# ON THE FREQUENCY OF MULTIPLE VALUES OF A MEROMORPHIC FUNCTION OF SMALL ORDER

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*For George Piranian on the occasion of his alleged retirement*

**Introduction.** We start from Nevanlinna's fundamental inequality

$$(1)\qquad \sum_{j=1}^{q} m(r,a_j) \le \{2+o(1)\}T(r,F) - N_1(r) \quad (r\to\infty,\ r\notin \mathcal{E})$$

and ask for estimates of

$$(2)\qquad N_1(r) = N_1(r;F) = N(r,0;F') + 2N(r,\infty;F) - N(r,\infty;F').$$

Here $F$ is meromorphic and non-rational in $\mathbf{C}$, the $a_j$ are distinct values in $\mathbf{C}\cup\{\infty\}$, and $\mathcal{E}=\mathcal{E}(F)\subset(0,\infty)$ has finite measure. The standard notations and results of value distribution theory used here are explained in the classic texts ([10], [13]). As usual we denote by

$$\delta(a,F) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,F)}$$

the Nevanlinna deficiency of $a$ for $F$.

We consider

$$(3)\qquad \Phi_1(F) = \inf_{A\in\mathcal{L}} \limsup_{\substack{r\to\infty\\ r\in A}} \frac{N_1(r)}{T(r,F)},$$

where $\mathcal{L}$ is the collection of sets $A\subset(0,\infty)$ of density one (cf. [9, p. 205]), rather than the usual index of total ramification $\Phi(F) = \liminf N_1(r)/T(r,F)$, and prove the following

THEOREM 1. *If $F$ has lower order $\mu<\frac{1}{2}$, then*

$$(4)\qquad \Phi_1(F) \ge \cos \pi\mu.$$

As a direct consequence of (4), and the simple inequality $T(r,F')/T(r,F) \le 2+o(1)$ $(r\to\infty,\ r\notin\mathcal{E})$, we have the following

COROLLARY. *If $F$ has only simple poles, then*

$$(5)\qquad \delta(0,F') \le 1-\tfrac{1}{2}\cos\pi\mu \quad (0\le\mu<\tfrac{1}{2}).$$

It is not difficult to achieve $\delta(0,F')=1$ for $F$ of any order $\mu\ge 0$, by allowing $F$ to have poles of arbitrarily high multiplicity.

Our estimates (4) and (5) are unlikely to be sharp: the simple examples $F_\mu(z) = 1/g(z;\mu)$, where $g$ is a Lindelöf function [13, p. 225], have

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$$\Phi_1(F_\mu) = \begin{cases} 1 & \text{for } 0 \le \mu < \frac{1}{2} \\ \sin \pi\mu & \text{for } \frac{1}{2} < \mu < 1 \end{cases}$$

and are probably extremal for these questions. (Examples in [4] suggest the corresponding conjecture for $\mu > 1$.)

Theorem 1 confirms a conjecture of Eremenko [1, p. 3] and others, for $F$ having $\mu < \frac{1}{2}$. We can restate this conjecture as follows: $\Phi_1(F) = 0$ implies

(6) $$2\mu \text{ is an integer} \ge 2, \quad \text{or} \quad \mu = \infty.$$

By (1),

(7) $$\sum \delta(a, F) + \Phi_1(F) \le 2.$$

Drasin [3] has recently succeeded in proving that $\sum \delta(a, F) = 2$ implies (6); but his methods do not seem to apply to Eremenko's problem.

Our conjecture $\Phi_1(F) \ge \Phi_1(F_\mu)$, for $F$ having lower order $\mu < 1$, is consistent with (7) and Edrei's solution [7] of the deficiency problem for these $F$.

Our method gives some information on a closely related problem concerning the Wronskian

(8) $$W = W(f, g) = fg' - f'g$$

of two linearly independent entire functions $f, g$. In case $f, g$ are solutions ($\not\equiv 0$) of

(9) $$y'' + z^m y = 0 \quad (m = \text{integer} \ge 0),$$

then $f$ and $g$ each have order $\lambda = 1 + m/2$, while $W = \text{constant}$. The problem is to show that $W(f, g)$ has growth as large as that of $f$ and $g$, except when $f, g$ both have the same half-integral (or infinite) order $\lambda \ge 1$.

We prove the following

THEOREM 2. *Let $f, g$ be linearly independent entire functions of orders $\lambda_1, \lambda$ respectively, with $\lambda_1 \le \lambda < \frac{1}{2}$ and $g$ transcendental. Then there exists a sequence $t_k \to \infty$ such that*

(10a) $$N(t_k, 0; W) \ge t_k^{\lambda - o(1)} \quad \textit{and}$$

(10b) $$N(t_k, 0; W) \ge (\cos \pi\lambda - o(1)) \max\{T(t_k, f), T(t_k, g)\}.$$

In particular, $W$ has order $\lambda$. Clunie had already shown (10a) (1963, unpublished) in case $\lambda < \frac{1}{3}$. His proof involved integrating $(f/g)' = W(g, f)/g^2$ on circles $|z| = r$ where $L(r, g) = \inf_\theta |g(re^{i\theta})|$ dominates $M(r, g)^{1/2}$. Such $r$ exist, by the classical $\cos \pi\lambda$ theorem, if $\lambda < \frac{1}{3}$. We refine his method by proving a variant of the $\cos \pi\lambda$ theorem more suited to our problem; see Theorem 3 below.

Finally, we recall that the solutions of equation (9) are also closely related to the problem considered in Theorem 1: quotients $F = f/g$ of such solutions have no multiple points at all; that is, $N_1(r; F) \equiv 0$ (cf. [12]).

**1. Proof of Theorem 1.** To avoid obscuring the argument we first carry out the proof when $F = f_1/f_2$ where the $f_j$ are entire functions of genus 0 without common zeros; a standard approximation argument, to be outlined at the end of this section, will complete our discussion.

Choose $a \in \mathbf{C} - \{F(z): F'(z) = 0\}$, $a \neq F(0)$, to satisfy

$$N(r, a; F) \sim T(r, F) \quad (r \to \infty) \tag{1.1}$$

(cf. [13, p. 276]), and put

$$G = \frac{1}{F-a} = \frac{f_2}{f_1 - af_2} = \frac{f}{g}, \quad g(0) = 1. \tag{1.2}$$

Then $f, g$ have no common zeros, $G$ has only simple poles, and

$$N_1(r; F) = N_1(r; G) = N(r, 0; G') = N(r, 0; W), \tag{1.3}$$

where $W = W(f_1, f_2) = W(g, f)$ is defined in (8). Since

$$T(r, F) \sim T(r, G) \sim N(r, \infty; G) = N(r, 0; g) \tag{1.4}$$

it is enough to study

$$N(r, 0; W)/N(r, 0; g) \sim N_1(r; F)/T(r, F) \quad (r \to \infty). \tag{1.5}$$

Let $\{x_k\}$ be a sequence of strong peaks ([11], [8]) of order $\mu$ for $N(r, 0; g)$. Then, by [8], there are $r_k \sim x_k$ and $\eta_k \to 0$ such that, simultaneously,

$$\log L(r_k, g)/T(r_k, g) > \pi\mu \cot \pi\mu - \eta_k \tag{1.6}$$

$$\log L(r_k, g)/\log M(r_k, g) > \cos \pi\mu - \eta_k \tag{1.7}$$

$$T(r_k, g) > T(t, g)(r_k/t)^{\mu}(1 - \eta_k) \tag{1.8}$$

$$\pi\mu \csc \pi\mu N(r_k, 0; g) > \log M(t, g)(r_k/t)^{\mu}(1 - \eta_k) \tag{1.9}$$

for all $t$ in

$$I_k = [\eta_k r_k, r_k/\eta_k]. \tag{1.10}$$

Inequality (1.7) measures the distortion by $g$ of circles $\{|z| = r_k\}$. We require an analogue of (1.7) for certain level curves $\{|g(z)| = R_k\}$.

THEOREM 3. *Let $g$ be entire, of lower order $\mu < \frac{1}{2}$. Then there exist $r_k, \eta_k$ as in* (1.6)-(1.10), *as well as $R_k \to \infty$ and Jordan curves $\gamma_k$ about the origin, such that*

$$|g(z)| \equiv R_k \quad \textit{for all } z \in \gamma_k, \tag{1.11}$$

*where*

$$\gamma_k \subset \{z : r_k > |z| > c_k r_k\}, \quad c_k \to (\cos \pi\mu)^{1/\mu}, \tag{1.12}$$

$$\text{length}\ (\gamma_k) = O(r_k T(r_k, g)^{1/2}), \tag{1.13}$$

*and*

$$R_k = L(r_k, g)^{1+o(1)} \quad (k \to \infty). \tag{1.14}$$

We interpret $(\cos \pi\mu)^{1/\mu} = 1$ when $\mu = 0$.

To prove Theorem 3, put $A_k = \log L(r_k, g)$, choose any

$$\delta_k \in (A_k^{-2/3}, A_k^{-1/3}), \tag{1.15}$$

and put

$$\alpha_k = \exp A_k(1 - A_k^{-1/3}), \quad \beta_k = \exp A_k(1 - A_k^{-2/3}),$$
$$D_k = \{z : |z| \le r_k, \ \log|g(z)| > (1-\delta_k)A_k\}.$$

Since $\delta_k > 0$, $D_k$ contains $\{|z| = r_k\}$. By the maximum principle, $D_k$ is connected. By (1.6) and (1.9) there exist $\xi_k \to 0$ such that

$$\log M(t, g) < (\cos \pi\mu - \xi_k)^{-1} A_k (t/r_k)^{\mu} \quad \text{for } t \in I_k;$$

see (1.10). Thus, if we define $\{c_k\}$ by $(\cos \pi\mu - \xi_k)^{-1} c_k^{\mu} = 1 - A_k^{-1/3}$, then

$$\log M(c_k r_k, g) < \log \alpha_k,$$

so that

$$D_k \subset \{z : c_k r_k < |z| \le r_k\}.$$

(If $\mu = 0$, we argue as above with $\mu$ replaced by a sequence $\mu_k$ decreasing slowly to zero.)

Let $O_k$ be the component of $\mathbf{C} - \bar{D}_k$ which contains $\{|z| < c_k r_k\}$. It is not difficult to see that the boundary $\gamma_k$ of $O_k$ is a Jordan curve (cf. [15, p. 123]), with (1.11) satisfied for $R_k = \exp A_k(1-\delta_k)$.

By Weitsman's estimate of Ahlfors' length-area inequality ([15, p. 120], cf. [14]), there exist $\delta_k$ in (1.15) such that the length $l(\gamma_k)$ of $\gamma_k$ satisfies

$$l(\gamma_k) \le 2\pi r_k T(e r_k, g)^{1/2} (1 - \alpha_k/\beta_k)^{-1/2}.$$

Thus (1.13) follows from (1.8).

To prove Theorem 1, we choose $r_k, \gamma_k$ as above and fix $z_k \in \gamma_k$. Integrating $G' = (f/g)' = W/g^2$ along $\gamma_k$, we have

$$G(z) - G(z_k) = \int_{z_k}^{z} \frac{W(\zeta)}{g(\zeta)^2}\, d\zeta \quad (z, \zeta \in \gamma_k).$$

Thus, by (1.11),

$$\begin{aligned} |g(z_k)f(z) - f(z_k)g(z)| &\le R_k^2 \int_{\gamma_k} \frac{|W(\zeta)|}{R_k^2} |d\zeta| \\ &\le M(r_k, W)\, l(\gamma_k) \end{aligned} \tag{1.16}$$

for all $z \in \gamma_k = \text{bdry}(O_k)$, and the inequality persists for all $z \in O_k \supset \{|z| \le c_k r_k\}$. Now, by (1.4) and (1.12), $g$ has a zero $a_1 \in O_k$ for all large $k$, and $f(a_1) \ne 0$ by (1.2). Thus (1.16) together with (1.13) and (1.14) imply

$$\log L(r_k, g) \le \log M(r_k, W)\{1 + o(1)\} \quad (k \to \infty)$$

and, by (1.6),

$$\liminf_{k \to \infty} \log M(r_k, W)/T(r_k, g) \ge \pi\mu \cot \pi\mu. \tag{1.17}$$

Now let $A \in \mathcal{L}$ be a given set of linear density one: $\int_0^r \chi(t)\, dt = o(r)$, where $\chi$ is the characteristic function of $(0, \infty) - A$, and put

$$C=\limsup_{\substack{r\to\infty\\ r\in A}}\frac{N(r,0;W)}{T(r,g)}. \tag{1.18}$$

By a standard inequality ([8], [11]),

$$\begin{aligned}(1-o(1))\log M(r_k,W)&\le r_k\int_{I_k}\frac{N(t,0;W)}{(t+r_k)^2}\,dt\\ &\le (C+o(1))r_k\int_{I_k\cap A}\frac{T(t,g)}{(t+r_k)^2}\,dt+r_k\int_{I_k-A}\frac{T(t,W)}{(t+r_k)^2}\,dt\\ &\le (C+o(1))T(r_k,g)\int_0^\infty\left(\frac{t}{r_k}\right)^\mu\frac{r_k\,dt}{(t+r_k)^2} \qquad (1.19)\\ &\quad+3T(r_k,g)\int_0^\infty\left(\frac{t}{r_k}\right)^\mu\frac{r_k\chi(t)}{(t+r_k)^2}\,dt\\ &=(\pi\mu\csc\pi\mu)(C+o(1))T(r_k,g)\quad (k\to\infty),\end{aligned}$$

where $I_k$ was defined in (1.10). Here we have used (1.18), (1.4), and (1.8) with the usual lemma on $m(r,g'/g)$ (see [10, p. 36]) to estimate

$$\begin{aligned}T(t,W)&=m(t,g^2G(g'/g-f'/f))\\ &\le 2T(t,g)+m(t,G)+4\log T(2t,g)+4\log T(2t,gG)+O(\log t)\\ &\le\{2+o(1)\}T(r_k,g)(t/r_k)^\mu \quad (t\in I_k).\end{aligned}$$

Using (1.17)-(1.19) we deduce $C\ge\cos\pi\mu$. This with (1.5) implies (4), since $A\in\mathcal{L}$ was arbitrary, and the proof of Theorem 1 is complete when $F$ has genus zero.

In the general case, we put $G=1/(F-a)$ where $a$ satisfies (1.1) and is not a multiple value of $F$. Thus $G$ has only simple poles $\{z_n\}$, and

$$N_1(r;F)=N_1(r;G)=N(r,0;G'),\qquad T(r,F)\sim T(r,G)\sim N(r,\infty;G).$$

Now let $\{x_k\}$ be a sequence of Pólya peaks of order $\mu$ for $N(r,\infty;G)$, so that

$$N(t,\infty;G)\le N(x_k,\infty;G)(t/x_k)^\mu(1+\alpha_k)$$

when $t\in[\alpha_k x_k,\,x_k/\alpha_k]$, for some $\alpha_k$ decreasing to zero. Put

$$g_k(z)=\prod_{|z_n|\le x_k/2\alpha_k}(1-z/z_n)$$

so that $N(r,0;g_k)=N(r,\infty;G)$ for $0<r\le x_k/2\alpha_k$. Notice that the proof of Theorem 1b in [8] (together with the comments at the start of Section 2 there) yields $r_k\sim x_k$ and $\eta_k\ge\alpha_k$, $\eta_k\to 0$ such that

$$\log L(r_k,g_k)/N(r_k,0;g_k)>\pi\mu\cot\pi\mu-\eta_k,$$

$$\pi\mu\csc\pi\mu N(r_k,0;g_k)>\log M(t,g_k)(r_k/t)^\mu(1-\eta_k),$$

$$N(r_k,0;g_k)>T(t,g_k)(r_k/t)^\mu(1-\eta_k)$$

for $t$ in intervals $I_k$ having the form (1.10).

Further, the proof of Theorem 3 given above yields, just as before, $R_k \to \infty$ and Jordan curves $\gamma_k$ about 0, such that (1.11)-(1.14) all hold with $g$ replaced by $g_k$.

We now put $f_k = Gg_k$ and $W_k = W(g_k, f_k)$, so that

$$G' = W_k/g_k^2, \qquad N(r,0;G') = N(r,0;W_k) \quad (0 < r < \infty).$$

Thus we can apply the arguments used for the genus zero case of Theorem 1 already considered, with $f, g, W$ replaced by $f_k, g_k, W_k$, to see that Theorem 1 holds as stated.

**2. Proof of Theorem 2.** We show how to modify the previous arguments to prove (10). Given $f, g$ as in Theorem 2, define

$$T_2(r) = \max\{T(r,f), T(r,g)\}.$$

By a lemma of Pólya [10, p. 103], there exist $x_k \to \infty$ and $\epsilon_k \to 0$ so that

$$T_2(t)/T_2(x_k) \le \begin{cases} (t/x_k)^{\lambda-\epsilon_k} & (1 \le t \le x_k), \\ (t/x_k)^{\lambda+\epsilon_k} & (t > x_k). \end{cases} \tag{2.1}$$

Choose a subsequence $K = \{k_j\}$ so that $T_2(x_k) = T(x_k, g)$ $(k \in K)$, say. By (2.1a), $g$ has order $\lambda$ on $\{x_k\}$, $k \in K$. Then define $F = f/g$ where it is not forbidden that $f, g$ have some common zeros. By [6] and [8],

$$N(x_k, 0; g) \sim T(x_k, g) \quad (k \to \infty,\ k \in K) \tag{2.2}$$

so that $\{x_k\}$ is a sequence of strong peaks of $N(r,0;g)$. Thus by [8] and Theorem 3 there exist $r_k \sim x_k$ so that (1.6)-(1.14) all hold (with $\mu$ replaced by $\lambda$). By increasing the $\eta_k$ in (1.6)-(1.14), if necessary, we can assume that the $\epsilon_k$ in (2.1) satisfy $\eta_k^{\epsilon_k} \to 1$.

The argument of Section 1 now gives (1.16) as before, when $k \in K$. Since (1.16) holds for $z$ inside $\gamma_k$, and $\gamma_k \to \infty$ $(k \in K)$, we can argue as before to obtain

$$\liminf_{\substack{k \to \infty \\ k \in K}} \log M(r_k, W)/T(r_k, g) \ge \pi\lambda \cot \pi\lambda \tag{2.3}$$

provided $g$ has a zero $a_1 \in \mathbf{C}$ that is not a zero of $f$. Otherwise, $f/g$ may still have a pole at some point $a_1$: in that case $F = f_1/g_1$, where $f(z) = (z-a_1)^m f_1(z)$, $g(z) = (z-a_1)^m g_1(z)$ with $f_1(a_1) \ne 0$, $g_1(a_1) = 0$. Since $f_1$, $g_1$, $W(f_1, g_1)$ differ from $f$, $g$, $W(f,g)$ by polynomial factors only, we can apply the previous argument to $f_1, g_1$ to get (2.3) again. Finally, if $F = f/g$ is an entire function, then $W(f,g) = -g^2F'$, where $F' \not\equiv 0$ since $f, g$ are linearly independent. Thus $N(r,0;W) = 2N(r,0;g) + N(r,0;F')$ and, by (2.2),

$$\liminf_{\substack{k \to \infty \\ k \in K}} N(r_k, 0; W)/T(r_k, g) \ge 2,$$

so that (2.3) holds in any case.

Now recall the intervals $I_k$ in (1.10), and define $t_k \in I_k$ by

$$\max_{r \in I_k} N(r,0;W)/r^\lambda = N(t_k, 0; W)/t_k^\lambda.$$

Using the first inequality in (1.19) again, we now deduce

$$(2.4)\qquad \begin{aligned}(1-o(1))\log M(r_k, W) &\le N(t_k, 0; W)\int_{I_k}\left(\frac{t}{t_k}\right)^{\lambda}\frac{r_k\,dt}{(t+r_k)^2}\\ &\le N(t_k, 0; W)(r_k/t_k)^{\lambda}\pi\lambda\csc\pi\lambda.\end{aligned}$$

Since $t_k \in I_k$ we can use (1.8) with (2.3) to see that

$$(1+o(1))\log M(r_k, W) \ge \pi\lambda\cot\pi\lambda T(t_k, g)(r_k/t_k)^{\lambda}$$

when $k\to\infty$ in $K$. This with (2.4) proves the second inequality in (10). Further, for $k\in K$, (2.3) and (2.4) imply

$$N(t_k, 0; W) \ge C_1 T(r_k, g)(t_k/r_k)^{\lambda}$$

with $C_1 = (\cos\pi\lambda)/2$, while (2.1) implies

$$T(r_k, g) \sim T(x_k, g) = T_2(x_k) > c_2 r_k^{\lambda-\epsilon_k},$$

so that

$$N(t_k, 0; W) > 2c t_k^{\lambda-\epsilon_k}(t_k/r_k)^{\epsilon_k} > c t_k^{\lambda-\epsilon_k}$$

for $k\ge k_0$, $k\in K$, since $t_k/r_k \ge \eta_k$.

This proves (10) in case the subsequence $K$ exists; if not, then $T_2(x_k) = T(x_k, f)$ for all large $k$, and we can argue exactly as above with $1/F = g/f$.

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