GENERATING NON-NOETHERIAN MODULES EFFICIENTLY

Raymond Heitmann

In their excellent 1973 paper [2], Eisenbud and Evans developed a unified treatment of a number of results relating the dimension of a ring to the generating sets of modules over that ring. Throughout, they assume the ring in question has Noetherian j-spectrum. Subsequent work by the present author [3], Vasconcelos and Wiegand [5], and Brumatti [1] has produced some of these results in a non-Noetherian setting, but many questions remained unanswered. In fact, with a few fairly minor modifications, virtually all of the results in [2] can be demonstrated without any Noetherian assumption. Moreover, the same unified presentation can be followed.

In §1, we shall introduce the notation we shall use and prove some elementary lemmas which shall be needed later. In §2, we present the results. This begins with a generalization of Bass's Stable Range Theorem (Theorem 2.1), which immediately allows us to extend Kronecker's Theorem that radical ideals (that are radicals of finitely generated ideals) are radicals of $(\dim R+1)$-generated ideals (Corollary 2.4). More importantly, (2.1) serves as the fundamental lemma needed to prove our version (Theorem 2.5) of the Basic Element Theorem [2, Theorem A, p. 282]. With this, we may extend the "corollaries" of Theorem A—Serre's Theorem (2.6), Bass's cancellation theorem (2.7), and the Forster–Swan Theorem (2.8, 2.9). In §3, we offer a few examples to illustrate the necessity of some of the modifications which have been made in the presentation.

The methods employed herein are not really new; primarily they are descended from the techniques introduced in [3]. The presentation is quite different however and no familiarity with the earlier paper will be required.

1. Throughout, R will be a commutative ring with identity and A will be a finite R-algebra (meaning finitely generated as an R-module). On first reading, the simplifying assumption $A=R$ may be helpful. All modules are unitary left A-modules.

Let $\mu(A,M)$ denote the minimal number of generators of M as an A-module. Following [2], we say a submodule $M'\subset M$ is basic at a prime P of R if $\mu(A_P,(M/M')_P)<\mu(A_P,M_P)$, and is t-fold basic if $\mu(A_P,(M/M')_P)\leqslant \mu(A_P,M_P)-t$. We say a set $m_1,\dots,m_u\in M$ is basic (resp. t-fold basic) at P if $A(m_1,\dots,m_u)$ is. We also use the terminology basic (resp. j-basic, X-basic) to mean basic at every prime $P\in \operatorname{Spec} R$ (resp. j-$\operatorname{spec} R$, X).

We make frequent use of $\operatorname{Spec} R$, the set of prime ideals of R with the usual Zariski topology. We will also need the patch topology; this has the same points as $\operatorname{Spec} R$ but has for a closed subbasis the Zariski-closed and Zariski quasi-

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compact open sets; that is, $V(J)$ and $D(I)$ are closed when I is finitely generated. A closed set in the patch topology is called a *patch*. The success of the methods in this paper depends on the two facts that patches occur conveniently and that the closure of a patch in the Zariski topology is the union of the closures of its points [4, p. 45]. Because of this reliance, the following revised definition seems in order.

DEFINITION. The j-spectrum of a ring R is the closure of the maximal spectrum of R in the patch topology. A prime in j-spec R is called a j-prime.

This definition coincides with the prevailing definition (primes which are intersections of maximal ideals) when the maximal spectrum is Noetherian, the only situation in which the j-spectrum has proved useful (or will without the revision). The necessity of the change is discussed in §3. As usual, the dimension of a subset of Spec R is the maximal length of a chain of primes in that subset. Of course, $\dim R = \dim(\operatorname{Spec} R)$ and $j\text{-}\dim R = \dim(j\text{-spec } R)$. We also (unfortunately) need a new dimension function.

DEFINITION. For any prime $P \in \operatorname{Spec} R$, we let

$$\delta(P) = \sup_{r \in R} \dim(j\text{-spec } R[r^{-1}] \cap V(P)).$$

DEFINITION. For $X \subset \operatorname{Spec} R$, we let $\delta\text{-}\dim(X) = \sup_{P \in X} \delta(P)$. We let $\delta\text{-}\dim(R) = \delta\text{-}\dim(\operatorname{Spec} R)$.

REMARKS. (i) $\delta(P) \leqslant \dim R/P$. All results and proofs in this paper will be valid (though weaker) if dim is used in place of δ-dim.

(ii) $j\text{-spec}(R/J) \subseteq (j\text{-spec } R) \cap V(J)$ but, unlike the Noetherian case, the reverse inclusion needn't hold.

(iii) $\delta\text{-}\dim R[r^{-1}] \leqslant \delta\text{-}\dim R$ for any $r \in R$. Krull dimension has this property also of course, but j-dimension doesn't.

Next we discuss the connection between μ, basic elements, and patches. We begin with the familiar.

LEMMA 1.1. *$\{P \mid \mu(A_P, M_P) > k\}$ is closed.*

Proof.

$$\{P \mid \mu(A_P, M_P) > k\} = \bigcap \{\text{support}(M/A(m_1, \ldots, m_k)) \mid m_1, \ldots, m_k \in M\}$$
$$= \bigcap \{\text{closed sets}\}$$

is closed. □

Of course, the complement of this set is open. Now in the Noetherian case all open sets are quasi-compact, and consequently both $\{P \mid \mu(A_P, M_P) > k\}$ and $\{P \mid \mu(A_P, M_P) \leqslant k\}$ are closed and open in the patch topology. So $\mu(A_-, M_-)$ partitions Spec R as a finite union of sets $\{P \mid \mu(A_P, M_P) = k\}$ which are likewise closed and open in the patch topology. The usefulness of this partition follows from two extremely elementary lemmas which formed the heart of [3]—and much of the present paper. Lemma 1.2 was somewhat obscured by notation and did not appear explicitly; Lemma 1.3 is [3, p. 118, Lemma 1] and is just a version of the Chinese Remainder Theorem.

LEMMA 1.2. *Suppose* $\dim X < \infty$. *If* C_1, C_2 *are disjoint patches* $\subset X \subset \operatorname{Spec} R$, *then either* $\bar{C}_1 \cap \bar{C}_2 \cap X = \varnothing$ *or* $\dim \bar{C}_1 \cap \bar{C}_2 \cap X < \dim X$. (Over a set, $\bar{\ }$ indicates closure in the Zariski topology.)

Proof. If $P \in \bar{C}_1 \cap \bar{C}_2$ then there must be $Q_1 \in C_1$, $Q_2 \in C_2$ with $Q_1, Q_2 \subset P$, since the closure of a patch is the union of its pointwise closures. Since $C_1 \cap C_2 = \varnothing$, Q_1, Q_2 can't both be P. Say $Q_1 \neq P$. Since $Q_1 \in X$, P is not minimal in X. Thus $\bar{C}_1 \cap \bar{C}_2$ contains no minimal members of X and the lemma follows. □

LEMMA 1.3. *If* B_1 *and* B_2 *are disjoint closed sets in* $\operatorname{Spec} R$, *then there exists an element* $r \in R$ *with* $B_1 \subset V(r)$ *and* $B_2 \subset D(r)$.

Proof. Write $B_1 = V(J_1)$ and $B_2 = V(J_2)$. Note $J_1 + J_2 = R$. So we may write $1 = r_1 + r_2$ with $r_i \in J_i$. Clearly $r = r_1$ has the desired properties. □

The basic technique is fairly easy to outline. Using (1.3), we can deal with disjoint closed sets one at a time. Meanwhile, (1.2) tells us that if we partition a set by patches, the closures will be disjoint except on a set of lower dimension which we can deal with using some sort of induction hypothesis. Of course, we cannot proceed unless we have the partition. Because $\mu(A_-, M_-)$ does so partition $\operatorname{Spec} R$ (or subsets thereof) under several different circumstances, it seems best to simply assume it and mention some of the conditions which imply our hypothesis.

DEFINITION. For a patch $X \subset \operatorname{Spec} R$, a finite R-algebra A is X-appropriate provided the open set $\{P \mid \mu(R_P, N_P) \leqslant k\} \cap X$ is quasi-compact for every k and every $N = A/B$, where B is a finitely generated right ideal of A. Of course, R is X-appropriate for any X. We say A is appropriate if it is $(\operatorname{Spec} R)$-appropriate.

We say a finitely generated A-module ${}_AM$ is X-appropriate provided A is appropriate and $\{P \mid \mu(A_P, N_P) \leqslant k\} \cap X$ is quasi-compact for every k and every $N = M/B$, where ${}_AB \subset {}_AM$ is finitely generated.

The remainder of §1 is quite transparent in the case $A = R$.

THEOREM 1.4. *For* ${}_AM$ *to be X-appropriate, either of the following conditions is sufficient:*

(i) $\operatorname{Spec} R$ *is Noetherian, or* $A = R$ *and X is Noetherian.*

(ii) *A is a finitely presented R-module and M is a finitely presented A-module.*

Proof. (i) has been noted earlier. We first prove (ii) in the special case where R/P is infinite for every $P \in X$. Let $m = \mu(R, A)$ and $A = R(a_1, \ldots, a_m)$ as an R-module. Let $N = M/B$. For $N = A/B$, a nearly identical proof works (as well as easier ones). It is easy to see M must be finitely presented as an R-module and so N is also. So $N \cong R^t/J$ where J is finitely generated. Suppose $x_1, \ldots, x_t$ generate N as an R-module. Now a set of elements $n_1, \ldots, n_k$ generates M as an A-module precisely where $\{a_i n_j \mid i \leqslant m, j \leqslant k\}$ generates N as an R-module. As each n_j must be an R-linear combination of the x_i's, we see that $\mu(A_P, N_P) \leqslant k$ if and only if there exists

$$\{r_{jl} \mid j \leqslant k,\ l \leqslant t\} \subset R \text{ with } P \notin \operatorname{support}\left(N/R\left(\left\{\sum_{l=1}^{t} r_{jl} a_i x_l \mid i \leqslant m,\ j \leqslant k\right\}\right)\right).$$

Let y_{il} be any preimage of $a_i x_l$ in R^t. Then, $\mu(A_P, N_P) \leqslant k$ if and only if there exists $\{r_{jl} \mid j \leqslant k, l \leqslant t\} \subset R$ with $P \notin \operatorname{support}(R^t/R(\{\sum_{l=1}^{t} r_{jl} y_{il} \mid i \leqslant m, j \leqslant k\}) + J)$. Suppose $J = R(z_1, \ldots, z_u)$; fixing a basis for R^t, we may regard the elements y_{il}, z_s as t-tuples. Let $\{\beta_{jl} \mid j \leqslant k, l \leqslant t\}$ be unknown; then $\sum_{l=1}^{t} \beta_{jl} y_{il} = \alpha_{ji}$ is a t-tuple in $R[\{\beta_{jl}\}]$. Consider the matrix over $R[\{\beta_{jl}\}]$ whose rows are $\{\alpha_{ji}\} \cup \{z_s\}$. Let I be the ideal generated by the $t \times t$ minors of this matrix. Then we see $\mu(A_P, N_P) \leqslant k$ if and only if there exists a homomorphism $\varphi: R[\{\beta_{jl}\}] \to R$ such that $\varphi(I) \not\subset P$. (Choosing φ is the same as picking $\{r_{jl}\}$ so that $\varphi(\beta_{jl}) = r_{jl}$.) Now, as I is generated by minors $\{f_e\}$, $\varphi(I) \not\subset P$ if and only if $\varphi(f_e) \notin P$ for some f_e. So $\mu(A_P, N_P) \leqslant k$ if and only if $P \in \bigcup_e \{Q \mid \varphi(f_e) \notin Q$ for some $\varphi\}$. This is a finite union and so it suffices to prove the intersection of each set with X is quasi-compact. Observe f_e is a polynomial in $\{\beta_{jl}\}$; as R/P is infinite for $P \in X$, we can find φ such that $\varphi(f_e) \notin P$ whenever any coefficient of f_e is not in P. Of course, if each coefficient is in P, we cannot. Thus $\{Q \mid \varphi(f_e) \notin Q$ for some $\varphi\} \cap X = D(\{g_i\}) \cap X$, where the g_i are the coefficients of f_e. This set is quasi-compact as desired.

Now consider the general case where R/P need not be infinite. Let T be an indeterminate and let $X' = \{P[T] \mid P \in X\} \subset \operatorname{Spec} R[T]$. By Lemma 3.1 of [6, p. 472], X' is a patch in $\operatorname{Spec} R[T]$ and the map $P \to P[T]$ is a homeomorphism. Now $R[T]/P[T]$ is infinite for every $P[T] \in X'$ and so the theorem follows from the special case, provided that $\mu(A_P, N_P) = \mu(A[T]_{P[T]}, N[T]_{P[T]})$ for every $P \in X$. This holds; in fact, if $R \to S$ is any homomorphism of commutative rings and P has a prime $Q \in \operatorname{Spec} S$ lying over it, then $\mu(A_P, N_P) = \mu((A \otimes_R S)_Q, (N \otimes_R S)_Q)$. To prove this, we may harmlessly replace R, S by the fields (R_P/PR_P), (S_Q/QS_Q) and A by the semisimple Artinian ring (A_P/(Jacobson radical of A_P)). It is therefore enough to show $\mu(A, N) = \mu(A \otimes_R S, N \otimes_R S)$ where A is simple and a finite dimensional vector space over R. Since $(\dim_R N/\dim_R A) \leqslant \mu(A, N) < (\dim_R N/\dim_R A) + 1$ and $\dim_S(N \otimes_R S) = \dim_R N$, the result follows. □

In [2], it is noted that while an element in a projective module is unimodular on an open set, the same is not true for basicness. What is true is the following.

THEOREM 1.5. *Suppose $_AM$ is X-appropriate and $m_1, \ldots, m_u \in M$. Then $\{P \in X \mid m_1, \ldots, m_u$ is t-fold basic in M at P$\}$ is open and closed in the patch topology on X.*

Proof. The set is precisely

$$\bigcup_k (\{P \mid \mu(A_P, M_P) \geqslant k\} \cap \{P \mid \mu(A_P, (M/A(m_1, \ldots, m_u))_P) \leqslant k - t\} \cap X),$$

which is a finite union of sets both open and closed in the patch topology on X. □

Noting that if I is a subset of R, $V(I) = \text{support}(R/IR)$ and $D(I)$ is its complement, we are led to the following definition.

DEFINITION. If I is a subset of A, set $\tilde{V}(I) = \text{support}(A/IA)$ and $\tilde{D}(I) = \operatorname{Spec} R - \tilde{V}(I)$.

Thus we may partition any patch $X = (\tilde{D}(I) \cap X) \cup (\tilde{V}(I) \cap X)$. If A is X-appropriate and IA is finitely generated (e.g., if I is finite), then both sets are open and closed in the patch topology on X and so X is represented as the disjoint union of two patches. It should be noted that, unlike the commutative situation, $\tilde{D}(I_1 \cup I_2)$ may properly contain $\tilde{D}(I_1) \cup \tilde{D}(I_2)$.

We conclude §1 with three technical lemmas which we shall need to deal with the case $A \neq R$.

LEMMA 1.6. *Suppose A is an X-appropriate R-algebra. For any $a \in A$, $\{P \in X \mid a \in \text{Jacobson radical of } A_P\}$ is open and closed in the patch topology on X.*

Proof. When $A = R$, this set is $V(a) \cap X$, but in general it is a proper subset of $\tilde{V}(a) \cap X$. Since A is finitely generated as an R-module, AaA is finitely generated as an R-module and so is $(AaA)^n$ for any n, in particular when $n = \mu(R, A)$. For any P, A_P/PA_P is a vector space over R_P/PR_P of dimension $\leqslant n$. So (Jacobson radical of $A_P)^n \subset PA_P$ and $a \in$ Jacobson radical of A_P if and only if $(AaA)^n A_P \subset PA_P$. Hence $a \in$ Jacobson radical of A_P if and only if $\mu(R_P, A_P) = \mu(R_P, (A/(AaA)^n)_P)$. So our set is just

$$\bigcup_k (\{P \mid \mu(R_P, A_P) = k\} \cap X \cap \{P \mid \mu(R_P, (A/(AaA)^n)_P) = k\}),$$

which is open and closed in the patch topology on X as desired. □

LEMMA 1.7. *Let A be an X-appropriate R-algebra, $c_1, c_2 \in A$, and let s be a unit in R. For each $P \in X$, let C_P denote A_P modulo its Jacobson radical. Then we may find a finite collection of disjoint patches $X_i \subset X$ and a corresponding set of elements $d_i \in A$ such that $X = \bigcup X_i$, and, for each $P \in X_i$, $(\bar{c}_1, \bar{c}_2)C_P = \bar{c}_1 \bar{d}_i(\bar{s} - \bar{c}_2)C_P \oplus \bar{c}_2 C_P = (\bar{c}_1 \bar{d}_i u(\bar{s} - \bar{c}_2) + \bar{c}_2)C_P$ for any central unit $u \in C_P$. Moreover, we may choose X_1 to be any set, open and closed in the patch topology, for which $d_1 \in A$ can be found to satisfy the equations.*

Proof. Throughout this proof, we will use only the patch topology. The X_i we seek must necessarily be clopen (closed and open) in X with this topology. Consider any $P \in X$. Since C_P is semisimple Artinian, we may decompose $(\bar{c}_1, \bar{c}_2)C_P$ into $\bar{c}_1 \alpha C_P \oplus \bar{c}_2 C_p$. Moreover, since $\bar{c}_1 \alpha C_P \oplus \bar{c}_2 C_P$ is a direct summand of C_P, we may find an orthogonal pair of idempotents which generates these right ideals, say $\bar{c}_1 \alpha$ and $\bar{c}_2 \beta$. We observe the equations $\bar{c}_1 = \bar{c}_1 \alpha \gamma_1 + \bar{c}_2 \gamma_2$ and $\bar{c}_2 = \bar{c}_2 \beta \delta$ must hold, and so $\bar{c}_1 \alpha \bar{c}_2 = 0$. Now lift $\alpha, \gamma_1, \gamma_2$ to elements $dr^{-1}, g_1 r^{-1}, g_2 r^{-1} \in A_p$, where $r \in R - P$. Letting $\theta_P \colon A \to C_P$ be the obvious map, we see that the following conditions are clearly satisfied: (i) $\theta_P((c_1 d)^2 - c_1 dr) = 0$; (ii) $\theta_P(c_1 d c_2) = 0$; (iii) $\theta_P(c_1 r^2 - c_1 d g_1 - c_2 g_2 r) = 0$; and (iv) $r \notin P$. Moreover, these four conditions guarantee $(\bar{c}_1, \bar{c}_2)C_P = \bar{c}_1 \bar{d} C_P \oplus \bar{c}_2 C_P$ and $\bar{c}_1 \overline{dc}_2 = 0$. So $\bar{c}_1 \bar{d}(\bar{s} - \bar{c}_2) = \bar{c}_1 \overline{ds}$ and we have $(\bar{c}_1, \bar{c}_2)C_P = \bar{c}_1 \bar{d}(\bar{s} - \bar{c}_2)C_P \oplus \bar{c}_2 C_P$. Further, let u be any central unit in C_P. Then

$$\bar{c}_1 \bar{d}(\bar{s} - \bar{c}_2) = (\bar{r}u)^{-1} \bar{c}_1 \bar{d}(\bar{c}_1 \bar{d} u(\bar{s} - \bar{c}_2) + \bar{c}_2),$$

from which $\bar{c}_1 \bar{d}(\bar{s} - \bar{c}_2)C_P \oplus \bar{c}_2 C_P = (\bar{c}_1 \bar{d} u(\bar{s} - \bar{c}_2) + \bar{c}_2)C_P$ quickly follows. Hence, for the set of primes satisfying (i)–(iv) and this particular $d \in A$, the equations are satisfied. Now, to say $\theta_P(a) = 0$ means a is in the Jacobson radical of A_P. Thus, by (1.6), each condition is satisfied on a set clopen in X. So there is a clopen set Y in X which contains P on which all conditions are satisfied. We repeat this procedure at every prime in X. Thus we cover X with sets clopen in X. Since X is compact, we may choose a finite subcover—$Y_1, \ldots, Y_n$. Adding an extra clopen set is harmless so if we wish to specify the first set in the cover, we let Y_1 be that set. Choosing $X_1 = Y_1$, $X_2 = Y_2 - Y_1$, etc., we produce our desired disjoint cover. □

Our final lemma of this section will be needed in the proof of Theorem 2.5 because Lemma 5 of [2] is not exactly what we need. As we shall need that result later and its statement will put (1.9) in the proper context, we state their Lemma 5 here.

PROPOSITION 1.8. *Let A be a semisimple Artinian ring, $a \in A$, and M a finitely generated A-module. If $m_1, \ldots, m_u$ is w-fold basic in M with $w < u$ and (a, m_1) is basic in $A \oplus M$, then there exist elements $a_1, \ldots, a_{u-1} \in A$ such that for all central units $r \in A$, $\{m_1 + aa_1 r m_u, m_2 + a_2 r m_u, \ldots, m_{u-1} + a_{u-1} r m_u\}$ is w-fold basic in M.*

Proof. [2, pp. 294–298].

LEMMA 1.9. *Let A be a semisimple Artinian ring, $a \in A$, and M a finitely generated A-module. If $m_1 \in M$ is such that (a, m_1) is basic in $A \oplus M$, then there exists $b \in A$ such that ab is idempotent, $ab \in \operatorname{ann}(m_1)$, and (ab, m_1) is basic in $A \oplus M$.*

Proof. We may handle each simple summand of A separately. While there may be summands on which (a, m_1) is not basic, such summands pose no difficulty. We can choose $b = 0$ as (ab, m_1) is not required to be basic. So we may harmlessly assume A is simple. Then conditions of basicness reduce to conditions of length. Letting $\lambda(A)$ be the length of A, we write $\lambda(A \oplus M) = n\lambda(A) + q$ for some n and $1 \leqslant q \leqslant \lambda(A)$. $\mu(A, A \oplus M) = n + 1$ and the condition (a, m_1) basic means precisely $\lambda(A(a, m_1)) \geqslant q$. Thus, for basicness, it suffices to find b such that $\lambda(A(ab, m_1)) = \lambda(A(a, m_1))$. We note $\lambda(A(a, m_1)) = \lambda(Am_1) + \lambda((\operatorname{ann} m_1)a)$. Now the A-module homomorphism $(\operatorname{ann} m_1) \to (\operatorname{ann} m_1)a$ has a splitting map $(\operatorname{ann} m_1)a \to \operatorname{ann}(m_1)$, which we may extend to a homomorphism $\varphi: A \to \operatorname{ann}(m_1)$ by sending the complementary summand of $(\operatorname{ann} m_1)a \to 0$. Now φ is right multiplication by $b = \varphi(1)$. As the map is injective on $(\operatorname{ann} m_1)a$, we have $\lambda((\operatorname{ann} m_1)ab) = \lambda((\operatorname{ann} m_1)a)$ and of course $ab \in Ab \subset \operatorname{ann}(m_1)$. As $bab = b$, ab is an idempotent and b is the desired element. □

2. We begin this section with a generalization of Bass's Stable Range Theorem. The statement will be in a form necessary to prove the Basic Element Theorem and so not very recognizable. Consequently, the statement will be followed by a succession of corollaries which are really special cases; these should seem familiar. Then Theorem 2.1 will be proved.

THEOREM 2.1. *Let A be an appropriate R-algebra, $s \in R$, and $d = j\text{-dim}\, R[s^{-1}]$. Suppose $a_1, \ldots, a_k \in A$ such that $k > d+1$ and*

$$D(s) \subset \tilde{D}(a_1, a_2) \cup \bigcup_{i=3}^{k} \tilde{D}(a_i).$$

Then there exist $b_2, \ldots, b_k \in A$ such that $D(s) \subset \tilde{D}(a_2 + a_1 b_2, \ldots, a_k + a_1 b_k)$. Moreover, we may take each $b_i \in s^2 A(s - a_i)$.

Of course, $\tilde{D}(a_2 + a_1 b_2, \ldots, a_k + a_1 b_k) \cap V(s) = \tilde{D}(a_2, \ldots, a_k) \cap V(s)$. In the case $A = R$, (2.1) immediately leads to

COROLLARY 2.2. *Let $s \in R$ and $d = j\text{-dim}\, R[s^{-1}]$. Suppose $r_1, \ldots, r_k \in R$ such that $k > d+1$ and $D(r_1) \subset D(s) \subset D(r_1, \ldots, r_k)$. Then there exist $t_2, \ldots, t_k \in sR$ such that $D(r_2 + r_1 t_2, \ldots, r_k + r_1 t_k) = D(r_1, \ldots, r_k)$.*

If $s = 1$, we get the usual Stable Range Theorem.

COROLLARY 2.3. *Let $d = j\text{-dim}\, R$ and $k > d+1$. If $(r_1, \ldots, r_k)R = R$, then there exist $t_2, \ldots, t_k \in R$ such that $(r_2 + r_1 t_2, \ldots, r_k + r_1 t_k)R = R$.*

On the other hand, the other extreme of (2.2), $s = r_1$, generalizes Kronecker's Theorem to the non-Noetherian setting.

COROLLARY 2.4. (i) *Suppose $j\text{-dim}\, R[r_1^{-1}] = d$ and $k > d+1$. Then there exist $t_2, \ldots, t_k \in r_1 R$ with $D(r_2 + r_1 t_2, \ldots, r_k + r_1 t_k) = D(r_1, \ldots, r_k)$.*

(ii) *If $d = \delta\text{-dim}\, R$, then any ideal which is the radical of a finitely generated ideal is the radical of an ideal requiring at most $(d+1)$-generators. In fact, $\sqrt{(a_1, \ldots, a_{d+1}, b_1, \ldots, b_m)} = \sqrt{(c_1, \ldots, c_{d+1})}$, where $c_i = a_i + \sum r_{ij} b_j$.*

Proof of Theorem 2.1. Since we are only concerned about primes in $D(s) = \operatorname{Spec} R[s^{-1}]$, we may work over this ring. So we assume s is a unit and $\operatorname{Spec} R \subset \tilde{D}(a_1, a_2) \cup \bigcup_{i=3}^{k} \tilde{D}(a_i)$. (Admittedly, this reduction will yield $b_i \in s^2(A \otimes R[s^{-1}])(s - a_i)$, which may not lift to $s^2 A(s - a_i)$. This concern will be addressed when b_i is actually chosen.)

Set $X = j\text{-spec}\, R$. We prove the theorem by induction on $\dim X$. Assume it holds for $\dim X < d$; we prove it for $\dim X = d$.

Next we want to construct a partition $X = \bigcup X_i$ and find a corresponding set of elements $d_i \in A$. Let C_P denote A_P modulo its Jacobson radical. (When $A = R$, C_P is the field R_P / PR_P. As our consideration of C_P will be restricted to its ideals, in the field case it matters only whether or not a particular element is zero. With this observation, the non-commutative notation can be circumvented.) Partition X according to (1.7), obtaining $\{d_i\} \subset A$ such that for $P \in X_i$, $(\bar{a}_1, \bar{a}_k)C_P = \bar{a}_1 \bar{d}_i (\bar{s} - \bar{a}_k) C_P \oplus \bar{a}_k C_P = (\bar{a}_1 \bar{d}_i u(\bar{s} - \bar{a}_k) + \bar{a}_k) C_P$ for any central unit $u \in C_P$. We may choose $X_1=\tilde{D}(a_k)\cap X$ since $d_1=0$ works there. (For $A=R$, $X=X_1\cup X_2$ where $X_2=V(a_k)\cap X$ and $d_2=1$.) Now let $Y=\bigcup_{i\neq j}(\bar{X}_i\cap\bar{X}_j)$. By (1.2), either $Y\cap X=\varnothing$ or $\dim Y\cap X<\dim X$.

To handle the case $Y\cap X=\varnothing$, it suffices to demonstrate a more general fact which we shall need later: If Z is closed and $Y\cap X\cap Z=\varnothing$, then we can find $b_k\in s^2A(s-a_k)$ such that $Z\subset\tilde{D}(a_2+a_1b_2,\ldots,a_k+a_1b_k)$ for any choice of $b_2,\ldots,b_{k-1}\in A$. Replacing R by R/J where $Z=V(J)$, we may assume $Z=\operatorname{Spec} R$. ($X\cap Z$ may properly contain j-spec(R/J) but this presents no problem; the important thing is that it contains max-spec R/J.) Since $\operatorname{Spec} R\subset \tilde{D}(a_1,\ldots,a_k)$, it suffices to show $A=(a_1,\ldots,a_k)A\subset(a_2+a_1b_2,\ldots,a_k+a_1b_k)A$. It is enough to show the right ideal contains a_1 and to do this we show that for each maximal P, necessarily in some X_i, $\bar{a}_1\in(\bar{a}_k+\bar{a}_1\bar{b}_k)C_P$. The condition $Y\cap X=\varnothing$ tells us that the closed set Y contains no maximal ideals. So $Y=\varnothing$ and the $\{\bar{X}_i\}$ are pairwise disjoint closed sets. By (1.3), find $\{r_i\}\subset R$ such that $\bar{X}_i\subset D(r_i)$ and $\bar{X}_j\subset V(r_i)$ for each $j\neq i$. Let $b_k=\sum r_is^ed_i(s-a_k)$ for some integer e. Then, for $P\in X_i$, $\bar{a}_1\in(\bar{a}_1,\bar{a}_k)C_P=\overline{a_1d_i(s-a_k)}C_P\oplus\bar{a}_kC_P=(\bar{a}_k+\bar{a}_1\overline{r_is^e}\bar{d}_i(\overline{s-a_k}))C_P$ since $\overline{r_is^e}$ is a central unit in C_P. This in turn equals $(a_k+a_1b_k)C_P$ since $\bar{r}_j=0$ for $j\neq i$. Here we must treat the concern that b_k should be in s^2A and not just $A[s^{-1}]$. The element $\sum r_id_i=as^{-n}$ for some $a\in A$ and $n\in\mathbf{Z}$. By choosing $e=n+2$, we get $b_k=s^2a(s-a_k)$ as desired.

Next we consider the case $\dim Y\cap X<\dim X$. Here $Y=\operatorname{Spec} R/I$ for some ideal I and j-spec$(R/I)\subset Y\cap X$. Then $k-1>1+\dim j$-spec R/I. Further, if $Q\in Y$, $Q\supset P$ for some $P\in X_j$, $j\neq 1$. Since $P\notin\tilde{D}(a_k)$ and $\tilde{D}(a_k)$ is open, $Q\notin\tilde{D}(a_k)$. Thus $\operatorname{Spec} R\subset\tilde{D}(a_1,a_2)\cup\bigcup_{i=3}^k\tilde{D}(a_i)$ implies $\operatorname{Spec}(R/I)\subset\tilde{D}(a_1,a_2)\cup\bigcup_{i=3}^{k-1}\tilde{D}(a_i)$. (Necessarily, $k\geqslant 3$ in this case.) Hence we may use the induction assumption to apply the theorem to R/I, thus obtaining suitable $b_2,\ldots,b_{k-1}\in A$ such that $Y\subset\tilde{D}(a_2+a_1b_2,\ldots,a_{k-1}+a_1b_{k-1})$. Finally, let $Z=\tilde{V}(a_2+a_1b_2,\ldots,a_{k-1}+a_1b_{k-1})$. It remains to choose b_k so that $Z\subset\tilde{D}(a_2+a_1b_2,\ldots,a_k+a_1b_k)$; we showed this was possible in the treatment of the case $Y\cap X=\varnothing$. □

We are now ready to state and prove our version of the Basic Element Theorem.

THEOREM 2.5. *Suppose* δ-dim $R=d<\infty$, *X a patch in* Spec R *which contains the maximal spectrum, A an R-algebra, and M an A-module such that ${}_AM$ is X-appropriate. Then the following hold.*

(i) *If $\mu(A_P,M_P)>d$ for every prime $P\in X$, then M contains an X-basic element.*

(ii)(a) *More generally, let M' be a finitely generated submodule of M. If, for every $P\in X$, M' is $(\delta(P)+1)$-fold basic in M at P, then M' contains an X-basic element of M.* (b) *If furthermore $M'=Am_1+M^*$ for some submodule $M^*\subset M'$, and if $a\in A$ is given such that $(a,m_1)\in A\oplus M$ is X-basic, then there is an X-basic element of M of the form m_1+am^*, where $m^*\in M^*$.*

Proof. (i) follows from (ii), and (ii)(a) is a consequence of (ii)(b), obtained by setting $a=1$. So we need only prove (ii)(b). In order to accomplish this, we shall make repeated use of the following reduction. Choose some $m^*\in M^*$ and set $m_1' = m_1 + am^*$. Now (a, m_1) is basic at P if and only if (a, m_1') is basic at P, because the automorphism φ on $A \oplus M$ given by $\varphi(b, m) = (b, m + bm^*)$ takes (a, m_1) to (a, m_1'). As $Am_1 + M^* = Am_1' + M^*$ and $m_1 + aM^* = m_1' + aM^*$, it suffices to prove (ii)(b) with m_1 replaced by the new element m_1'. By repeatedly deforming m_1 in this manner, we shall force m_1 to satisfy an increasing number of nice properties. After each replacement, we shall feel free to reuse the symbols m^* and m_1'. (The m^* promised in the theorem will never be displayed explicitly.)

Let $k = \inf \mu(A_P, M_P)$ and $k + n = \sup(A_P, M_P)$. We use induction on n, assuming the theorem holds when $\sup \mu(A_P, M_P) - \inf(A_P, M_P) < n$. Let $Y = \{P \mid \mu(A_P, M_P) > k\}$. We shall deform m_1 in order to make it satisfy the property "m_1 is $X \cap Y$-basic". If $n = 0$, $Y = \varnothing$ and m_1 already is. If $n > 0$, Y is closed and nonempty and so for some ideal I of R, $Y = V(I) \cong \operatorname{Spec} R/I$. We apply the induction hypothesis to R/I and M/IM to obtain $m_1' = m_1 + am^*$, which is $X \cap Y$-basic. Replace m_1 by m_1'. Next let $Z = \{P \in X \mid m_1 \text{ is not basic in } M \text{ at } P\}$. Since ${}_AM$ is X-appropriate, Z is a patch. Of course, for each $P \in Z$, $\mu(A_P, M_P) = k$. For each t, we define a set $B_t = \{P \mid M' \text{ is } t\text{-fold basic, but not } (t+1)\text{-fold basic, in } M \text{ at } P\}$. We will now prove the following statement for every integer N, $1 \leqslant N \leqslant k$.

(*) *If m_1 is $X \cap Y$-basic and $B_t \cap Z = \varnothing$ for all $t < N$, then we can find $m_1' = m_1 + am^*$ with $m^* \in M^*$ such that m_1' is $X \cap Y$-basic and $B_t \cap Z' = \varnothing$ for all $t \leqslant N$, where $Z' = \{P \in X \mid m_1' \text{ is not basic in } M \text{ at } P\}$.*

This statement will be sufficient to prove the theorem since it allows us to repeatedly deform m_1, starting with $B_t \cap Z = \varnothing$ for $t < 1$ and finishing with $B_t \cap Z' = \varnothing$ for $t \leqslant k$. As $Z' \subset \bigcup_{t \leqslant k} B_t$, $Z' = \varnothing$ and m_1' is X-basic. Now we must prove the statement.

Let $s \in R$, $a_1 \in A$. We say $\langle s, a_1 \rangle$ is an m_1-suitable pair provided we can find $a_2, \ldots, a_k \in A$ and local generators $m_1 + \tilde{m}, m_2, \ldots, m_k$ of M with $\tilde{m}, m_2, \ldots, m_N \in M^*$ such that $sM \subset A(m_1 + \tilde{m}, m_2, \ldots, m_k)$, $sm_1 = a_1(m_1 + \tilde{m}) + a_2 m_2 + \cdots + a_k m_k$, and $D(s) \subset \tilde{D}(a, a_1)$. An open set D is called an m_1-suitable set if there exists an m_1-suitable pair $\langle s, a_1 \rangle$ with $D = D(s)$. We shall demonstrate that, given any $P \in B_N \cap Z$, we can find an m_1-suitable pair $\langle s, a_1 \rangle$ with $P \in D(s)$, and hence we will cover $B_N \cap Z$ with m_1-suitable sets. To accomplish this, first note that generating M locally at P is the same as generating $C_P \otimes M$, where C_P is A_P modulo its Jacobson radical. (For $A = R$, we get a vector space and it is easy to find generators of the right form and lift to M; since m_1 is not basic at P, $\bar{m}_1 = 0 \in C_P \otimes M$ and causes no difficulty. Since (a, m_1) is basic at P and m_1 isn't, $a \notin P$. So we can choose $s \in aR$ to force $D(s) \subset \tilde{D}(a, a_1)$.) In general, C_P is a semisimple Artinian ring and we can utilize Lemma 5 of [2, p. 295], that is, Proposition (1.8). First, by (1.9), since $(\bar{a}, \bar{m}_1)$ is basic in $C_P \oplus C_P \otimes M$, we can find $\beta \in C_P$ such that $(\bar{a}\beta)^2 = (\bar{a}\beta) \in \operatorname{ann} \bar{m}_1$ and $(\bar{a}\beta, \bar{m}_1)$ is basic. Choose $n_2, \ldots, n_u \in M^*$ so that $m_1, n_2, \ldots, n_u$ is N-fold basic in M at P (recalling $P \in B_N$). If $N < u$, we apply (1.8) to the sequence $\bar{m}_1, \bar{n}_2, \ldots, \bar{n}_u$ and the element $(\bar{a}\beta, \bar{m}_1) \in C_P \oplus C_P \otimes M$ to find $\alpha_1, \ldots, \alpha_{u-1} \in C_P$ such that for all central units $\gamma \in C_P$, $\{\bar{m}_1 + \bar{a}\beta\alpha_1\gamma\bar{n}_u, \ldots, \bar{n}_{u-1} + \bar{a}\beta\alpha_{u-1}\gamma\bar{n}_u\}$ is N-fold basic. Choosing γ

properly, we can lift $\beta\alpha_i\gamma$ to A for each i. Thus, we can lift our new set to a subset of M that is N-fold basic in M at P and has the form $m_1+\tilde{n}, n_2', \ldots, n_{u-1}'$ with $\tilde{n}, n_2', \ldots, n_{u-1}' \in M^*$ and $\bar{\tilde{n}} \in \bar{a}\beta C_P \otimes M^*$. If $u-1>N$, we may repeat the process until we have produced a set of N elements, $m_1+\tilde{m}, m_2, \ldots, m_N$ with $\tilde{m}, m_2, \ldots, m_N \in M^*$ and $\bar{\tilde{m}} \in \bar{a}\beta C_P \otimes M^*$. As $(\bar{a}\beta)^2=(\bar{a}\beta)$ and $\bar{a}\beta\bar{m}_1=0$, $\bar{m}_1=(1-\bar{a}\beta)(\bar{m}_1+\bar{\tilde{m}})$. Lifting this equation to $A_P \otimes M$, we get

$$m_1-(1-abr^{-1})(m_1+\tilde{m}) \in (\text{Jacobson radical of } A_P)(A_P \otimes M),$$

where br^{-1} is a lifting of β. Next choose $m_{N+1}, \ldots, m_k \in M$ to complete the local generating set. We can now find a pair $s \in R-P$, $a_1 \in A$ such that $sM \subset A(m_1+\tilde{m}, m_2, \ldots, m_k)$, $sm_1=a_1(m_1+\tilde{m})+\cdots+a_k m_k$, and $a_1 \equiv (1-abr^{-1})s$ modulo the Jacobson radical of A_P. Noting $P \in \tilde{D}(a, a_1)$ if and only if $(\bar{a}, \bar{a}_1)C_P=C_P$, we observe $\bar{s}C_P=C_P$ and $\bar{s}=\bar{a}_1+\bar{a}(\overline{br}^{-1}\bar{s})$, yielding $P \in D(s) \cap \tilde{D}(a, a_1)$. Next we find a basic open set $D(t)$ with $P \in D(t) \subset D(s) \cap \tilde{D}(a, a_1)$. $\langle st, a_1 t\rangle$ is the desired m_1-suitable pair.

Thus we can cover $B_N \cap Z$ with m_1-suitable sets. By Theorem (1.5), $B_N \cap X$ is a patch and so then is $B_N \cap Z$. Hence it is quasi-compact and can be covered by finitely many m_1-suitable sets. We shall now prove (*) by induction on the number of m_1-suitable sets needed to cover $B_N \cap Z$. The case of no sets (i.e., $B_N \cap Z = \varnothing$) is trivial. Otherwise, let J be an ideal such that $V(J)=\overline{B_N \cap X}$; we note $\delta\text{-dim}(R/J) \leqslant \delta\text{-dim } V(J)=\sup\{\delta(P) \mid P \in B_N \cap X\}$, the equality holding since every prime in $V(J)$ contains a prime in $B_N \cap X$. If $P \in B_N \cap X$, M' is not $(N+1)$-fold basic in M at P and so $\delta(P)<N$. Hence $\delta\text{-dim}(R/J)<N$. Now choose some m_1-suitable set $D(s)$ from a minimal cover and let $\langle s, a_1\rangle$ be the corresponding m_1-suitable pair. Since $D(s) \subset \tilde{D}(a, a_1)$, we may apply (2.1) to the algebra $R/J \otimes A$ and the sequence $a, a_1, \ldots, a_N$. Lifting the result to A, we obtain elements $b_i \in s^2A(s-a_i)$ such that $V(J) \cap D(s) \subset \tilde{D}(a_1+ab_1, \ldots, a_N+ab_N)$. Write $b_1=sc_1(s-a_1)$ and $b_2=sc_2, \ldots, b_N=sc_N$ where each $c_i \in sA$. Now set $m^*=sc_1\tilde{m}+(c_2+c_1a_2)m_2+\cdots+(c_N+c_1a_N)m_N \in sM^*$ and, as usual, $m_1'=m_1+am^*$. As $m_1'-m_1 \in sM$, we have, for every $P \in V(s)$, m_1' is basic at P if and only if m_1 is basic at P. Now $sM \subset A(m_1+\tilde{m}, m_2, \ldots, m_k)$ tells us that $\mu(A_P, M_P)=k$ for $P \in D(s)$ and M' is N-fold basic in M at P for $P \in D(s)$. Thus $X \cap Y \subset V(s)$ and $X \cap B_t \subset V(s)$ for $t<N$; it follows that m_1' is $X \cap Y$-basic and $X \cap B_t$-basic for $t<N$. Letting $Z'=\{P \in X \mid m_1' \text{ is not basic in } M \text{ at } P\}$, we will have proved (*) and so Theorem (2.5), provided we can show $B_N \cap Z'$ can be covered by fewer m_1'-suitable sets than the number of m_1-suitable sets in a minimal cover of $B_N \cap Z$. We already know $B_N \cap Z' \cap V(s)=B_N \cap Z \cap V(s)$; we shall show $B_N \cap Z' \cap D(s)=\varnothing$, that is, m_1' is basic at P for each $P \in B_N \cap D(s)$. It suffices to prove sm_1' is basic. Noting $as^2c_1\tilde{m}=as^2c_1(m_1+\tilde{m})-as^2c_1m_1$, we see

$$\begin{aligned}
sm_1' &= sm_1+sam^* \\
&= (1-asc_1)sm_1+as^2c_1(m_1+\tilde{m})+a(sc_2+sc_1a_2)m_2+\cdots+a(sc_N+sc_1a_N)m_N \\
&= (1-asc_1)(a_1(m_1+\tilde{m})+\cdots+a_km_k)+as^2c_1(m_1+\tilde{m}) \\
&\quad +(ab_2+asc_1a_2)m_2+\cdots+(ab_N+asc_1a_N)m_N \\
&= (a_1+asc_1(s-a_1))(m_1+\tilde{m})+(a_2+ab_2)m_2+\cdots+(a_N+ab_N)m_N \\
&\quad +(1-asc_1)a_{N+1}m_{N+1}+\cdots+(1-asc_1)a_km_k \\
&= (a_1+ab_1)(m_1+\tilde{m})+\cdots+(a_N+ab_N)m_N+\cdots+(1-asc_1)a_km_k.
\end{aligned}$$

If we let F be the free module on the generators $(m_1+\tilde{m}),\ldots,m_k$, the map $F_P \to M_P$ is onto for $P \in D(s)$. Thus, to show sm_1' is basic at P it suffices to find a preimage of sm_1' which is basic in F_P. For $P \in B_N \cap X \cap D(s)$, the choice of $\{b_i\}$ guarantees the obvious preimage is basic, since $P \in \tilde{D}(a_1+ab_1,\ldots,a_N+ab_N)$ gives $1=(a_1+ab_1)d_1+\cdots+(a_N+ab_N)d_N$ for some $d_i \in A_P$. So the map $F_P \to A_P$ which sends $m_1+\tilde{m} \to d_1,\ldots,m_N \to d_N$, $m_{N+1} \to 0,\ldots,m_k \to 0$ is onto, sending our preimage to 1. Thus m_1' is basic at each prime in $B_N \cap X \cap D(s)$, that is, $B_N \cap Z' \cap D(s)=\varnothing$.

Suppose $B_N \cap Z \subset D(s) \cup (\bigcup D(s_i))$ where $\langle s_i, g_i\rangle$ is m_1-suitable for some $\{g_i\} \subset A$. Now $B_N \cap Z' = B_N \cap Z \cap V(s) \subset \bigcup D(s_i)$, so we will be done if for each i we can find an m_1'-suitable pair $\langle t_i, b_i\rangle$ with $B_N \cap Z \cap V(s) \cap D(s_i) \subset D(t_i)$. Let $m_1+\tilde{n}, n_2,\ldots,n_k$ be a local generating set on $D(s_i)$ corresponding to $\langle s_i, g_i\rangle$. To show a pair is m_1'-suitable, we can use this same generating set because $m_1+\tilde{n} \in m_1'+M^*$. As $m_1'=m_1+am^*$ with $m^* \in sM^*$, we may instead write $m_1'=m_1+s\hat{m}$. We have $s_i m_1 = g_1(m_1+\tilde{n})+\cdots+g_k n_k$ and $s_i\hat{m}=h_1(m_1+\tilde{n})+\cdots+h_k n_k$. Thus $s_i m_1' = (g_1+sh_1)(m_1+\tilde{n})+\cdots+(g_k+sh_k)n_k$. Clearly $\tilde{D}(a,g_1) \cap V(s) = \tilde{D}(a, g_1+sh_1) \cap V(s)$. As $D(s_i) \subset \tilde{D}(a, g_i)$, we obtain $D(s_i) \cap V(s) \subset \tilde{D}(a, g_1+sh_1)$. Hence, in Spec $R[s_i^{-1}]$, we get disjoint closed sets $D(s_i) \cap V(s)$ and $D(s_i) \cap \tilde{V}(a, g_1+sh_1)$. Applying Lemma (1.3), we find $r \in R$ such that $V(s) \cap D(s_i) \subset D(r)$ and $\tilde{V}(a, g_1+sh_1) \cap D(s_i) \subset V(r)$. We set $t_i = rs_i$ and $b_i = r(g_1+sh_1)$. Clearly $B_N \cap Z \cap V(s) \cap D(s_i) \subset D(t_i)$, and since $D(t_i) \subset \tilde{D}(a, g_1+sh_1) \cap D(r) \subset \tilde{D}(a,(g_1+sh_1)r)$ we see that $\langle t_i, b_i\rangle$ is an m_1'-suitable pair. □

REMARKS. Theorem 2.5 actually contains new information even in the Noetherian case. In Theorem A of [2, p. 282], the hypothesis needed to guarantee an element basic on all of Spec R is that the set M' be $(\dim(P)+1)$-fold basic in M at each prime P. Here we require only that M' be $(\delta(P)+1)$-fold basic in M. This may be smaller, and will be, for example, if R is local and not zero-dimensional.

On the other hand, the remark made in [2, p. 282] following Theorem A has no analogue here. Since the essence of our proof is to start at maximal primes and work down, restricting to primes of low height yields nothing.

Now we are ready to derive the corollaries, analogues of Corollaries 1, 4, and 5 to Theorem A. As the proofs require no alteration, they will not be repeated here. Assume $X = j$-spec R in the statements.

COROLLARY 2.6 (Serre's Theorem). *Let* δ-dim $R=d$.

(a) *If P is a finitely generated projective R-module whose rank at each localization is at least $d+1$, then P has a direct summand.*

(b) *With R and P as above, if P is generated by elements $m_1,\ldots,m_u$, then the generator of the free direct summand may be chosen to be of the form $m=m_1+a_2m_2+\cdots+a_um_u$ with all $a_i \in R$.*

Proof. See [2, p. 283].

COROLLARY 2.7 (Bass's Cancellation Theorem). *Let* δ-dim $R=d$ *and let P be a finitely generated projective R-module whose rank at each localization is at* *least $d+1$. Let Q be any finitely generated projective R-module and M any R-module. If $Q \oplus P \cong Q \oplus M$, then $P \cong M$.*

Proof. See [2, p. 285].

COROLLARY 2.8 (Forster–Swan). *Let N be a finitely presented R-module and suppose that $t = \max_{P \in X}(\delta(P) + \mu(R_P, N_P))$. Then N can be generated by t elements.*

Proof. See [2, p. 286 and note on p. 304].

COROLLARY 2.9. *Let N be a finitely generated R-module and suppose $n = \max \mu(R_P, N_P)$. Then N may be generated by δ-dim $R+n$ elements.*

Proof. Start with the exact sequence $0 \to M \to R^k \to N \to 0$. M is $(k-n)$-fold basic in R^k at each prime P. We select a $(k-n)$-fold basic subset consisting of finitely many elements for each P. Such a subset is $(k-n)$-fold basic on an open set. These open sets cover Spec R. Take a finite subcover; the union of the corresponding subsets of M is finite and $(k-n)$-fold basic in R^k at every prime P. This union generates a submodule $M^* \subset M$. Since $R^t/M^* \twoheadrightarrow N$, $\mu(R,N) \leqslant \mu(R, R^t/M^*)$, and since R^t/M^* is finitely presented, (2.8) yields $\mu(R, R^t/M^*) \leqslant \delta\text{-dim}\, R + n$. □

Actually, this proof encompasses a weakened version of [5, Lemma 1.1, p. 2] which asserts that any finitely generated module M is an image of a finitely presented module N (where we can assume that for finitely many quasi-compact sets X_i, if $\mu(R_P, M_P) \leqslant n_i$ for $P \in X_i$ then $\mu(R_P, N_P) \leqslant n_i$ for $P \in X_i$). However, since $\{P \mid \delta(P) = k\}$ needn't be quasi-compact, (2.9) can't be strengthened.

COROLLARY 2.10. *Let A be an R-algebra which is finitely presented as an R-module. Then corollaries* (2.6)–(2.9) *hold for A-modules.*

Proof. By (1.4), finitely presented A-modules will be X-appropriate and so we may apply (2.5), which is really all we need. □

3. In this section we shall attempt to justify the hypotheses we have "added" to the theorems—precisely, the redefinition of j-spectrum, use of δ-dim instead of j-dim, and the restriction to appropriate modules.

Treating the last item first, appropriate modules are simply a generalization of finitely presented R-modules; the necessity of a restriction of this type was observed in [5]. Of course, all modules are appropriate in the Noetherian case anyway. We include an example of a 1-dimensional ring, and a module which requires n generators locally at each $P \in \text{Spec}\, R = j\text{-spec}\, R$ but does not contain a basic element.

EXAMPLE 3.1. *Let F be a countable field, $\{T_i\}$ a countable set of indeterminates and $R = F[\{T_i\}]/(\{T_iT_j \mid i \neq j\})$. Let $\{m_i\}$ be a countable enumeration of the elements of the free module $M = R^{n+1}$ and set $N = M/(\{T_i m_i\})R$. Then $\mu(R_P, N_P) \geqslant n$ for all P but no element of N is basic on all of* Spec R.

Proof. Any basic element must be $\bar{m}_i$ for some i. However, if $P_i = (\{T_j \mid T_i \neq T_j\})R$, we see $\bar{m}_i R_{P_i} = 0$ and so $\bar{m}_i$ is not basic at P_i. For the other half, note that every prime Q contains some P_i since $\{P_i\}$ is the set of minimal primes of R. Hence $\{T_j m_j \mid j \neq i\}R \subset QN$ and so $\mu(R_Q, N_Q) = \mu(R_Q, (M/T_i m_i R)_Q) \geqslant \mu(R_Q, M_Q) - 1 = n$. □

Next we discuss why we have altered the definition of j-dimension. First we shall make some philosophical observations and then offer an example showing the failure of the theorems when the classical definition of j-dimension is used.

A topological space is spectral if it is T_0 and quasi-compact, if the quasi-compact open sets are closed under finite intersection and form an open basis, and if every nonempty irreducible closed subset has a generic point [4, p. 43]. The maximal spectrum is always T_0 and quasi-compact but the other two properties may fail. If the maximal spectrum is Noetherian, all open sets are quasi-compact and so the maximal spectrum satisfies the third property, making the absence of generic points its only topological deficiency. Classically, the j-spectrum was defined by adjoining the missing generic points (which is certainly the right thing to do if that was the only problem). However, the results in this area are inherently topological and it is not sufficient to add generic points when the maximal spectrum does not have a quasi-compact basis. In Example 3.2, we exhibit a ring whose maximal spectrum satisfies the generic point property but which has no quasi-compact open subsets except the entire spectrum. Moreover, the theorems in this paper fail when the classical definition of j-spectrum is used. Note that with the new definition, j-spec R is a spectral topology and the injection j-spec $R \to$ spec R is a spectral map.

EXAMPLE 3.2. *There exists a ring R whose maximal spectrum is isomorphic to the unit circle with the Euclidean topology and which has an invertible ideal which is not free. Since the only nonempty irreducible closed sets in* max-spec R *are points, the classical* j-dim $R = 0$ *and so* (2.5), (2.6), *and* (2.8) *are immediately contradicted with this definition.*

Proof. Let T be the ring of continuous functions from the unit circle $(x^2 + y^2 = 1)$ to the reals and let

$$R = \{f \in T \mid f(-x, -y) = f(x, y)\}.$$

Let $I = (x^2, xy)R$. It is well known that max-spec $R = \{M_{a,b}\}$, where $M_{a,b} = \{f \in R \mid f(a, b) = 0\}$. To see this, note $R/M_{a,b} \cong \mathbf{R}$ and so each $M_{a,b}$ is maximal. On the other hand, if M is maximal and $M \neq M_{a,b}$ for any (a, b) on the circle, we employ compactness to find $f_1, \ldots, f_k \in M$ such that for each (a, b), some $f_i(a, b) \neq 0$. Then $f_1^2 + \cdots + f_k^2$ is never zero and so M contains a unit. Now $M_{a,b} = M_{-a,-b}$, but otherwise the maximals are distinct and so max-spec $R \cong$ unit circle with antipodal points identified $\cong$ unit circle. Clearly the correspondence is a homeomorphism. Now let $I = (xy, x^2)R$. Since $(xy, x^2)(xy, y^2) = (x^2y^2, x^3y, xy^3) = xyR$ and xy is not a zero divisor, I is invertible. As $IT = xT$, I is principal if and only if there exists $u \in T$ such that $xu \in R$ and u is a unit in T.

Since x is antisymmetric and nonzero on a dense set, xu is symmetric if and only if u is antisymmetric. However, an antisymmetric function must be zero somewhere and so cannot be a unit.

Finally, we must consider the use of δ-dimension. Some of our results certainly require it. In our results, we talk about every prime in Spec R, not just primes in j-spec R. This may often be unimportant to our ends but can have interesting consequences, for example, Kronecker's Theorem (2.4). Without δ-dimension, we cannot do this. In a local ring, the fact that j-dim $R=0$ gives information about the maximal ideal only. However, if we are only concerned about j-spec R, is j-dim good enough? The author does not know; the techniques used here will not work. In fact, Vasconcelos and Wiegand obtained a bound on the number of generators of a module of $(\sup\{\mu(R_P, M_P)\})(j\text{-dim}\, R+1)$. This bound is not always higher than that given by (2.8). If j-dim $R<\delta$-dim R, this bound will be better for modules requiring few generators locally. For j-dim $R=0$, the [5]-bound is optimal; that bound also follows from this paper. To see this, note that while δ-dim $R>0$ can happen when j-dim $R=0$, it suffices to work over $R/\bigcap(\text{Max})$, a zero-dimensional ring. This illustrates another point—take homomorphic images when you can, for example, work modulo annihilator of M. The δ-dimension may go down.

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Department of Mathematics
The University of Texas
Austin, Texas 78712