TENSOR PRODUCTS OF REFLEXIVE SUBSPACE LATTICES

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Is the tensor product of two reflexive subspace lattices again reflexive? The dual question, whether the tensor product of reflexive operator algebras is reflexive, has been the subject of much recent investigation. This note will address the lattice question and the relationship between the two problems. Several special cases in which the lattice question has an affirmative answer will be discussed. In particular, the answer is affirmative if the subspace lattices are the full projection lattices of two injective von Neumann algebras.

Throughout this paper, all Hilbert spaces are separable and all projections are orthogonal projections. For a set $\mathfrak{A}$ of bounded operators on $\mathfrak{H}$ and a set $\mathfrak{L}$ of orthogonal projections, we use the standard notation, $\operatorname{Lat}\mathfrak{A}$ and $\operatorname{Alg}\mathfrak{L}$, to denote the lattice of all projections left invariant by each operator in $\mathfrak{A}$ and the algebra of all operators which leave invariant each projection in $\mathfrak{L}$. Lattices and algebras which satisfy $\mathfrak{L} = \operatorname{Lat}\operatorname{Alg}\mathfrak{L}$ and $\mathfrak{A} = \operatorname{Alg}\operatorname{Lat}\mathfrak{A}$ are called *reflexive.* Reflexive algebras form a subclass of the class of weakly closed algebras and reflexive lattices form a subclass of the class of subspace lattices. (A *subspace lattice* is a lattice of projections which contains 0 and I and which is closed in the strong operator topology.) If a subspace lattice $\mathfrak{L}$ consists of mutually commuting projections, it is called a *commutative subspace lattice* (CSL) and the corresponding algebra, $\operatorname{Alg}\mathfrak{L}$, is called a *CSL-algebra.* By a result in [1], every commutative subspace lattice is reflexive.

If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two weakly closed algebras, $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ will denote the weakly closed algebra generated by all elementary tensors $A_1 \otimes A_2$, where $A_i \in \mathfrak{A}_i$. When needed, the algebra generated by the elementary tensors (the algebraic tensor product) will be denoted by $\mathfrak{A}_1 \odot \mathfrak{A}_2$. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are subspace lattices, $\mathfrak{L}_1 \otimes_s \mathfrak{L}_2$ will denote the smallest subspace lattice which contains all elementary tensors $P_1 \otimes P_2$, where $P_i \in \mathfrak{L}_i$.

The reflexivity of the tensor product of two reflexive algebras would be assured if a stronger result, the algebra tensor product formula,

$$\text{(ATPF)} \qquad \operatorname{Alg}\mathfrak{L}_1 \otimes \operatorname{Alg}\mathfrak{L}_2 = \operatorname{Alg}(\mathfrak{L}_1 \otimes_s \mathfrak{L}_2),$$

were known to be true. Similarly, the analogous problem for reflexive lattices would follow from a lattice tensor product formula,

$$\text{(LTPF)} \qquad \operatorname{Lat}\mathfrak{A}_1 \otimes_s \operatorname{Lat}\mathfrak{A}_2 = \operatorname{Lat}(\mathfrak{A}_1 \otimes \mathfrak{A}_2).$$

Received May 14, 1984.

This research was supported in part by grants from the National Science Foundation, the United States Educational Foundation (Norway), and the Science and Engineering Research Council of the United Kingdom.

Michigan Math. J. 31 (1984).

For self-adjoint reflexive algebras (i.e. von Neumann algebras) the ATPF is known to be valid: it is just a reformulation of Tomita's Tensor Product Commutation Theorem. The non-self-adjoint case has been studied in a series of papers [4, 6, 7, 8, 9, 10, 11] and the formula has been verified under fairly general circumstances, at least for CSL-algebras. For example, if $\mathcal{L}_1$ is a completely distributive CSL and $\mathcal{L}_2$ is an arbitrary subspace lattice, then the ATPF is valid for $\mathcal{L}_1$ and $\mathcal{L}_2$ [11]. ($\mathcal{L}$ is *completely distributive* if it satisfies distributive laws for families of projections of arbitrary cardinality. For a CSL, it is shown in [12] that $\mathcal{L}$ is completely distributive if, and only if, the finite sums of rank-one operators in Alg $\mathcal{L}$ are dense in Alg $\mathcal{L}$ in any of the following topologies: weak, strong, ultraweak, ultrastrong.) Another example is obtained by taking for $\mathcal{L}_1$ the subspace lattice join of a CSL which is generated by finitely many nests and the full projection lattice of an injective von Neumann algebra which commutes with each of the nests. Here again the ATPF is satisfied with arbitrary second factor [7].

The LTPF and the ATPF do not appear to be equivalent statements. In order to elucidate the relationship between the two formulae, it is convenient to define several different lattice tensor products. For that, in turn, we consider four types of lattice join. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of orthogonal projections. (Normally, $\mathcal{L}_1$ and $\mathcal{L}_2$ will be subspace lattices, but that is not necessary for the definitions.) Then:

$\mathcal{L}_1 \vee_a \mathcal{L}_2$ denotes the *algebraic* join: the smallest lattice containing both $\mathcal{L}_1$ and $\mathcal{L}_2$;

$\mathcal{L}_1 \vee_c \mathcal{L}_2$ denotes the *complete* join: the smallest complete lattice containing both $\mathcal{L}_1$ and $\mathcal{L}_2$;

$\mathcal{L}_1 \vee_s \mathcal{L}_2$ denotes the *subspace* join: the smallest subspace lattice containing both $\mathcal{L}_1$ and $\mathcal{L}_2$; and

$\mathcal{L}_1 \vee_r \mathcal{L}_2$ denotes the *reflexive* join: the smallest reflexive lattice containing both $\mathcal{L}_1$ and $\mathcal{L}_2$.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are CSL's which commute with each other, then

$$\mathcal{L}_1 \vee_c \mathcal{L}_2 = \mathcal{L}_1 \vee_s \mathcal{L}_2 = \mathcal{L}_1 \vee_r \mathcal{L}_2.$$

This follows from two facts [1]: a lattice of commuting projections is complete if, and only if, it is a subspace lattice and every CSL is reflexive. In general, the various lattice joins are distinct and only the containments,

$$\mathcal{L}_1 \vee_a \mathcal{L}_2 \subseteq \mathcal{L}_1 \vee_c \mathcal{L}_2 \subseteq \mathcal{L}_1 \vee_s \mathcal{L}_2 \subseteq \mathcal{L}_1 \vee_r \mathcal{L}_2,$$

are always valid. There is one other case of interest in which it is known that $\mathcal{L}_1 \vee_c \mathcal{L}_2 = \mathcal{L}_1 \vee_r \mathcal{L}_2$. This occurs when $\mathcal{L}_1$ is a nest, $\mathcal{L}_2$ is a reflexive, orthocomplemented lattice and $\mathcal{L}_1$ and $\mathcal{L}_2$ commute with each other [5].

On the other hand, it is easy to check that each of the four joins has the same associated reflexive algebra. Indeed,

$$\begin{aligned}\text{Alg } \mathcal{L}_1 \cap \text{Alg } \mathcal{L}_2 &= \text{Alg}(\mathcal{L}_1 \vee_a \mathcal{L}_2) = \text{Alg}(\mathcal{L}_1 \vee_c \mathcal{L}_2) \\ &= \text{Alg}(\mathcal{L}_1 \vee_s \mathcal{L}_2) = \text{Alg}(\mathcal{L}_1 \vee_r \mathcal{L}_2).\end{aligned}$$

From this, it follows immediately that

$$\mathcal{L}_1 \vee_r \mathcal{L}_2 = \text{Lat}(\text{Alg}\ \mathcal{L}_1 \cap \text{Alg}\ \mathcal{L}_2).$$

Let $\mathcal{L}\otimes I = \{P\otimes I \mid P \in \mathcal{L}\}$. It is immediate that if $\mathcal{L}$ is a lattice, a complete lattice, or a subspace lattice, then so is $\mathcal{L}\otimes I$. We then define four tensor products:

$$\mathcal{L}_1 \otimes_\alpha \mathcal{L}_2 = (\mathcal{L}_1\otimes I) \vee_\alpha (I\otimes \mathcal{L}_2), \quad \text{where } \alpha = a, c, s, r.$$

Using the elementary facts about joins which are stated above, we obtain

$$(*) \qquad \mathcal{L}_1\otimes_s \mathcal{L}_2 \subseteq \mathcal{L}_1 \otimes_r \mathcal{L}_2 = \text{Lat}(\text{Alg}(\mathcal{L}_1\otimes_s \mathcal{L}_2)) \subseteq \text{Lat}(\text{Alg}\ \mathcal{L}_1 \otimes \text{Alg}\ \mathcal{L}_2).$$

Thus the LTPF is split by (*) into two separate problems:

(i) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are reflexive lattices, does $\mathcal{L}_1 \otimes_s \mathcal{L}_2 = \mathcal{L}_1 \otimes_r \mathcal{L}_2$?

(ii) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are reflexive lattices, does

$$\text{Lat}(\text{Alg}(\mathcal{L}_1\otimes_s \mathcal{L}_2)) = \text{Lat}(\text{Alg}\ \mathcal{L}_1 \otimes \text{Alg}\ \mathcal{L}_2)?$$

The answer to question (ii) is "yes" if the ATPF holds for $\mathcal{L}_1$ and $\mathcal{L}_2$, so we obtain:

$$\text{(a)} \qquad \text{ATPF} + \mathcal{L}_1\otimes_s \mathcal{L}_2 = \mathcal{L}_1 \otimes_r \mathcal{L}_2 \Rightarrow \text{LTPF} \quad (\text{for } \mathcal{L}_1 \text{ and } \mathcal{L}_2).$$

For a "reverse" implication of this nature, we need to recall the definition of synthetic from [1]: a reflexive lattice $\mathcal{L}$ is *synthetic* if the only ultraweakly closed algebra $\mathcal{A}$ which satisfies Lat $\mathcal{A} = \mathcal{L}$ and $\mathcal{A}\cap\mathcal{A}^* = \mathcal{L}'$ (the commutant of $\mathcal{L}$) is Alg $\mathcal{L}$.

Now, suppose we know that $\mathcal{L}_1$ and $\mathcal{L}_2$ are reflexive lattices which satisfy the LTPF and that $\mathcal{L}_1\otimes_r\mathcal{L}_2$ is synthetic. Let

$$\mathcal{A} = \text{Alg}\ \mathcal{L}_1 \otimes \text{Alg}\ \mathcal{L}_2 \quad \text{and} \quad \mathcal{L} = \mathcal{L}_1\otimes_r \mathcal{L}_2 = \mathcal{L}_1 \otimes_s \mathcal{L}_2.$$

The equality of the two tensor products follows from the hypothesis that $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the LTPF. Actually, the LTPF yields more: we obtain Lat $\mathcal{A} = \mathcal{L}$ from it. We claim that $\mathcal{A}\cap\mathcal{A}^* = \mathcal{L}'$. The containment $\mathcal{A}\cap\mathcal{A}^* \subseteq \mathcal{L}'$ is automatic any time $\mathcal{L} = \text{Lat}\ \mathcal{A}$. The reverse containment is

$$\begin{aligned}
\mathcal{L}' &= (\mathcal{L}_1\otimes_s\mathcal{L}_2)' = \mathcal{L}_1'\otimes\mathcal{L}_2' \\
&= [\text{Alg}\ \mathcal{L}_1 \cap (\text{Alg}\ \mathcal{L}_1)^*] \otimes [\text{Alg}\ \mathcal{L}_2 \cap (\text{Alg}\ \mathcal{L}_2)^*] \\
&\subseteq [\text{Alg}\ \mathcal{L}_1 \otimes \text{Alg}\ \mathcal{L}_2] \cap [(\text{Alg}\ \mathcal{L}_1)^* \otimes (\text{Alg}\ \mathcal{L}_2)^*] \\
&= [\text{Alg}\ \mathcal{L}_1 \otimes \text{Alg}\ \mathcal{L}_2] \cap [\text{Alg}\ \mathcal{L}_1 \otimes \text{Alg}\ \mathcal{L}_2]^* = \mathcal{A}\cap\mathcal{A}^*.
\end{aligned}$$

Now use the hypothesis that $\mathcal{L}_1\otimes_r\mathcal{L}_2$ is synthetic to obtain the ATPF. Thus we have

$$\text{(b)} \qquad \text{LTPF} + \mathcal{L}_1\otimes_r\mathcal{L}_2 \text{ synthetic} \Rightarrow \text{ATPF} \quad (\text{for } \mathcal{L}_1 \text{ and } \mathcal{L}_2).$$

As mentioned earlier, the ATPF is always valid for von Neumann algebras (Alg $\mathcal{L}_1 \otimes$ Alg $\mathcal{L}_2 = \text{Alg}(\mathcal{L}_1\otimes\mathcal{L}_2)$ is a trivial reformulation of $\mathcal{M}_1'\otimes\mathcal{M}_2' = (\mathcal{M}_1\otimes\mathcal{M}_2)'$). It is not known if the LTPF is valid for all von Neumann algebras.

(We shall see below that it is valid for approximately finite dimensional algebras.) It is easy to see that every reflexive, orthocomplemented lattice is synthetic. Indeed, if $\mathcal{L}$ is reflexive and orthocomplemented, then $\mathcal{L} = \text{Proj } \mathfrak{M}$, the lattice of all projections in $\mathfrak{M}$, for some von Neumann algebra $\mathfrak{M}$. If $\mathfrak{A}$ is an ultraweakly closed algebra such that $\text{Lat } \mathfrak{A} = \mathcal{L}$ and $\mathfrak{A} \cap \mathfrak{A}^* = \mathcal{L}' = \mathfrak{M}'$, then $\mathfrak{M}' = \mathfrak{A} \cap \mathfrak{A}^* \subseteq \mathfrak{A} \subseteq \mathfrak{M}'$. (The second containment uses the fact that $\mathcal{L}$ is orthocomplemented.) Thus $\mathfrak{A} = \mathfrak{M}' = \text{Alg } \mathcal{L}$ and $\mathcal{L}$ is synthetic.

Since the ATPF is known for von Neumann algebras we may use (a) to reformulate the LTPF in the von Neumann algebra setting as follows:

$$\text{Proj } \mathfrak{M}_1 \otimes_s \text{Proj } \mathfrak{M}_2 = \text{Proj}(\mathfrak{M}_1 \otimes \mathfrak{M}_2).$$

Thus, while it is elementary to show that $\text{Proj } \mathfrak{M}_1 \otimes_s \text{Proj } \mathfrak{M}_2$ generates $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ as a von Neumann algebra, the question of whether $\text{Proj } \mathfrak{M}_1 \otimes_s \text{Proj } \mathfrak{M}_2$ is the full projection lattice of $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is not at all obvious.

The following results describe several general situations in which the LTPF is valid.

PROPOSITION 1. *Let* $\mathfrak{A}_1 = \text{Alg } \mathcal{L}_1$ *and* $\mathfrak{A}_2 = \text{Alg } \mathcal{L}_2$ *be* CSL-*algebras* (*each with* $\mathcal{L}_i = \text{Lat } \mathfrak{A}_i$). *Assume that one of the lattices is either finite width or completely distributive. Then*

$$\text{Lat}(\text{Alg } \mathcal{L}_1 \otimes \text{Alg } \mathcal{L}_2) = \mathcal{L}_1 \otimes_s \mathcal{L}_2.$$

Proof. Since all commutative subspace lattices are reflexive [1], $\mathcal{L}_1 \otimes_s \mathcal{L}_2 = \mathcal{L}_1 \otimes_r \mathcal{L}_2$. By results in [7] and [11], the ATPF is satisfied for $\text{Alg } \mathcal{L}_1$ and $\text{Alg } \mathcal{L}_2$. Hence, by (a), the LTPF is satisfied. □

PROPOSITION 2. *Let* $\mathcal{L}$ *be an arbitrary reflexive lattice acting on a Hilbert space* $\mathcal{K}$. *Then*

$$\text{Lat}(\text{Alg } \mathcal{L} \otimes \mathcal{B}(\mathcal{H})) = \mathcal{L} \otimes I.$$

Proof. Fix a set of matrix units E_{ij} for $\mathcal{B}(\mathcal{H})$. Every operator T in $\mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathcal{H})$ may be written in a unique way as a matrix of operators with entries in $\mathcal{B}(\mathcal{K})$; we express this as $T = \sum_{i,j} T_{ij} \otimes E_{ij}$ (see [3], I, 2, 3.).

The containment $\mathcal{L} \otimes I \subseteq \text{Lat}(\text{Alg } \mathcal{L} \otimes \mathcal{B}(\mathcal{H}))$ is obvious. For the reverse, let $P \in \text{Lat}(\text{Alg } \mathcal{L} \otimes \mathcal{B}(\mathcal{H}))$. Write $P = \sum_{i,j} P_{ij} \otimes E_{ij}$. For each pair q, r, $I \otimes E_{qr} \in \text{Alg } \mathcal{L} \otimes \mathcal{B}(\mathcal{H})$ and hence leaves P invariant. Compute:

$$\begin{aligned}(I \otimes E_{qr})P &= (I \otimes E_{qr})\Big(\sum_{s,j} P_{sj} \otimes E_{sj}\Big)\\ &= \sum_{s,j} P_{sj} \otimes E_{qr}E_{sj} = \sum_j P_{rj} \otimes E_{qj}\end{aligned}$$

and

$$\begin{aligned}P(I \otimes E_{qr})P &= \Big(\sum_{i,s} P_{is} \otimes E_{is}\Big)(I \otimes E_{qr})\Big(\sum_{t,j} P_{tj} \otimes E_{tj}\Big)\\ &= \sum_{i,j,s,t} P_{is}P_{tj} \otimes E_{is}E_{qr}E_{tj} = \sum_{i,j} P_{iq}P_{rj} \otimes E_{ij}.\end{aligned}$$

Since these two operators are equal, we obtain:

(i) $P_{qq}P_{rj}=P_{rj}$ for all q, r, j.
(ii) $P_{iq}P_{rj}=0$ for all i, q, r, j with $i\neq q$.
Since P is a projection, it is self-adjoint; hence $P_{ij}^*=P_{ji}$ for all i,j (and, in particular, each P_{ii} is self-adjoint). Now, from (ii) we obtain, for $i\neq j$, $0=P_{ji}P_{ij}=P_{ij}^*P_{ij}$. Thus $P_{ij}=0$ for all $i\neq j$ and P is a diagonal matrix. From (i) we get $P_{ii}P_{jj}=P_{jj}$ for all i,j. This immediately yields $P_{ii}=P_{jj}$ for all i,j and $P_{ii}^2=P_{ii}$ for all i. Let Q denote the common value P_{ii}, so that we have $P=Q\otimes I$, where Q is a projection. All that remains is to show that $Q\in\mathcal{L}$.

For any $A\in\operatorname{Alg}\mathcal{L}$, $A\otimes I\in\operatorname{Alg}\mathcal{L}\otimes\mathcal{B}(\mathcal{H})$. So $A\otimes I$ leaves $P=Q\otimes I$ invariant. Thus $QAQ\otimes I=AQ\otimes I$, from which we may conclude that $QAQ=AQ$, for all $A\in\operatorname{Alg}\mathcal{L}$. Since $\mathcal{L}$ is reflexive, $Q\in\mathcal{L}$. □

In Propositions 3 and 4 we shall use the Arveson representation for commutative subspace lattices and Arveson's description of $\operatorname{Lat}(\mathcal{A}\otimes I)$ for certain operator algebras $\mathcal{A}$. (The reference for both these items, which are described briefly below, is [1].)

Let X be a locally compact metric space, let $\leq$ be a transitive and symmetric relation (hereafter called an *order*) on X whose graph is a closed subset of $X\times X$, and let μ be a finite Borel measure on X. If E is a Borel subset of X, P_E will denote the corresponding orthogonal projection on $L^2(X,\mu)$; namely, the multiplication operator associated with the characteristic function of E. A set E is *increasing* if $x\in E$ and $x\leq y$ imply $y\in E$. Let $\mathcal{L}(X,\leq,\mu)=\{P_E\colon E$ is an increasing Borel set$\}$. $\mathcal{L}(X,\leq,\mu)$ is a commutative subspace lattice and every CSL acting on a separable Hilbert space is unitarily equivalent to one of the form $\mathcal{L}(X,\leq,\mu)$.

If $\mathcal{L}$ is a commutative subspace lattice, then the family of ultraweakly closed algebras for which $\operatorname{Lat}\mathcal{A}=\mathcal{L}$ and $\mathcal{A}\cap\mathcal{A}^*=\mathcal{L}'$ has a maximal and a minimal element. The maximal element is, of course, $\operatorname{Alg}\mathcal{L}$. The minimal element, denoted by $\mathcal{A}_{\min}$, is described in detail in [1]. When $\mathcal{L}=\mathcal{L}(X,\leq,\mu)$, $\mathcal{A}_{\min}$ can also be described as the smallest ultraweakly closed algebra $\mathcal{A}$ which contains the L^∞-multiplication algebra on $L^2(X,\mu)$ and for which $\operatorname{Lat}\mathcal{A}=\mathcal{L}$. When $\mathcal{L}$ is synthetic, we have $\mathcal{A}_{\min}=\operatorname{Alg}\mathcal{L}$.

Let $\mathcal{L}$ be a commutative subspace lattice. A projection of the form $E=P-Q$, where $P,Q\in\mathcal{L}$, $Q<P$ is called an *atom from* $\mathcal{L}$ if every projection in $\mathcal{L}$ which is not orthogonal to E contains E. The set of atoms from $\mathcal{L}$ is an at most countable family of mutually orthogonal projections; if I is the sum of all the atoms from $\mathcal{L}$ then $\mathcal{L}$ is said to be *totally atomic*. By results in [8], any totally atomic commutative subspace lattice is synthetic.

With the help of some standard identifications, $\operatorname{Lat}(\mathcal{A}_{\min}\otimes I)$ can be described. We shall assume that $\mathcal{L}=\mathcal{L}(X,\leq,\mu)$ and that I acts on the Hilbert space $\mathcal{H}$. $L^2(X,\leq,\mu)\otimes\mathcal{H}$ is identified with the set of square integrable weakly measurable functions mapping X into $\mathcal{H}$. If $\mathcal{M}$ is the L^∞-multiplication algebra, $(\mathcal{M}\otimes I)'$ can be identified with the multiplication operators associated with bounded $\mathcal{B}(\mathcal{H})$-valued Borel functions defined on X. Further, $\operatorname{Proj}((\mathcal{M}\otimes I)')=\operatorname{Lat}(\mathcal{M}\otimes I)$ can be identified with the projection valued Borel functions on X. Finally, we will say that a projection valued function R on X is *increasing* if there is a subset $N\subseteq X$ with measure 0 so that for $x,y\in X\setminus N$, $x\leq y$ implies $R(x)\leq R(y)$.

If R is a projection valued function, let L_R denote the multiplication operator on $L^2(X,\mu)\otimes\mathcal{H}$; namely, $(L_RF)(x)=R(x)F(x)$, for all $F\in L^2(X,\mu)\otimes\mathcal{H}$. L_R is a projection in $\mathcal{B}(L^2(X,\mu)\otimes\mathcal{H})$. Arveson has proven that $\mathrm{Lat}(\mathcal{A}_{\min}\otimes I)$ is equal to $\{L_R: R \text{ is an increasing projection valued Borel function}\}$. In particular, when $\mathcal{L}$ is synthetic, this gives a description of $\mathrm{Lat}(\mathrm{Alg}\,\mathcal{L}\otimes I)$.

Now suppose $\mathcal{L}=\mathcal{L}(X,\le,\mu)$ and $L_R\in\mathrm{Lat}(\mathcal{A}_{\min}\otimes I)$. Let $N\subseteq X$ be a Borel set for which $\mu(N)=0$ and $x,y\in X\backslash N$, $x\le y$ imply $R(x)\le R(y)$. Suppose Q is an arbitrary projection in $\mathcal{B}(\mathcal{H})$. Let $Y=\{x\in X\backslash N: R\{x\}\ge Q\}$. Since R is a Borel function, Y is a Borel subset of X. If $x\in Y$, $y\in X\backslash N$ and $y\ge x$ then $R(y)\ge R(x)\ge Q$, whence $y\in Y$. It is easy to check that Y differs by a null set (a subset of N, in fact) from an increasing Borel set. Thus $P_Y\in\mathcal{L}(X,\le,\mu)$. The function $\chi_Y(\cdot)Q$ from $X\to\mathrm{Proj}\,\mathcal{B}(\mathcal{H})$ is increasing, hence lies in $\mathrm{Lat}(\mathcal{A}_{\min}\otimes I)$. (In fact, this function corresponds to the elementary tensor $P_Y\otimes Q$.) Finally, note that $\chi_Y(\cdot)Q\le R(\cdot)$, that is $P_Y\otimes Q\le L_R$.

PROPOSITION 3. *Let $\mathcal{L}$ be a totally atomic commutative subspace lattice acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{P}=\mathrm{Proj}\,\mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is another Hilbert space. Then*

$$\mathrm{Lat}(\mathrm{Alg}\,\mathcal{L}\otimes I)=\mathcal{L}\otimes_s\mathcal{P}.$$

Proof. Let $\mathcal{E}=\{E_1,E_2,\dots\}$ be the set of atoms from $\mathcal{L}$. For each $P\in\mathcal{L}$ and $E_n\in\mathcal{E}$, either $PE_n=0$ or $E_n\le P$. As a consequence, for any pair E_n,E_m in $\mathcal{E}$ we have that either every operator of the form $E_nTE_m\in\mathrm{Alg}\,\mathcal{L}$ or that no non-zero operator of the form E_nTE_m lies in $\mathrm{Alg}\,\mathcal{L}$. If $E_nTE_m\in\mathrm{Alg}\,\mathcal{L}$ for all $T\in\mathcal{B}(\mathcal{H})$, write $E_n\ll E_m$. For each n, let X_n be a set whose cardinality is equal to $\dim E_n$, so chosen that $X_n\cap X_m=\varnothing$ whenever $n\ne m$. Let $X=\bigcup X_n$. Then X is a countable set, to which we give the discrete topology. Let μ be a finite measure whose support is all of X. Finally, put an order on X as follows: we shall say $x\le y$ in two situations: when both $x,y\in X_n$ for some n and when $x\in X_n$, $y\in X_m$, and $E_m\ll E_n$. It is easy to check that $\mathcal{L}$ is unitarily equivalent to the subspace lattice $\mathcal{L}(X,\le,\mu)$. Without loss of generality, then, we assume $\mathcal{L}=\mathcal{L}(X,\le,\mu)$.

Since $\mathcal{L}\otimes_s\mathcal{P}\subseteq\mathrm{Lat}(\mathrm{Alg}\,\mathcal{L}\otimes I)$ is obvious, we only have to prove the reverse containment. We use the fact that $\mathcal{L}$ is synthetic (since it is totally atomic) and the description of $\mathrm{Lat}(\mathrm{Alg}\,\mathcal{L}\otimes I)$ given above. Let $L_R\in\mathrm{Lat}(\mathrm{Alg}\,\mathcal{L}\otimes I)$, where $R: X\to\mathrm{Proj}\,\mathcal{B}(\mathcal{K})$ is an increasing projection valued function defined on X. Note that R is constant on each set X_n. (If $x,y\in X_n$ then both $x\le y$ and $y\le x$.) Let R_n be the common value of R on X_n and let $Y_n=\{x\in X: R(x)\ge R_n\}$. Then Y_n is an increasing set. Let $Q_n: X\to\mathrm{Proj}\,\mathcal{B}(\mathcal{K})$ be given by $Q_n(x)=\chi_{Y_n}(x)R_n$. Since L_{Q_n} corresponds to $P_{Y_n}\otimes R_n$, $L_{Q_n}\in\mathcal{L}\otimes_s\mathcal{P}$. Finally, for each x let $Q(x)=\bigvee_n Q_n(x)$. Then Q is increasing and $L_Q=\bigvee_n L_{Q_n}$. It is easy to check that $L_R=L_Q$; thus $L_R\in\mathcal{L}\otimes_s\mathcal{P}$. □

REMARKS. (1) We have actually shown that, when $\mathcal{L}$ is totally atomic, each projection in $\mathcal{L}\otimes_s\mathcal{P}=\mathrm{Lat}(\mathrm{Alg}\,\mathcal{L}\otimes I)$ is a (countable) join of elementary tensors in $\mathcal{L}\otimes_s\mathcal{P}$.

(2) The technique employed in the proof of Proposition 3 does not work for all synthetic lattices. Consider the following example. Let $X=[0,1]$, let μ be

Lebesgue measure, and let $\leq$ be the trivial order (in which x is related to y if, and only if, x equals y). Then $\mathcal{L} = \mathcal{L}(X, \leq, \mu)$ is just the lattice of all projections in the L^∞-multiplication algebra, a synthetic lattice. Now let $R: X \to \mathcal{B}(\mathbf{C}^2)$ be the projection valued function given by:

$$R(x) = \begin{bmatrix} \cos^2 \frac{\pi x}{2} & \cos \frac{\pi x}{2} \sin \frac{\pi x}{2} \\ \cos \frac{\pi x}{2} \sin \frac{\pi x}{2} & \sin^2 \frac{\pi x}{2} \end{bmatrix}.$$

Since every Borel subset of X is increasing, R is an increasing function and hence lies in $\mathrm{Lat}(\mathrm{Alg}\ \mathcal{L} \otimes I)$. But, for any projection $Q \in \mathcal{B}(\mathbf{C}^2)$, $Y = \{x \in X: R(x) \geq Q\}$ is a null set. Thus we cannot write L_R as a join of elementary tensors $P_Y \otimes Q$. All the same, it is true in this example that $\mathrm{Lat}(\mathrm{Alg}\ \mathcal{L} \otimes I) = \mathcal{L} \otimes \mathcal{P}$. We will prove below that the LTPF holds for any two approximately finite dimensional von Neumann algebras. Whenever $\mathcal{L}$ is an orthocomplemented commutative subspace lattice, Alg $\mathcal{L}$ is a von Neumann algebra with abelian commutant and so is, in particular, approximately finite dimensional. Of course, $\mathcal{B}(\mathcal{H})$ is also approximately finite dimensional.

(3) The LTPF may fail if one of the algebras is not reflexive. Suppose that $\mathcal{L}$ is a commutative subspace lattice which is not synthetic. Then $\mathcal{A}_{\min} \neq \mathrm{Alg}\ \mathcal{L}$. From Lemma 2 in [2] or the remarks on page 471 in [1], it follows that $\mathrm{Lat}(\mathcal{A}_{\min} \otimes I) \neq \mathrm{Lat}(\mathrm{Alg}\ \mathcal{L} \otimes I)$. Therefore:

$$\mathrm{Lat}\ \mathcal{A}_{\min} \otimes_s \mathcal{P} = \mathcal{L} \otimes_s \mathcal{P} \subseteq \mathrm{Lat}(\mathrm{Alg}\ \mathcal{L} \otimes I) \subsetneq \mathrm{Lat}(\mathcal{A}_{\min} \otimes I),$$

and the LTPF fails for $\mathcal{A}_{\min}$ and $\mathcal{B}(\mathcal{H})$.

PROPOSITION 4. *Let $\mathcal{L}$ be a nest and let $\mathcal{P} = \mathcal{B}(\mathcal{K})$. Then* $\mathrm{Lat}(\mathrm{Alg}\ \mathcal{L} \times I) = \mathcal{L} \otimes_s \mathcal{P}$.

Proof. The technique used in Proposition 3 will work in this case also. In the interest of clarity, we shall ignore exceptional null sets in the argument which follows. Write $\mathcal{L} = \mathcal{L}(X, \leq, \mu)$, where, by Corollary 1 of Theorem 1.2.2 of [1], we may assume that $\leq$ is a linear ordering on X. Let $R: X \to \mathcal{P}$ be an arbitrary increasing projection valued Borel function. (So L_R is a typical element of $\mathrm{Lat}(\mathrm{Alg}\ \mathcal{L} \otimes I)$. Note that any nest is synthetic.)

Let $\mathfrak{N}$ be the closure in the strong operator topology of the range of R. Observe that $\mathfrak{N}$ is a nest. Let $\mathcal{E} \subseteq \mathfrak{N}$ be a *countable* subset of $\mathfrak{N}$ with the property that each element of $\mathfrak{N}$ is a join of elements of $\mathcal{E}$. Write $\mathcal{E} = \{Q_1, Q_2, \ldots\}$. For each Q_n, let $Y_n = \{x \in X: R(x) \geq Q_n\}$ and let S_n be given by $S_n(x) = \chi_{Y_n}(x) Q_n$. As before, S_n is increasing for all n and each $L_{S_n} = P_{Y_n} \otimes Q_n \in \mathcal{L} \otimes_s \mathcal{P}$. Let $S(x) = \vee_n S_n(x)$, for all x. Then S is an increasing Borel function and it is easy to check that $S(x) = R(x)$, for all x. Since $L_S = \vee_n L_{S_n}$, this shows that $L_R \in \mathcal{L} \otimes_s \mathcal{P}$. Thus $\mathrm{Lat}(\mathrm{Alg}\ \mathcal{L} \otimes I) = \mathcal{L} \otimes_s \mathcal{P}$. □

We now turn to the main result of this paper, the verification of the LTPF in the form $\mathrm{Proj}\ \mathfrak{M} \otimes_s \mathrm{Proj}\ \mathfrak{N} = \mathrm{Proj}(\mathfrak{M} \otimes \mathfrak{N})$ for approximately finite dimensional von Neumann algebras. A von Neumann algebra is *approximately finite* *dimensional* if it is the weak closure of an ascending union of finite dimensional von Neumann algebras. It is known that this condition is equivalent to injectivity and to semi-discreteness, but here it is the presence of the finite dimensional subalgebras which is relevant. The proof will proceed in several steps; the last step, the reduction from the approximately finite dimensional case to the finite dimensional case, follows an argument suggested by E. Størmer, to whom the author wishes to express his thanks.

In the first, and crucial, step we prove $\operatorname{Proj} M_n \otimes_a \operatorname{Proj} M_m = \operatorname{Proj}(M_n \otimes M_m)$ for full matrix algebras. In particular, for matrix algebras the algebraic, the complete, the subspace and the reflexive tensor products all coincide.

M_n acts on the Hilbert space $\mathbf{C}^n$, M_n acts on $\mathbf{C}^m$ and $M_n \otimes M_m$ acts on $\mathbf{C}^n \otimes \mathbf{C}^m \approx \mathbf{C}^{nm}$. It will be convenient to write a vector $x \in \mathbf{C}^n \otimes \mathbf{C}^m$ as an $n \times m$ matrix

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix},$$

where x_{ij} is the coefficient of the basis vector $e_i \otimes e_j$. If $v_1, \ldots, v_k$ are vectors in $\mathbf{C}^n \otimes \mathbf{C}^m$, then $[v_1, \ldots, v_k]$ will denote the orthogonal projection onto the linear span of $\{v_1, \ldots, v_n\}$. If a vector $x \in \mathbf{C}^n \otimes \mathbf{C}^m$ has the special form

$$x = \begin{bmatrix} y_1 z_1 & \cdots & y_1 z_m \\ \vdots & & \vdots \\ y_n z_1 & \cdots & y_n z_m \end{bmatrix},$$

then the one-dimensional projection $[x]$ is an elementary tensor product of two one-dimensional projections in M_n and M_m; namely, $[x] = [y] \otimes [z]$, where $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_m)$. In particular, if x written as a matrix as above has only one non-zero column or only one non-zero row then $[x]$ is an elementary tensor (in fact, one of the form $[y] \otimes [e_k]$ or $[e_j] \otimes [z]$). We shall use this terminology and these elementary facts in the following proposition.

PROPOSITION 5. $\operatorname{Proj} M_n \otimes_a \operatorname{Proj} M_m = \operatorname{Proj}(M_n \otimes M_m)$.

Proof. As usual, we need only prove that each projection in $M_n \otimes M_m$ actually lies in $\operatorname{Proj} M_n \otimes_a \operatorname{Proj} M_m$. Since every projection in $M_n \otimes M_m$ is a finite join of rank-one projections, it suffices to prove that every rank-one projection in $M_n \otimes M_m$ lies in $\operatorname{Proj} M_n \otimes_a \operatorname{Proj} M_m$.

Let $\mathcal{S} = \{x \in \mathbf{C}^n \otimes \mathbf{C}^m : [x] \in \operatorname{Proj} M_n \otimes_a \operatorname{Proj} M_m\}$. It will be sufficient to prove that $\mathcal{S} = \mathbf{C}^n \otimes \mathbf{C}^m$. It is evident that $0 \in \mathcal{S}$ and, from the remarks above, if x has only one non-zero row or column then $x \in \mathcal{S}$. The following induction step plus a routine induction argument yields $\mathcal{S} = \mathbf{C}^n \otimes \mathbf{C}^m$. Let $p \in \{2, 3, \ldots, n\}$ and $q \in \{1, \ldots, m\}$. Assume $\mathcal{S}$ contains every vector whose non-zero entries are confined to the first $p-1$ rows and the first $q-1$ entries of the pth row. We shall prove that $\mathcal{S}$ contains any vector whose non-zero entries are confined to the first $p-1$ rows and the first q entries in the pth row.

Let

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1q} & & \cdots & x_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{p1} & \cdots & x_{pq} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $x_{pq} \neq 0$. As soon as we show that $x \in \mathcal{S}$, the proposition is proven.

At first, assume $x_{iq} \neq 0$ for some $i < p$. Consider four vectors, which we describe as follows:

a: the pq entry is 0, all other entries are the same as the entries in x.

b: all entries are 0 except for the pq entry, which is x_{pq}.

c: all entries in the qth column are zero; all other entries are the same as in x.

d: all entries in the qth column are the same as in x; all other entries are 0.

The induction hypothesis guarantees that $a \in \mathcal{S}$ and $c \in \mathcal{S}$. Since b and d each possess only one non-zero column, they too lie in $\mathcal{S}$. Note that the linear span of $\{a, b\}$ differs from the linear span of $\{c, d\}$. (For example, $b \notin \operatorname{sp}\{c, d\}$—this uses the assumption that some $x_{iq} \neq 0$, $i < p$.) Also, x is an element of both linear spans. Consequently, $[x] = ([a] \vee [b]) \wedge ([c] \vee [d])$. Since $[a]$, $[b]$, $[c]$ and $[d]$ all lie in $\operatorname{Proj} M_n \otimes_a \operatorname{Proj} M_n$, so does $[x]$. Thus $x \in \mathcal{S}$.

If we only wanted to prove $\operatorname{Proj} M_n \otimes_s \operatorname{Proj} M_m = \operatorname{Proj}(M_n \otimes M_m)$, then the proof would be virtually complete. For in the case in which $x_{iq} = 0$ for $i = 1, \dots, p-1$, let y_n be the vector whose entries are the same as in x except for the iq location, where the entry in y_n is $1/n$. Then $y_n \to x$ and so $[y_n] \to [x]$. Since $[y_n] \in \operatorname{Proj} M_n \otimes_s \operatorname{Proj} M_m$ by the considerations above, we obtain

$$[x] \in \operatorname{Proj} M_n \otimes_s \operatorname{Proj} M_m.$$

We now complete the proof for the algebraic tensor product. If $q > 1$ and one of the entries $x_{pj} \neq 0$, $1 \leq j < q$, then we can proceed much as above. Indeed, let a and b be the same vectors as described above; for c, take instead the vector whose entries in the pth row are zero and whose other entries are the same as in x; and for d take the vector whose entries in the pth row are the same as in x and whose other entries are all 0. Once again, $a, b, c, d \in \mathcal{S}$ and

$$[x] = ([a] \vee [b]) \wedge ([c] \vee [d]),$$

whence $x \in \mathcal{S}$ also.

This leaves the case in which x_{pq} is the only non-zero entry in the pth row and in the qth column. If all other $x_{ij} = 0$ then $[x]$ is an elementary tensor and $x \in \mathcal{S}$. So fix i, j with $1 \leq i < p$ and $j \neq q$ and assume that $x_{ij} \neq 0$.

First suppose that there are no other non-zero entries in x and that $j > q$. To simplify notation, the 2×2 matrix $\left[\begin{smallmatrix} y_{iq} & y_{ij} \\ y_{pq} & y_{pj} \end{smallmatrix}\right]$ will denote the $n \times m$ matrix with at most 4 non-zero entries, located in the iq, ij, pq, and pj places. Observe that both $\begin{bmatrix} 1 & x_{ij} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ x_{pq} & -1 \end{bmatrix}$ lie in $\mathcal{S}$. Indeed, if

$$a=\begin{bmatrix} 1 & x_{ij} \\ 0 & 0 \end{bmatrix},\quad b=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\quad c=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad\text{and}\quad d=\begin{bmatrix} 0 & x_{ij} \\ 0 & 1 \end{bmatrix},$$

then each of these vectors lies in $\mathcal{S}$ and the orthogonal projection determined by $\begin{bmatrix} 1 & x_{ij} \\ 0 & 1 \end{bmatrix}$ is equal to $([a]\vee[b])\wedge([c]\vee[d])$. The vector $\begin{bmatrix} -1 & 0 \\ x_{pq} & -1 \end{bmatrix}$ is handled in the same way. With these two special vectors in $\mathcal{S}$, we obtain $\begin{bmatrix} 0 & x_{ij} \\ x_{pq} & 0 \end{bmatrix} \in \mathcal{S}$ by letting

$$a=\begin{bmatrix} 0 & x_{ij} \\ 0 & 0 \end{bmatrix},\quad b=\begin{bmatrix} 0 & 0 \\ x_{pq} & 0 \end{bmatrix},\quad c=\begin{bmatrix} 1 & x_{ij} \\ 0 & 1 \end{bmatrix} \quad\text{and}\quad d=\begin{bmatrix} -1 & 0 \\ x_{pq} & -1 \end{bmatrix}.$$

In case $j<q$ we do the same sort of thing to obtain $\begin{bmatrix} x_{ij} & 0 \\ 0 & x_{pq} \end{bmatrix} \in \mathcal{S}$.

All that remains is the case in which there are additional non-zero entries in x (though not in the pth row or qth column). Here, take a and b as we did originally, that is a is the same as x except for 0 in the pq entry and b has x_{pq} as its only non-zero entry. Using the convention of the preceding paragraph, let $c=\begin{bmatrix} 0 & x_{ij} \\ x_{pq} & 0 \end{bmatrix}$ or $c=\begin{bmatrix} x_{ij} & 0 \\ 0 & x_{pq} \end{bmatrix}$, as appropriate. Let d be the vector whose ij and pq entries are 0 and whose other entries are the same as in x. Then once again $a,b,c,d\in\mathcal{S}$ and $[x]=([a]\vee[b])\wedge([c]\vee[d])$, whence $x\in\mathcal{S}$. This, at last, completes the proof. □

COROLLARY. *If* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are *-isomorphic to* M_n *and* M_m *then*

$$\text{Proj } \mathfrak{M}\otimes_a \text{Proj } \mathfrak{N} = \text{Proj}(\mathfrak{M}\otimes\mathfrak{N}).$$

Proof. $\mathfrak{M}\otimes\mathfrak{N}$ is *-isomorphic to $M_n\otimes M_m$ and the *-isomorphism preserves the lattice operations. □

PROPOSITION 6. *Let* $\{\mathfrak{M}_i\}_{i\in I}$ *and* $\{\mathfrak{N}_j\}_{j\in J}$ *be two families of von Neumann algebras and let* $\mathfrak{M}=\sum^{\oplus}\mathfrak{M}_i$ *and* $\mathfrak{N}=\sum^{\oplus}\mathfrak{N}_j$. *Assume that Proj* $\mathfrak{M}_i\otimes_s \text{Proj } \mathfrak{N}_j = \text{Proj}(\mathfrak{M}_i\otimes\mathfrak{N}_j)$ *for each pair* i,j. *Then* $\text{Proj } \mathfrak{M}\otimes_s \text{Proj } \mathfrak{N} = \text{Proj}(\mathfrak{M}\otimes\mathfrak{N})$.

Proof. For each i and j let $\mathcal{H}_i$ and $\mathcal{K}_j$ be the Hilbert spaces on which $\mathfrak{M}_i$ and $\mathfrak{N}_j$ act and let $\mathcal{H}=\sum^{\oplus}\mathcal{H}_i$; $\mathcal{K}=\sum^{\oplus}\mathcal{K}_j$. Let E_i and F_j be the projections of $\mathcal{H}$ and $\mathcal{K}$ onto $\mathcal{H}_i$ and $\mathcal{K}_j$. Each E_i lies in the center of $\mathfrak{M}$ and each F_j, in the center of $\mathfrak{N}$. Let $P\in\text{Proj}(\mathfrak{M}\otimes\mathfrak{N})$. Then

$$P=\sum_{i,j}(E_i\otimes F_j)P(E_i\otimes F_j)=\bigvee_{i,j}(E_i\otimes F_j)P(E_i\otimes F_j).$$

For each pair i,j, $(E_i\otimes F_j)P(E_i\otimes F_j)\,|\,_{\mathcal{H}_i\otimes\mathcal{K}_j}$ lies in $\text{Proj}(\mathfrak{M}_i\otimes\mathfrak{N}_j)$, hence in $\text{Proj } \mathfrak{M}_i\otimes_s \text{Proj } \mathfrak{N}_j$. Thus $(E_i\otimes F_j)P(E_i\otimes F_j)\in \text{Proj } \mathfrak{M}\otimes_s \text{Proj } \mathfrak{N}$, which in turn implies that $P\in \text{Proj } \mathfrak{M}\otimes_s \text{Proj } \mathfrak{N}$.

COROLLARY. *If* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are finite dimensional von Neumann algebras then* $\text{Proj } \mathfrak{M}\otimes_a \text{Proj } \mathfrak{N} = \text{Proj}(\mathfrak{M}\otimes\mathfrak{N})$.

THEOREM. *Let* $\mathfrak{M}$ and $\mathfrak{N}$ *be two approximately finite dimensional von Neumann algebras. Then* $\text{Proj } \mathfrak{M}\otimes_s \text{Proj } \mathfrak{N} = \text{Proj}(\mathfrak{M}\otimes\mathfrak{N})$.

Proof. Let $\mathfrak{M}_i$ and $\mathfrak{N}_j$ be two nested sequences of finite dimensional sub-von Neumann algebras of $\mathfrak{M}$ and $\mathfrak{N}$ such that $\bigcup_i \mathfrak{M}_i$ and $\bigcup_j \mathfrak{N}_j$ are strongly dense in $\mathfrak{M}$ and $\mathfrak{N}$. Then the algebraic tensor product $(\bigcup_i \mathfrak{M}_i) \odot (\bigcup_j \mathfrak{N}_j)$ is strongly dense in the von Neumann algebra $\mathfrak{M} \otimes \mathfrak{N}$. Note that if

$$T = \sum_{k=1}^{n} A_k \otimes B_k \in \left(\bigcup_i \mathfrak{M}_i\right) \odot \left(\bigcup_j \mathfrak{N}_j\right)$$

then, since the $\mathfrak{M}_i$ and $\mathfrak{N}_j$ are nested, $T \in \mathfrak{M}_i \otimes \mathfrak{N}_j$ for some i, j.

Let $P \in \operatorname{Proj}(\mathfrak{M} \otimes \mathfrak{N})$. As usual, we need to show that $P \in \operatorname{Proj} \mathfrak{M} \otimes_s \operatorname{Proj} \mathfrak{N}$. By the Kaplansky density theorem, there is a net T_ν of positive contractions in $(\bigcup_i \mathfrak{M}_i) \odot (\bigcup_j \mathfrak{N}_j)$ such that $T_\nu \to P$ in the strong operator topology.

For each ν, let P_ν be the spectral projection for T_ν associated with the interval $[1/2, 1]$; that is $P_\nu = \chi_{[1/2,1]}(T_\nu)$. We claim that $P_\nu \to P$ in the strong operator topology. We freely use the following facts, valid for all ν: $T_\nu P_\nu = P_\nu T_\nu$; $\|T_\nu P_\nu x\| \geq \frac{1}{2}\|P_\nu x\|$, for all vectors x; and $\|T_\nu (I - P_\nu)x\| \leq \frac{1}{2}\|(I - P_\nu)x\|$, for all vectors x.

First suppose $x \in P^\perp$. Then we have $T_\nu x \to Px = 0$. Since $\frac{1}{2}\|P_\nu x\| \leq \|T_\nu P_\nu x\| \leq \|T_\nu x\|$, we obtain $P_\nu x \to 0$ also.

Now suppose $x \in P$. So, we have $T_\nu x \to Px = x$. Observe that

$$\begin{aligned} \|T_\nu x\|^2 &= \|T_\nu P_\nu x\|^2 + \|T_\nu (I - P_\nu)x\|^2 \leq \|P_\nu x\|^2 + \tfrac{1}{4}\|(I - P_\nu)x\|^2 \\ &= \|P_\nu x\|^2 + \|(I - P_\nu)x\|^2 - \tfrac{3}{4}\|(I - P_\nu)x\|^2 = \|x\|^2 - \tfrac{3}{4}\|(I - P_\nu)x\|^2 \leq \|x\|^2. \end{aligned}$$

Since $\|T_\nu x\|^2 \to \|x\|^2$, we have $\|(I - P_\nu)x\|^2 \to 0$. Thus $P_\nu x \to x$.

Now, if x is arbitrary, simply write $x = Px + P^\perp x$. Then

$$P_\nu x = P_\nu Px + P_\nu^\perp Px \to Px + 0 = Px.$$

Thus $P_\nu \to P$ strongly, as desired.

Since each T_ν lies in some $\mathfrak{M}_i \otimes \mathfrak{N}_j$, we also have $P_\nu \in \mathfrak{M}_i \otimes \mathfrak{N}_j$, some i, j. Thus $P_\nu \in \operatorname{Proj}(\mathfrak{M}_i \otimes \mathfrak{N}_j)$. But $\mathfrak{M}_i$ and $\mathfrak{N}_j$ are finite dimensional von Neumann algebras, so

$$P_\nu \in \operatorname{Proj} \mathfrak{M}_i \otimes_a \operatorname{Proj} \mathfrak{N}_j \subseteq \operatorname{Proj} \mathfrak{M} \otimes_a \operatorname{Proj} \mathfrak{N}.$$

Thus $P \in \operatorname{Proj} \mathfrak{M} \otimes_s \operatorname{Proj} \mathfrak{N}$ and the proof is complete. □

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