(c_6)=x$.

If G has order 6, it must be $Z/6$. There must be an epimorphism onto $Z/6$ having kernel $(1, +, [\text{—}], \{(\text{—})(\text{—})\})$. So the unique possible signature is $(0, +, [2, 6], \{(\text{—})\})$, and the epimorphism is $\tilde{\theta}(x_1)=x^3$, $\tilde{\theta}(x_2)=x$, $\tilde{\theta}(e_1)=x^2$, $\tilde{\theta}(c_1)=1$. The epimorphism θ from $(0, +, [2, 6], \{(\text{—})\})$ onto $G/\rho=Z/3$ is $\theta(x_1)=1$, $\theta(x_2)=x$, $\theta(e_1)=x^2$, $\theta(c_1)=1$.

If G has order 4, G/ρ is $Z/2$. The possible signatures are

$$(0, +, [\text{—}], \{(2,2,2,2,2,2)\}),\quad (0, +, [2], \{(2,2,2,2)\}),$$
$$(0, +, [\text{—}], \{(2,2,2,4,4)\}),\quad (0, +, [\text{—}], \{(2,2,4,2,4)\}),$$
$$(0, +, [2,2], \{(2,2)\}),\quad (0, +, [4], \{(2,2,2)\}),\quad (0, +, [\text{—}], \{(2,2)(\text{—})\}),$$
$$(1, -, [\text{—}], \{(2,2)\}),\quad (0, +, [2,2,2], \{(\text{—})\}),\quad (0, +, [4,4], \{(\text{—})\}),$$
$$(1, -, [2], \{(\text{—})\}),\quad \text{and}\quad (0, +, [2], \{(\text{—})(\text{—})\}).$$

Only four of these groups provide epimorphisms onto $Z/2$ with kernel

$$(0, +, [2,2,2,2], \{(\text{—})\});$$

they are

$$(0, +, [\text{—}], \{(2,2,2,2,2,2)\}),\quad (0, +, [2], \{(2,2,2,2)\}),$$
$$(0, +, [2,2], \{(2,2)\}),\quad \text{and}\quad (0, +, [2,2,2], \{(\text{—})\}).$$

As all these groups are generated by elements of order 2, G may not be $Z/4$. If G is $Z/2\oplus Z/2$, we have θ from $(0, +, [2, 2], \{(2, 2)\})$ onto $Z/2$, given by $\theta(x_1)=\theta(x_2)=\theta(e_1)=1$, $\theta(c_1)=x$, $\theta(c_2)=1$, $\theta(c_3)=x$. The epimorphism $\tilde{\theta}$ onto $Z/2\oplus Z/2$, compatible with θ, is $\tilde{\theta}(x_1)=\tilde{\theta}(x_2)=y$, $\tilde{\theta}(e_1)=1$, $\tilde{\theta}(c_1)=x$, $\tilde{\theta}(c_2)=1$, $\tilde{\theta}(c_3)=x$.

The group $Z/2$ is a group of automorphisms of D/Γ because it is a hyperelliptic Klein surface, and the corresponding signature is $(0, +, [2, 2, 2, 2], \{(\text{—})\})$.

Using the list of [26], we prove that $Z/2$, $Z/2\oplus Z/2$, and $Z/2\oplus Z/2\oplus Z/2$ are the full group of automorphisms. By other side, $Z/2\oplus D_4$ is the full group because no other group contains them.

Finally, in the same way of Theorem 5.1, checking the hypotheses of Theorem 4.1, whenever $Z/2\oplus Z/4$ (respectively $Z/6$) is a group of automorphisms, so is $Z/2\oplus D_4$ (respectively D_6).

THEOREM 5.3. *Let D/Γ be a hyperelliptic Klein surface, Γ having signature* $(3, -, [—], \{(—)\})$. *The group of automorphisms of D/Γ is $Z/2$ or $Z/2\oplus Z/2$. Both groups are realized as the group of automorphisms of such a surface.*

Proof. By Theorem 3.1 the orders of the automorphisms of D/Γ are 2 or 4, and by [6] the group of automorphisms is cyclic or dihedral. Thus G may be $Z/2$, $Z/2\oplus Z/2$, $Z/4$, or D_4.

If $G=D_4$, $G/\rho=Z/2\oplus Z/2$. The possible signatures are

$$(0,+,[—],\{(2,2,2,2,2)\}),\ (0,+,[2],\{(2,2,2)\}),$$
$$(0,+,[—],\{(2,2,4,4)\}),\ (0,+,[4],\{(2,2)\}),$$
$$(0,+,[—],\{(2)(—)\}),\quad \text{and} \quad (0,+,[2,4],\{(—)\}).$$

The unique one that gives us an epimorphism θ onto G/ρ with kernel

$$(0,+,[2,2,2],\{(2,2)\})$$

is $(0, +, [—], \{(2, 2, 4, 4)\})$, and this epimorphism is $\theta(c_1)=\theta(c_2)=x$, $\theta(c_3)=1$, $\theta(c_4)=y$, $\theta(c_5)=x$. But there is no epimorphism $\tilde{\theta}$ from

$$(0,+,[—],\{(2,2,4,4)\})$$

onto D_4 compatible with θ.

If $G=Z/2\oplus Z/2$, $G/\rho=Z/2$. The possible signatures are

$$(0,+,[—],\{(2,2,2,2,2,2)\}),\ (0,+,[2],\{(2,2,2,2)\}),$$
$$(0,+,[—],\{(2,2,2,4,4)\}),\ (0,+,[—],\{(2,2,4,2,4)\}),$$
$$(0,+,[2,2],\{(2,2)\}),\ (0,+,[4],\{(2,2,2)\}),$$
$$(0,+,[—],\{(2,2)(—)\}),\quad \text{and} \quad (1,-,[—],\{(2,2)\}).$$

The groups having one of the last five signatures may not be mapped onto $Z/2$ with kernel $(0, +, [2, 2, 2], \{(2, 2)\})$. If we have $(0, +, [—], \{(2, 2, 2, 2, 2, 2)\})$, the epimorphism θ onto $Z/2$ is given by $\theta(c_1)=x$, $\theta(c_2)=\theta(c_3)=1$, $\theta(c_4)=\theta(c_5)=\theta(c_6)=\theta(c_7)=x$; the epimorphism $\tilde{\theta}$ onto $Z/2\oplus Z/2$, compatible with θ, is $\tilde{\theta}(c_1)=xy$, $\tilde{\theta}(c_2)=1$, $\tilde{\theta}(c_3)=y$, $\tilde{\theta}(c_4)=x$, $\tilde{\theta}(c_5)=xy$, $\tilde{\theta}(c_6)=x$, $\tilde{\theta}(c_7)=xy$.

If $G=Z/4$, the three unique groups not deleted by the above paragraph are generated by elements of order two, and so have no epimorphisms onto $Z/4$.

If $G=Z/2$, it is a group of automorphisms of D/Γ because this is a hyperelliptic Klein surface, and the corresponding signature is $(0, +, [2, 2, 2], \{(2, 2)\})$.

As before, the list of [26] allows us to assure that both $Z/2$ and $Z/2\oplus Z/2$ are the full group of automorphisms. □

THEOREM 5.4. *Let D/Γ be a hyperelliptic Klein surface,* Γ *having signature* $(2, -, [—], \{(—)(—)\})$. *The group of automorphisms of D/Γ is one of the following: $Z/2$, $Z/2 \oplus Z/2$, and $Z/2 \oplus Z/2 \oplus Z/2$. Each of these groups may be realized as the group of automorphisms of such a surface.*

Proof. By Theorem 3.1 the orders of the automorphisms are 2 or 4. So the order of the group is a power of two.

If G has order 16, G/ρ has order 8. So G/ρ is $Z/2\oplus Z/2\oplus Z/2$, D_4, $Z/2\oplus Z/4$, or Q. So there must be an epimorphism θ from $(0, +, [—], \{(2, 2, 2, 4)\})$ onto G/ρ with kernel $(0, +, [2, 2], \{(2, 2, 2, 2)\})$. Thus G/ρ may be generated by elements of order 2, and so we can delete $Z/2\oplus Z/4$ and Q. No epimorphism from $(0, +, [—], \{(2, 2, 2, 4)\})$ onto $Z/2\oplus Z/2\oplus Z/2$ or D_4 has the desired kernel. So G may not have order 16.

When G has order 8, G/ρ must be $Z/4$ or $Z/2\oplus Z/2$. The possible signatures are

$$(0,+,[—],\{(2,2,2,2,2)\}),\quad (0,+,[2],\{(2,2,2)\}),$$
$$(0,+,[—],\{(2,2,4,4)\}),\quad (0,+,[4],\{(2,2)\}),$$
$$(0,+,[—],\{(2)(—)\}),\quad \text{and}\quad (0,+,[2,4],\{(—)\}).$$

We are looking for epimorphisms from these groups onto G/ρ having kernel $(0, +, [2, 2], \{(2, 2, 2, 2)\})$. The last two signatures may not provide proper periods and non-empty period-cycles in the signature of the kernel. As $Z/4$ is not generated by elements of order 2, the three first signatures are deleted. Besides there is no epimorphism from $(0, +, [4], \{(2, 2)\})$ onto $Z/4$ with the stated kernel. If G/ρ is $Z/2\oplus Z/2$, there is no epimorphism for the signatures $(0, +, [2], \{(2, 2, 2)\})$ and $(0, +, [4], \{(2, 2)\})$. Finally we have the epimorphism θ from $(0, +, [—], \{(2, 2, 2, 2, 2)\})$ onto $Z/2\oplus Z/2$ given by $\theta(c_1)=\theta(c_2)=x$, $\theta(c_3)=\theta(c_4)=1$, $\theta(c_5)=y$, $\theta(c_6)=x$; and θ' from

$$(0,+,[—],\{(2,2,4,4)\})$$

onto $Z/2\oplus Z/2$ given by $\theta'(c_1)=\theta'(c_2)=y$, $\theta'(c_3)=x$, $\theta'(c_4)=1$, $\theta'(c_5)=y$. Moreover, G may be generated by elements of order 2, and so G is $Z/2\oplus Z/2\oplus Z/2$ or D_4. There is not an epimorphism $\tilde{\theta}$ nor $\tilde{\theta}'$ from the groups

$$(0,+,[—],\{(2,2,2,2,2)\})\quad\text{and}\quad(0,+,[—],\{(2,2,4,4)\})$$

onto D_4, compatible with θ or θ'. In order to prove that $Z/2\oplus Z/2\oplus Z/2$ is a group of automorphisms of D/ρ, we construct the epimorphism $\tilde{\theta}$ from $(0, +, [—], \{(2, 2, 2, 2, 2)\})$ onto $Z/2\oplus Z/2\oplus Z/2$ given by $\tilde{\theta}(c_1)=x$, $\tilde{\theta}(c_2)=xz$, $\tilde{\theta}(c_3)=z$, $\tilde{\theta}(c_4)=1$, $\tilde{\theta}(c_5)=y$, $\tilde{\theta}(c_6)=x$, that is compatible with θ.

When G has order 4, G/ρ is $Z/2$. The possible signatures are, as formerly,

$$(0,+,[—],\{(2,2,2,2,2,2)\}),\quad(0,+,[2],\{(2,2,2,2)\}),$$
$$(0,+,[—],\{(2,2,2,4,4)\}),\quad(0,+,[—],\{(2,2,4,2,4)\}),$$
$$(0,+,[2,2],\{(2,2)\}),\quad(0,+,[4],\{(2,2,2)\}),$$
$$(0,+,[—],\{(2,2)(—)\}),\quad\text{and}\quad(1,-,[—],\{(2,2)\}),$$

but the two last ones may be deleted because they do not provide proper periods and non-empty period-cycles. We construct the following epimorphisms: θ from $(0, +, [—], \{(2, 2, 2, 2, 2, 2)\})$ onto $Z/2$ given by $\theta(c_1)=\theta(c_2)=\theta(c_3)=x$, $\theta(c_4)=\theta(c_5)=\theta(c_6)=1$, $\theta(c_7)=x$; θ' from $(0, +, [2], \{(2, 2, 2, 2)\})$ onto $Z/2$ defined by $\theta'(x_1)=\theta'(e_1)=1$, $\theta'(c_1)=x$, $\theta'(c_2)=\theta'(c_3)=\theta'(c_4)=1$, $\theta'(c_5)=x$; θ'' from $(0, +, [—], \{(2, 2, 4, 2, 4)\})$ onto $Z/2$, such that $\theta''(c_1)=\theta''(c_2)=\theta''(c_3)=x$, $\theta''(c_4)=\theta''(c_5)=1$, $\theta''(c_6)=x$; and θ''' from $(0, +, [2, 2], \{(2, 2)\})$ onto $Z/2$ given by $\theta'''(x_1)=1$, $\theta'''(x_2)=\theta'''(e_1)=x$, $\theta'''(c_1)=\theta'''(c_2)=\theta'''(c_3)=1$. The two other signatures do not provide epimorphisms. For none of these signatures is there an epimorphism onto $Z/4$ because this group is not generated by elements of order 2. Finally, if $G=Z/2\oplus Z/2$, we have $\tilde{\theta}$ from

$$(0, +, [—], \{(2, 2, 2, 2, 2, 2)\})$$

onto $Z/2\oplus Z/2$, given by $\tilde{\theta}(c_1)=x$, $\tilde{\theta}(c_2)=xy$, $\tilde{\theta}(c_3)=x$, $\tilde{\theta}(c_4)=1$, $\tilde{\theta}(c_5)=y$, $\tilde{\theta}(c_6)=1$, $\tilde{\theta}(c_7)=x$, that is compatible with θ.

The group $Z/2$ is a group of automorphisms of D/Γ, because it is a hyperelliptic Klein surface, and the corresponding signature is

$$(0, +, [2, 2], \{(2, 2, 2, 2)\}).$$

Again, we use the list of [26] to check that

$$Z/2, \quad Z/2\oplus Z/2, \quad \text{and} \quad Z/2\oplus Z/2\oplus Z/2$$

are the full group of automorphisms. □

THEOREM 5.5. *Let D/Γ be a hyperelliptic Klein surface, Γ having signature $(1, -, [—], \{(—)(—)(—)\})$. The group of automorphisms of D/Γ is one of the following: $Z/2$, $Z/2\oplus Z/2$, and D_6. Each of these groups is realized as the group of automorphisms of such a surface.*

Proof. From Theorem 3.1 we know that each automorphism of D/Γ has order 2, 3, 4, or 6.

If G has order 24, G/ρ has order 12. There must be an epimorphism from $(0, +, [—], \{(2, 2, 2, 3)\})$ onto G/ρ; consequently G/ρ must be D_6. But there is no epimorphism from $(0, +, [—], \{(2, 2, 2, 3)\})$ onto D_6 having kernel $(0, +, [2], \{(2, 2, 2, 2, 2, 2)\})$.

When G has order 16, G/ρ has order 8; and since the unique groups of order 8 generated by elements of order 2 are $Z/2\oplus Z/2\oplus Z/2$ and D_4, these are the only groups of order 8 onto which there is an epimorphism from

$$(0, +, [—], \{(2, 2, 2, 4)\}).$$

In both cases the kernel of the epimorphism has at least two proper periods. So G may not have order 16.

If the order of G is 12, G/ρ may be $Z/6$ or D_3. The possible signatures are $(0, +, [2, 3], \{(—)\})$, $(0, +, [3], \{(2, 2)\})$, $(0, +, [—], \{(2, 2, 3, 3)\})$, and $(0, +, [—], \{(2, 2, 2, 6)\})$. With the three first signatures, there is no epimorphism onto G/ρ having a unique proper period in the kernel. As $Z/6$ is not generated by elements of order 2, the unique remaining case is the construction of an epimorphism from (0, +, [—], {(2, 2, 2, 6)}) onto D_3. This is the following: $\theta(c_1)=x$, $\theta(c_2)=\theta(c_3)=1$, $\theta(c_4)=y$, $\theta(c_5)=x$. If G/ρ is D_3, then necessarily G is D_6, and we have the following epimorphism $\tilde{\theta}$ compatible with θ: $\tilde{\theta}(c_1)=x$, $\tilde{\theta}(c_2)=(xy)^3$, $\tilde{\theta}(c_3)=1$, $\tilde{\theta}(c_4)=y$, $\tilde{\theta}(c_5)=x$.

When G has order 8, G/ρ is $Z/4$ or $Z/2\oplus Z/2$. The possible signatures are (0, +, [—], {(2, 2, 2, 2, 2)}), (0, +, [2], {(2, 2, 2)}), (0, +, [—], {(2, 2, 4, 4)}), (0, +, [4], {(2, 2)}), (0, +, [—], {(2)(—)}), and (0, +, [2, 4], {(—)}). Only a group whose signature is (0, +, [—], {(2, 2, 4, 4)}) may provide a unique proper period in the kernel of the epimorphism onto G/ρ, and naturally G/ρ must then be $Z/2\oplus Z/2$. We have the epimorphism θ, given by $\theta(c_1)=y$, $\theta(c_2)=\theta(c_3)=1$, $\theta(c_4)=x$, $\theta(c_5)=y$. As G may be generated by elements of order 2, there are two possibilities, $Z/2\oplus Z/2\oplus Z/2$ and D_4. In both cases there does not exist $\tilde{\theta}$ from (0, +, [—], {(2, 2, 4, 4)}) onto G, compatible with θ, and with kernel (1, −, [—], {(—)(—)(—)}). Thus G does not have order 8.

When G has order 6, G is $Z/6$ and G/ρ is $Z/3$. Thus the unique possible signature having the desired epimorphisms onto G and G/ρ is (0, +, [6], {(2, 2)}). The epimorphism θ from (0, +, [6], {(2, 2)}) onto $Z/3$ is defined by $\theta(x_1)=x$, $\theta(e_1)=x^2$, $\theta(c_1)=\theta(c_2)=\theta(c_3)=1$, and the epimorphism $\tilde{\theta}$, compatible with θ, is $\tilde{\theta}(x_1)=x$, $\tilde{\theta}(e_1)=x^5$, $\tilde{\theta}(c_1)=x^3$, $\tilde{\theta}(c_2)=1$, $\tilde{\theta}(c_3)=x^3$.

If G has order 4, G/ρ is $Z/2$. The possible signatures are

(0, +, [—], {(2, 2, 2, 2, 2, 2)}), (0, +, [2], {(2, 2, 2, 2)}),

(0, +, [—], {(2, 2, 2, 4, 4)}), (0, +, [—], {(2, 2, 4, 2, 4)}),

(0, +, [2, 2], {(2, 2)}), (0, +, [4], {(2, 2, 2)}),

(0, +, [—], {(2, 2)(—)}), and (1, −, [—], {(2, 2)}).

Only with a group having the first signature we may have an epimorphism onto $Z/2$ with kernel (0, +, [2], {(2, 2, 2, 2, 2, 2)}). This epimorphism, θ, is $\theta(c_1)=\theta(c_2)=x$, $\theta(c_3)=\theta(c_4)=\theta(c_5)=\theta(c_6)=1$, $\theta(c_7)=x$. G may not be $Z/4$, because this is not generated by elements of order 2. Then $\tilde{\theta}$ from

(0, +, [—], {(2, 2, 2, 2, 2, 2)})

onto $Z/2\oplus Z/2$, compatible with θ, is $\tilde{\theta}(c_1)=x$, $\tilde{\theta}(c_2)=xy$, $\tilde{\theta}(c_3)=1$, $\tilde{\theta}(c_4)=y$, $\tilde{\theta}(c_5)=1$, $\tilde{\theta}(c_6)=y$, $\tilde{\theta}(c_7)=x$.

$Z/2$ is a group of automorphisms of D/Γ and the corresponding signature is (0, +, [2], {(2, 2, 2, 2, 2, 2)}).

The usual argument assures that $Z/2$ and $Z/2\oplus Z/2$ are the full group of automorphisms. There are not groups containing D_6 and so this is also the full group.

Applying again Theorem 4.1, $Z/6$ is not the full group of automorphisms, because when $Z/6$ is a group of automorphisms of D/Γ, so is D_6.

The results of this section are displayed in Table 2, where * denotes that the respective group is achieved as the group of automorphisms of the surface.

	$Z/2$	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/2 \oplus Z/2$	D_6	$Z/2 \oplus D_4$
Sphere with 4 holes	*	*	*		*
Torus with 2 holes	*	*	*	*	*
Connected sum of 3 projective planes with 1 hole	*	*			
Connected sum of 2 projective planes with 2 holes	*	*	*		
Projective plane with 3 holes	*	*		*	

Table 2.

6. **Real algebraic curves.** We shall call C an irreducible real projective algebraic curve. If we denote $\tilde{C}$ the complexified of C, we will use the following two facts:

(1) A birational smooth model C' of C is homeomorphic to the boundary of the Klein surface $X(C)$ associated to C [2].

(2) $\tilde{C} \setminus C$ is connected if and only if $X(C)$ is nonorientable [24].

The topological type of the curve is so determined by the knowledge of the connectedness of $\tilde{C} \setminus C$ and the number of connected components of C' since these data allow us to know the orientability, the topological genus, and the number of connected components of the boundary of $X(C)$.

So we translate the results of §5, obtaining the group of automorphisms of a real hyperelliptic algebraic curves of genus 3, classified according with its topological type. Table 2 becomes now the following Table 3:

$\tilde{C} \setminus C$	Number of connected components of C'	$Z/2$	$Z/2 \oplus Z/2$	$Z/2 \oplus Z/2 \oplus Z/2$	D_6	$Z/2 \oplus D_4$
Non-connected	4	*	*	*		*
Non-connected	2	*	*	*	*	*
Connected	1	*	*			
Connected	2	*	*	*		
Connected	3	*	*		*	

Table 3.

Notice that only those curves C such that $\tilde{C} \setminus C$ is not connected can have $Z/2 \oplus D_4$ as the group of automorphisms, and we can distinguish among them the M-curves [24] because those are the only ones that may have D_6 as group of automorphisms.

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