ON THE UNITARY EQUIVALENCE OF CLOSE C^*-ALGEBRAS

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Introduction. A central question in the theory of perturbations of C^*-algebras is to determine which C^*-algebras A satisfy the following property: Every C^*-algebra B "sufficiently close" to A is unitarily equivalent to it (cf. [3], [4], [11]). In this paper we use "Ext" theory in order to find new C^*-algebras with this property.

Let D be a separable C^*-subalgebra of a C^*-algebra C and suppose that D is an extension of a C^*-algebra A by a C^*-algebra I. Under certain assumptions on A and I we show that if D' is a C^*-subalgebra of C, "sufficiently close" to D, then D and D' are unitarily equivalent. To that end, we prove that D' is also an extension of A by I and show that these two extensions are unitarily equivalent. This second problem is dealt with by viewing the two extensions of A by I given by D and D' through the six-term exact sequence of K-theory associated with the two extensions, using the Rosenberg and Schochet universal coefficient formula (cf. [13]) and Theorem 2.11 of [9].

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NOTATIONS. Throughout this paper H will denote a separable infinite dimensional Hilbert space. $\mathcal{L}(H)$ is the C^*-algebra of bounded linear operators on H and $K(H)$ is the C^*-algebra of compact operators. If A is a C^*-algebra $M(A)$ denotes the multiplier algebra of A.

The distance between the C^*-subalgebras A and B of a C^*-algebra C is defined by

$$d(A,B)=\operatorname{Max}\left\{\sup_{a\in A_1}\inf_{b\in B_1}\|a-b\|;\ \sup_{b\in B_1}\inf_{a\in A_1}\|a-b\|\right\},$$

where A_1 and B_1 denote the unit balls of A and B respectively.

1. Some results from the theory of extensions. Here we recall some facts about Kasparov's bi-functor $\operatorname{Ext}(A,B)$ (cf. [8]).

1.1. Let A be a separable nuclear C^*-algebra and B a C^*-algebra with countable approximate unit. An (A,B) extension is a short exact sequence

$$0\to B\otimes K(H)\to D\xrightarrow{\phi}A\to 0.$$

Such an extension will be denoted by the pair (D,ϕ). We note that (cf. [2]) such extensions are in one-to-one correspondence with *-homomorphisms $\sigma: A\to M(B\otimes K(H))/B\otimes K(H)$. Two extensions σ_1 and σ_2 are said to be unitarily equivalent (write $\sigma_1 \underset{u}{\sim} \sigma_2$) if there exists a unitary $u\in M(B\otimes K(H))$ such

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that $\sigma_1(a)=u\sigma_2(a)u^*$ for every $a\in A$. An extension σ is said to be trivial if it has a lifting $\pi: A\to M(B\otimes K(H))$. The sum $\sigma_1\oplus\sigma_2$ is defined to be the direct sum $\sigma_1\oplus\sigma_2(a)=\sigma_1(a)\oplus\sigma_2(a)$ (with the identification of $M_2(M(B\otimes K(H)))$ with $M(B\otimes K(H))$. Now $\operatorname{Ext}(A,B)$ is the set of equivalence classes of (A,B) extensions with respect to the relation: $\sigma_1\sim\sigma_2$ if and only if there exist trivial extensions τ_1,τ_2 such that $\sigma_1\oplus\tau_1 \underset{u}{\sim} \sigma_2\oplus\tau_2$.

1.2. In 1.1 if A and B are unital one may define $\operatorname{Ext}_s(A,B)$ to be the set of unital extensions divided by the equivalence relation $\phi_1\sim\phi_2$ if and only if there exist unitarily trivial extensions τ_1 and τ_2 such that $\phi_1\oplus\tau_1\underset{u}{\sim}\phi_2\oplus\tau_2$. Here an extension τ is said to be unitarily trivial if it has a unital lifting $\pi: A\to M(B\otimes K(H))$. In [15] G. Skandalis studies the bi-functor $\operatorname{Ext}_s(A,B)$ and shows that it is a group that is homotopy invariant in both variables and that $\operatorname{Ext}_s(A,B)\cong \operatorname{Ext}(A_c, B\otimes C_0(\mathbf{R}))$, where

$$A_c=\{f:[0,1]\to A: f(1)=0,\ f(0)\in\mathbf{C}\}.$$

In the case that $B=\mathbf{C}$ it follows from Voiculescu's theorem [16] that $\operatorname{Ext}_s(A,B)=\operatorname{Ext}_s(A)$, where $\operatorname{Ext}_s(A)$ is the Brown-Douglas-Fillmore strong Ext group [1]—that is, the group of unitary equivalence classes of unital essential extensions of A by $K(H)$.

1.3. If A is a unital C^*-algebra and X is a compact finite dimensional metrizable space, then by the result of Pimsner, Popa, and Voisculescu [12] $\operatorname{Ext}(A, C(X))$ is the group of homogeneous extensions $0\to C(X)\otimes K(H)\to D\xrightarrow{\phi} A\to 0$ [12, Definition 1.7] divided by unitary equivalence.

1.4. To each extension $0\to B\otimes K(H)\to D\xrightarrow{\phi} A\to 0$ one can associate a six-term exact sequence of abelian groups

$$K_1(B)\to K_1(D)\xrightarrow{\phi_*} K_1(A)\xrightarrow{\delta_1} K_0(B)\to K_0(D)\xrightarrow{\phi_*} K_0(A)\xrightarrow{\delta_0} K_1(B).$$

The pair (δ_0,δ_1) defines a homomorphism

$$\gamma:\operatorname{Ext}(A,B)\to \operatorname{Hom}(K_0(A),\ K_1(B))\oplus\operatorname{Hom}(K_1(A),\ K_0(B)).$$

J. Rosenberg and C. Schochet [13] showed that for a large class n of C^*-algebras, the homomorphism γ is onto; they established the following "universal coefficient" formula

$$\begin{aligned}0&\to \operatorname{Ext}(K_0(A),\ K_0(B))\oplus\operatorname{Ext}(K_1(A),\ K_1(B))\\ &\to\operatorname{Ext}(A,B)\xrightarrow{\gamma}\operatorname{Hom}(K_0(A),\ K_1(B))\oplus\operatorname{Hom}(K_1(A),\ K_0(B))\to 0.\end{aligned}$$

1.5. We recall that a subgroup H of an abelian group G is said to be a pure subgroup if for every positive integer n and $h\in H$ the equation $nx=h$ is solvable in H whenever it has a solution in G [7, §23]. Equivalently $H\otimes\Gamma\xrightarrow{i\otimes id} G\otimes\Gamma$ is injective for every abelian group Γ, where $i: H\to G$ is the inclusion map. If H is a pure subgroup of G we say $0\to H\to G\to G/H\to 0$ is a pure extension of G/H by H.

2. Some lemmas.

2.1. DEFINITION. We say that a nuclear C^*-algebra A has property P_ϵ if for each pair of C^*-algebras, $B \subseteq C$, and for every *-monomorphism $i: A \to C$ the relation $d(i(A), B) < \epsilon$ implies that there exists a *-isomorphism $\rho: A \to B$. If moreover $d(i(A), B) < 1/38$, we require that $\rho_*: K_*(A) \to K_*(B)$ is the closeness isomorphism given by proposition 2.4 of [9]. This closeness isomorphism is obtained by mapping a projection (or a unitary) in $M_n(A^+)$ to a nearby projection (or a unitary) in $M_n(B^+)$.

Commutative C^*-algebras, separable unital continuous trace C^*-algebras, and ideal C^*-algebras all have this property for suitable ϵ's (cf. [3], [11]).

2.2. REMARK. We will use Lemma 1.2 and part of Lemma 2.6 of [10] in several places. Using the estimate of [9, Lemma 1.10] in the proof of these lemmas one obtains the following: If A and B are C^*-subalgebras of a C^*-algebra D such that $d(A, B) = k < 1/11$ and I is a closed ideal in A, then there exists a unique closed ideal in B such that $d(I, J) \le \alpha(2k) + 3k$. Moreover, there are C^*-subalgebras A_0 and B_0 of a C^*-algebra D_0, respectively *-isomorphic to A/I and B/J, such that

$$d(A_0, B_0) \le \frac{\alpha(2k)}{2} + k, \quad \text{where } \alpha(k) = 2 \sin \frac{\arcsin k}{2}.$$

We are going to use the function $\alpha: [0,1] \to [0, \sqrt{2}]$ frequently in this paper.

2.3. LEMMA. *Let A and B be separable C^*-algebras in the class n* (see 1.4) *such that every pure extension* (see 1.5) *of $K_*(B)$ by $K_*(A)$ splits. Let $0 \to B \to D \xrightarrow{\phi} A \to 0$ be an extension such that for every nuclear C^*-algebra C the connecting maps of K-theory given by the extension $0 \to B \otimes C \to D \otimes C \xrightarrow{\phi \otimes id} A \otimes C \to 0$ are zero. Then (D, ϕ) defines the zero element of* $\operatorname{Ext}(A, B)$.

Proof. By virtue of the universal coefficient formula (1.4) it suffices to show that

$$0 \to K_*(B) \to K_*(D) \xrightarrow{\phi_*} K_*(A) \to 0$$

splits. If this does not split the hypothesis of the Lemma implies that $K_*(B)$ is not a pure subgroup of $K_*(D)$ (see 1.5). This is turn implies that there exists a positive integer n such that $K_*(B) \otimes \mathbf{Z}_n \to K_*(D) \otimes \mathbf{Z}_n$ is not injective. This, together with the fact that $K_*(0_{n+1}) = \mathbf{Z}_n$ (cf. [6]) and the Künneth formula [14, Theorem 4.1], imply that $K_*(B \otimes 0_{n+1}) \to K_*(D \otimes 0_{n+1})$ is not injective (0_m denotes the Cuntz algebra [5]). But this last statement contradicts the hypothesis by letting $C = 0_{n+1}$, and the proof is complete. □

2.4. LEMMA. *Let D and D' be C^*-subalgebras of a C^*-algebra C such that $d(D, D') = k < 1/100$. Suppose that $0 \to I \xrightarrow{j} D \xrightarrow{\phi} A \to 0$ is exact, where A and I are C^*-algebras having property P_{ϵ_1} and P_{ϵ_2} respectively. If $\alpha(2k)/2 + k < \epsilon_1$ and $\alpha(2k) + 3k < \epsilon_2$, then there exist *-homomorphisms j' and ϕ' making $0 \to I \xrightarrow{j'} D' \xrightarrow{\phi'} A \to 0$ exact and the following diagram commutative.*

$$\begin{array}{ccccccccccccc} K_1(I) & \to & K_1(D) & \xrightarrow{\phi_*} & K_1(A) & \xrightarrow{\delta_1} & K_0(I) & \to & K_0(D) & \xrightarrow{\phi_*} & K_0(A) & \xrightarrow{\delta_0} & K_1(I). \\ \| & & \tau\downarrow & & \| & & \| & & \tau\downarrow & & \| & & \| \\ K_1(I) & \to & K_1(D') & \xrightarrow{\phi'_*} & K_1(A) & \xrightarrow{\delta'_1} & K_0(I) & \to & K_0(D') & \xrightarrow{\phi'_*} & K_0(A) & \xrightarrow{\delta'_0} & K_1(I) \end{array}$$

Proof. With no loss of generality we may assume that I is contained in D as a closed ideal. Then since $d(D,D')=k$ by Remark 2.2 there exists a unique closed ideal I' in D' such that $d(I,I')\leq 3k+\alpha(2k)$. Also there are C^*-algebras D_0 and D'_0 *-isomorphic to D/I and D'/I' such that $d(D_0,D'_0)<k+\alpha(2k)/2$. Now since I has property P_{ϵ_2} and $d(I,I')\leq 3k+\alpha(2k)<\epsilon_2$, there exists an *-isomorphism $j':I\to I'$. Also since A has property P_{ϵ_1} and $A\cong D/I\cong D_0$ the relation $d(D_0,D'_0)\leq \alpha(2k)/2+k<\epsilon_1$ implies that D_0 and D'_0 are *-isomorphic. This and the fact that D_0 and D'_0 are respectively *-isomorphic to D/I and D'/I' give a *-isomorphism $\rho: D/I\to D'/I'$. Now define $\phi':D'\to A$ by $\phi'(d')=\phi(d)$ if $\rho(d+I)=d'+I'$. It is routine to check that ϕ' is well-defined and that it is a *-homomorphism making $0\to I\xrightarrow{j'} D'\xrightarrow{\phi'} A\to 0$ exact. Since $k<1/100$ the commutativity of the diagram follows from [9, Theorem 2.11] and the fact that ρ and j' induce the isomorphism τ mentioned in 2.1.

2.5. LEMMA. *Let D, D', I, ϕ, ϕ' and k be as in* 2.4. *Moreover, assume that A and I belong to the class n* (see 1.4).

(i) *If (D,ϕ) defines the zero element of* $\mathrm{Ext}(A,I)$, *then so does (D',ϕ').*

(ii) *If $k<1/200$, $\alpha(4k)+6k<\epsilon_1$, $\alpha(4k)/2+2k<\epsilon_2$, and if A and D are unital and (D,ϕ) defines the zero element of* $\mathrm{Ext}_s(A,I)$, *then so does (D',ϕ').*

Proof. (i) Since (D,ϕ) is a trivial extension the connecting maps δ_0 and δ_1 are zero and the extension (D,ϕ) is given by the split short exact sequence $0\to K_*(I)\to K_*(D)\xrightarrow{\phi_*} K_*(A)\to 0$. Now the commutativity of the diagram given in 2.4 obviously implies that δ'_0 and δ'_1, the connecting maps given by the extension (D',ϕ'), are also zero and that $0\to K_*(I)\to K_*(D')\xrightarrow{\phi'_*} K_*(A)\to 0$ splits. Now Rosenberg, Schochet's universal coefficient formula (see 1.4) shows that (D',ϕ') is trivial in $\mathrm{Ext}(A,I)$.

(ii) We recall our comment in 1.2, that $\mathrm{Ext}_s(A,I)\cong \mathrm{Ext}(A_c, I\otimes C_0(\mathbf{R}))$. This isomorphism is given by the map that sends an extension $0\to I\to E\xrightarrow{\psi} A\to 0$ to the extension $0\to I\otimes C_0(0,1)\to E_c\xrightarrow{\hat\psi} A_c\to 0$, where $(\hat\psi f)(t)=\psi(f(t))$ for every $f\in E_c$ and $t\in[0,1]$ (see 1.2 for the notation). Now by [4, Theorem 3.4] $d(D_c,D'_c)\leq 2d(D,D')$ and we can apply the first part of the lemma to the extensions $(D_c,\hat\phi)$ and $(D'_c,\hat\phi')$. This implies that $(D'_c,\hat\phi')$ is trivial in $\mathrm{Ext}(A_c, I\otimes C_0(\mathbf{R}))$ which in turn shows that (D',ϕ') is trivial in $\mathrm{Ext}_s(A,I)$ as desired. □

2.6. LEMMA. *Let D, D', I, A, ϕ, ϕ' and k be as in* 2.5. *Furthermore let $k<1/2400$. If every pure extension of $K_*(I)$ by $K_*(A)$ splits, then*

(i) *$(D,\phi)\sim(D',\phi')$ in* $\mathrm{Ext}(A,I)$; *and*

(ii) *when A and D are unital, then $(D,\phi)\sim_s(D',\phi')$ in* $\mathrm{Ext}_s(A,I)$.

Proof. (i) Let x in $\mathrm{Ext}(A,I)$ be the difference of the two extensions (D,ϕ) and (D',ϕ'). Since $d(D,D')=k$ by [4, Theorem 3.1], $d(D\otimes B, D'\otimes B)\leq 12k<$

1/100 for every nuclear C^*-algebra B. Therefore by [9, Theorem 2.11] the two extensions

$$0 \to I \otimes B \otimes K(H) \to D \otimes B \to A \otimes B \to 0 \quad \text{and}$$
$$0 \to I \otimes B \otimes K(H) \to D' \otimes B \to A \otimes B \to 0$$

have the same connecting maps δ_0, δ_1. Using this and the universal coefficient formula (1.4) we deduce that the hypothesis of 2.3 holds for the extension x. Hence x is trivial which shows that $(D, \phi) \sim (D', \phi')$ in $\operatorname{Ext}(A, I)$.

(ii) This follows simply by applying (i) to $(D_c, \hat{\phi})$ and $(D_c', \hat{\phi}')$, noting that by [4, Theorem 3.2] $d(D_c, D_c') \leq 2d(D, D')$, where $(D_c, \hat{\phi})$ and $(D_c', \hat{\phi}')$ are described in the proof of Lemma 2.5 (ii). □

2.7. LEMMA. *Let D, D' be C^*-subalgebras of a C^*-algebra C and let I, J be closed ideals in D and I', J' closed ideals in D'. Let $K = \{x \in D \mid xJ \subseteq I\}$ and $K' = \{x \in D' \mid xJ' \subseteq I'\}$. Then*

(i) *If $I' \subseteq J'$ and $d(I, I') + d(J, J') < 1$, then $I \subseteq J$.*

(ii) $d(K, K') \leqq 3d(D, D') + 2d(I, I') + 2d(J, J')$.

Proof. Let $d(I, I') = \gamma$, $d(J, J') = \delta$ and $d(D, D') = k$. (i) Let $x \in I$, $\|x\| \leq 1$. Then there exists $x' \in I'$ such that $\|x'\| \leq 1$ and $\|x - x'\| \leq \gamma$. As $I' \subseteq J'$ there exists $y \in J$ with $\|x' - y\| \leq \delta$. Hence $\|x - y\| \leq \delta + \gamma$. Let $\pi: I \to D/J$ be the projection. We get $\|\pi\| \leqq \gamma + \delta < 1$. This implies that $d(I \cap J, I) < 1$, which shows that $I \cap J = I$, that is, $I \subseteq J$.

(ii) Let $\epsilon > 0$. Let $x \in K$ with $\|x\| \leq 1$. Let $x' \in D'$ with $\|x - x'\| \leqq k + \epsilon$, and $\|x\| \leqq 1$. Let $z' \in J'$, $\|z'\| \leqq 1$ and choose $z \in J$ such that $\|z - z'\| \leqq \delta$ and $\|z\| \leqq 1$. Then

$$\|x'z' - xz\| \leqq \|x'(z' - z)\| + \|(x' - x)z\| \leqq k + \delta + \epsilon.$$

Now $xz \in I$. Hence there exists $y \in I'$ with $\|xz - y\| \leqq \gamma$. We get $\|x'z' - y\| \leqq k + \delta + \gamma + \epsilon$. Let $p: D' \to M(J'/I')$ be the natural map (given by $p(a)\bar{b} = \overline{ab}$, $a \in D'$, $b \in J'$, where $\bar{b}$ denotes the class of b modulo I'). We have $\|p(x')\| \leqq k + \delta + \gamma + \epsilon$. Hence there exists $x'' \in \ker p = K'$ with $\|x' - x''\| \leqq k + \delta + \gamma + \epsilon$. Put $\hat{x} = x''/\sup(1, \|x''\|)$. We have $\|x' - \hat{x}\| \leqq 2(k + \delta + \gamma + \epsilon)$. Hence $\|x - \hat{x}\| \leqq 3k + 2\delta + 2\gamma + 3\epsilon$. By symmetry for every $y \in K'$, $\|y\| \leqq 1$ we can find $\hat{y} \in K$ such that $\|\hat{y}\| \leq 1$ and $\|y - \hat{y}\| \leqq 3k + 2\delta + 2\gamma + 3\epsilon$. Hence $d(K, K') \leq 3k + 2\delta + 2\gamma + 3\epsilon$. But ϵ was arbitrary and we must have $d(K, K') \leq 3k + 2\delta + 2\gamma$.

2.8. DEFINITION. An extension $0 \to I \to D \to A \to 0$ is said to be a homogeneous extension if for every closed ideal J in I the homomorphism $D/I \to M(I/J)$ is injective.

We note that if $I = C(X) \otimes K(H)$ this definition coincides with the definition of homogeneous X-extension given by Pimsner, Popa, and Voiculescu (cf. [12]).

2.9. LEMMA. *Let D and D' be C^*-subalgebras of a C^*-algebra C such that $d(D, D') = k < 1/11$. Let I be a closed ideal in D and I' the closed ideal in D' such that $d(I, I') \leq 3k + \alpha(2k)$* (see 2.2). *Let $\phi: D \to D/I$ and $\phi': D' \to D'/I'$ be the quotient maps.*

(i) *If I is an essential ideal in D and $9k + 2\alpha(2k) < 1$, then I' is an essential ideal in D.*

(ii) *If the extension* (D, ϕ) *is homogeneous and* $15k+4\alpha(2k)<1$, *then* (D', ϕ') *is also homogeneous.*

Proof. (i) If I is an essential ideal, then $\mathrm{Ann}(I,D)=0$. Now 2.7 (ii) implies that $d(\mathrm{Ann}(I,D),\ \mathrm{Ann}(I',D'))\leq 9k+2\alpha(2k)<1$. Hence $\mathrm{Ann}(I',D')=0$.

(ii) This also follows by applying 2.7 and definition 2.8.

3. Main results.

3.1. THEOREM. *Let* D *be a* C^**-subalgebra of a* C^**-algebra* C *such that* $0\to K(H)\to D\xrightarrow{\phi} A\to 0$ *is an essential extension. Let* D' *be a* C^**-subalgebra of* C *and let* $d(D,D')=k$. *If* A *is commutative and* $\alpha(4.2.299(3k+\alpha(2k)))<1/200$ *or* A *is separable unital with continuous trace and*

$$3k+\alpha(2k)<(10^4.86.35.12)^{-2}(2.299)^{-1},$$

then $D=uD'u^*$ *for a unitary operator* u.

Proof. Since $d(D,D')=k<1/11$ by 2.2 there exists a unique closed ideal I' in D' such that $d(I',K(H))\leq 3k+\alpha(2k)=\delta$. As $\delta<1/600$ by [3, Theorem 5.1] there exists a unitary operator v such that $K(H)=vI'v^*$ and $\|v-1\|<299\delta$. Now $d(D,vD'v^*)<2.299\delta$ and replacing D' by $vD'v^*$ we may and will assume that $K(H)\subset D'$. Then $d(D/K(H),D'/K(H))<2.299\delta$ and $D/K(H)\cong A$. Hence there exists a *-isomorphism $\rho\colon D/K(H)\to D'/K(H)$ such that $\|\rho-\mathrm{id}_{D/K(H)}\|\leqq 2.\alpha(4.2.299\delta)<1/100$ when A is commutative [4, Theorem 5.3] and

$$\|\rho-\mathrm{id}_{D/K(H)}\|<100.86.35.12(2.299)^{1/2}<1/100$$

in the case that A is unital separable with continuous trace [11, Theorem 4.22]. Let $0\to K(H)\to D'\xrightarrow{\phi'} A\to 0$ be the extension obtained as described in 2.4 (here $I=I'=K(H)$) and let $\sigma,\sigma'\colon A\to L(H)/K(H)$ be the *-homomorphisms associated with (D,ϕ) and (D',ϕ') respectively. Then it is easy to verify that $\|\sigma-\sigma'\|=\|\rho-\mathrm{id}_{D/K(H)}\|$. Choose an extension σ'' such that $\sigma\oplus\sigma''$ is unitarily trivial. Then since $\|\sigma\oplus\sigma''-\sigma'\oplus\sigma''\|=\|\sigma-\sigma'\|$ we have

$$d(q^{-1}(\sigma\oplus\sigma''(A)),\ q^{-1}(\sigma'\oplus\sigma''(A)))<\|\rho-\mathrm{id}_{D/K(H)}\|<\frac{1}{100}$$

($q\colon L(H)\to L(H)/K(H)$ is the quotient map). Now Lemma 2.5 (ii) when applied to the extensions $\sigma\oplus\sigma''$ and $\sigma'\oplus\sigma''$ implies that $\sigma'\oplus\sigma''$ is also unitarily trivial. This obviously means that σ and σ' belongs to the same class in $\mathrm{Ext}_s(A,K(H))$, that is, $\sigma\underset{s}{\sim}\sigma'$. Now since (D,ϕ) is an essential extension it follows from 2.9 (i) that (D',ϕ') is also an essential extension. Hence $\sigma\underset{s}{\sim}\sigma'$ implies that $\sigma\underset{u}{\sim}\sigma'$ (see 1.2). This obviously shows that $D=uD'u^*$ as desired. □

3.2. COROLLARY. *Let* D *and* D' *be* C^**-algebras acting on* H *and suppose that* D *is generated by the identity operator* $K(H)$ *and a countable family of essentially commutative, essentially normal operators. If* $d(D,D')=k$ *and* $\alpha(4.2.299(3k+\alpha(2k))<1/200$, *then* $D=uD'u^*$ *for some unitary operator* u.

Proof. The assumption implies that D is an extension of $C(X)$ by $K(H)$ for some compact metrizable space X [1]. Now the corollary follows directly from 3.1.

It is desirable to iterate 3.1 as this would imply that property P_ϵ (see 2.1) is preserved under extension by $K(H)$ for a certain class of C^*-algebras. However, the argument given in 3.1 can not be repeated since the unitary u obtained there may not be close to 1. This problem (although not in full generality) is avoided in the following.

3.3. THEOREM. *Let D be a C^*-subalgebra of a C^*-algebra C such that $0 \to K(H) \to D \to A \to 0$ is an essential extension. Let D' be a C^*-subalgebra of C and let $d(D, D') = k$. Suppose that A is in the class n* (see 1.4) *and has property P_{ϵ_0}. If $\alpha(4k)/2 + 2k < \epsilon_0$ and $\alpha(4k) + 6k < 1/600$ and $K_0(A)$ is the direct sum of a torsion group with a free abelian group, then $D = uD'u^*$ for some unitary operator u.*

Proof. Since $K(H)$ has the property P_ϵ with $\epsilon = 1/600$ [4, Theorem 5.1], by using 2.4 we obtain a second essential extension $0 \to I \to D' \xrightarrow{\phi'} A \to 0$. Now the assumption on $K_0(A)$ implies that pure extensions of $\mathbf{Z} = K_0(K(H))$ by $K_0(A)$ split. Hence by 2.6 (ii), $(D, \phi) \underset{\mathcal{S}}{\sim} (D', \phi')$. Now the argument given at the end of the proof of 3.1 may be repeated to show that $(D, \phi) \underset{\mathcal{U}}{\sim} (D', \phi')$, which implies $D = uD'u^*$. □

Note that if $K_0(A)$ is finitely generated and $0 \to K(H) \to D \to A \to 0$ is an extension, then $K_0(D)$ is also finitely generated (hence the direct sum of a torsion group with a free abelian group), and Theorem 3.4 can be iterated in this case.

The following theorem applies to a more general situation.

3.4. THEOREM. *Let D be a unital C^*-subalgebra of a C^*-algebra C such that $0 \to C(X) \otimes K(H) \to D \xrightarrow{\phi} A \to 0$ is an homogeneous extension, for some unital C^*-algebra A which has property P_{ϵ_0} and belongs to the class n. Let D' be a C^*-subalgebra C and $d(D, D') = k$ with $\alpha(4k)/2 + 2k < \epsilon_0$ and $\alpha(4k) + 6k <$* 1/103. *If every pure extension of $K_*(X)$ by $K_*(A)$ splits, then $D = uD'u^*$ for some unitary operator u.*

Proof. First we note that by [11, Theorem 3.8] $C(X) \otimes K(H)$ has property P_ϵ for $\epsilon = 1/103$. Then let (D', ϕ') be the extension given by 2.4. Now usual arguments show that D' is unital and by construction (D', ϕ') is also a unital extension. Now by 2.6 (ii), $(D, \phi) \underset{\mathcal{S}}{\sim} (D', \phi')$ and this would imply that $(D, \phi) \underset{\mathcal{U}}{\sim} (D', \phi')$ if we show that (D', ϕ') is also a homogeneous extension (see 1.3). But this follows from 2.9 (ii) and the remark made after 2.8. This ends the proof of the theorem. □

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