POLYNOMIAL RINGS OVER A HILBERT RING

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The object of the present paper is to clarify the literature surrounding an incorrect result, Theorem 3.3 of [2], on just when polynomial rings in infinitely many variables over a Hilbert ring are again Hilbert.

A ring R (commutative with unity) is a Hilbert ring if every prime ideal of R is an intersection of maximal ideals. The concept of a Hilbert ring (also called a Jacobson ring) was introduced by Goldman in [4] and Krull in [9], where it is shown that if R is a Hilbert ring, then the polynomial ring $R[X]$ is again a Hilbert ring. In particular, the fact that for k a field, the polynomial ring $k[X_1,\dots,X_n]$ is a Hilbert ring yields a ring-theoretic formulation of the Hilbert Nullstellensatz. Krull also showed in [9] that a polynomial ring $k[\{X_i\}_{i=1}^{\infty}]$ in a countably infinite number of variables over a field k is a Hilbert ring if and only if the field k has uncountable cardinality. For $\{X_\lambda\}_{\lambda\in\Lambda}$ an infinite set of indeterminates, Gilmer in [2] considers the general question of when the polynomial ring $R[\{X_\lambda\}]$ is a Hilbert ring. Since a homomorphic image of a Hilbert ring is again a Hilbert ring, it is clear from Krull's result that if $R[\{X_i\}_{i=1}^{\infty}]=S$ is a Hilbert ring, then for each maximal ideal m of R, the field R/m must have uncountable cardinality (for $S/mS\cong(R/m)[\{X_i\}_{i=1}^{\infty}]$). In Theorem 3.3 of [2], Gilmer asserts that if R is a Hilbert ring and if $\{X_\lambda\}_{\lambda\in\Lambda}$ is an infinite set of indeterminates such that for each maximal ideal m of R, the cardinality of the field R/m is greater than that of the set Λ, then $S=R[\{X_\lambda\}]$ is a Hilbert ring. However, this assertion is incorrect as can be seen, for example, by taking R to be a 1-dimensional Noetherian domain containing an uncountable field and having a countably infinite number of maximal ideals. For R with this property, (0) is the intersection of the maximal ideals of R so that R is Hilbert. But for any maximal ideal m of R, the local ring R_m is a non-Hilbert ring that is a countably generated R-algebra, and hence a homomorphic image of the polynomial ring $R[\{X_i\}_{i=1}^{\infty}]$. Therefore $R[\{X_i\}_{i=1}^{\infty}]$ is not Hilbert. A specific example of such a ring R is the example given in [2, p. 211]. Let $\mathbf{C}$ be the field of complex numbers and let R be the localization of the polynomial ring $\mathbf{C}[X]$ at the multiplicative system generated by $\{X-\alpha\mid\alpha\in\mathbf{C}\setminus\mathbf{Z}\}$. Contrary to what is asserted in [2] and repeated in [6, Example 174, p. 145], for this ring R the polynomial ring $R[\{X_i\}_{i=1}^{\infty}]$ is not a Hilbert ring.

Gilmer informs me that the error in the proof of Theorem 3.3 of [2] occurs on page 210, 15 lines from the bottom, where it is stated that P_σ is an ideal. The above example also shows that the sufficiency assertion in Corollary 3.4 of [2] in order that $S=R[\{X_\lambda\}]$ be Hilbert is incorrect; and since Corollary 3.4 is used in the proof of Theorem 3.5 of [2], the status of this result is in need of clarification. A ring R is Hilbert if and only if each finitely generated R-algebra that is a

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field is an integral R-algebra, or equivalently if and only if for each maximal ideal M of the polynomial ring $R[X]$, $M\cap R$ is a maximal ideal of R. If $\{X_\lambda\}_{\lambda\in\Lambda}$ is an infinite set of indeterminates over R, then Gilmer in Theorem 3.5 of [2] observes the equivalence of the following statements:

(1) For each maximal ideal M of $S=R[\{X_\lambda\}]$, the ideal $M\cap R$ is maximal in R, and S/M is algebraic over $R/(M\cap R)$.

(2) For each maximal ideal M of $S=R[\{X_\lambda\}]$, the residue field S/M is integral over $R/(M\cap R)$.

The incorrect Corollary 3.4 is then used to conclude that condition (1) or (2) implies that $S=R[\{X_\lambda\}]$ is a Hilbert ring. It is correct that (1) or (2) implies S is Hilbert, and this can be seen as follows. Since $\{X_\lambda\}$ is infinite, S is R-isomorphic to the polynomial ring $S[Y]$. If S is not Hilbert, then there exists a maximal ideal M of $S[Y]$ such that $M\cap S$ is not maximal in S. We then have $R/(M\cap R)\subset S/(M\cap S)\subset S[Y]/M$, where $S/(M\cap S)$ is not a field. This contradicts condition (2).

It is stated in Theorem 3.5 of [2] that if $S=[\{X_\lambda\}]$ is Hilbert this does not imply conditions (1) or (2). But, as noted above, in the example given in [2, p. 311] to show this, the ring $S=R[\{X_i\}]$ is not Hilbert. In fact, for a Noetherian ring R, or more generally for a ring R that satisfies the descending chain condition (d.c.c.) on prime ideals, conditions (1) and (2) above are equivalent to the statement that $S=R[\{X_\lambda\}]$ is a Hilbert ring. To show this, let us fix some terminology. Let $|\Lambda|$ denote the cardinality of the infinite set $\{X_\lambda\}$. We will say that an R-algebra T is $|\Lambda|$-generated if T can be generated as an R-algebra by a set of cardinality $\leqslant|\Lambda|$. If $S=R[\{X_\lambda\}]$ is Hilbert, then for any maximal ideal m of R, $S/mS\cong(R/m)[\{X_\lambda\}]$ is Hilbert. By [2, Theorem 2.9], the field R/m has cardinality $>|\Lambda|$. Hence for any proper ideal I of R, the ring R/I has cardinality $>|\Lambda|$. It follows that if $S=R[\{X_\lambda\}]$ is Hilbert and M is a maximal ideal of S, then the field $L=S/M$ is algebraic over the quotient field K of $D=R/(M\cap R)$. Since K has cardinality $>|\Lambda|$, it follows from [2, Theorem 2.5] that a transcendental field extension of K cannot be $|\Lambda|$-generated as a K-algebra, and hence, *a fortiori,* as a D-algebra. Hence L is algebraic over K. We note also that K is $|\Lambda|$-generated as a D-algebra since L is $|\Lambda|$-generated as a D-algebra. For if $L=D[\{t_\lambda\}]$ and if we choose $a_\lambda\in D\setminus(0)$ such that t_λ is integral over $D[a_\lambda^{-1}]$, then L is integral over $D[\{a_\lambda^{-1}\}]$ so that $D[\{a_\lambda^{-1}\}]=K$.

If R had d.c.c. on prime ideals and $D=R/(M\cap R)$ is not a field, then D has a prime ideal P of height one. The following lemma completes a proof that conditions (1) and (2) hold for any ring R satisfying the d.c.c. on prime ideals and such that $R[\{X_\lambda\}]$ is a Hilbert ring.

LEMMA. *Let D be an integral domain with quotient field K and assume that K is $|\Lambda|$-generated as a D-algebra. If D contains a height-one prime ideal P, then there exists a non-Hilbert ring E between D and K such that E is $|\Lambda|$-generated as a D-algebra. In fact, E can be constructed so that K is a simple ring extension of E.*

Proof. Since K is $|\Lambda|$-generated as a D-algebra, there exists $\{t_\lambda\}_{\lambda\in\Lambda}\subset D\setminus(0)$ such that $K=D[\{t_\lambda^{-1}\}]$. Let $a\in P$, $a\neq0$, and let $V=D_P$. Then V is a 1-dimensional quasi-local domain, and for each t_λ there exists a positive integer n_λ such that $a^{n_\lambda} \in t_\lambda V$. Let $E = D[\{a^{n_\lambda}/t_\lambda\}]$. Then $E \subset V$, so $E \neq K$ but $E[a^{-1}] = D[\{t_\lambda^{-1}\}] = K$. Hence a is in every nonzero prime ideal of E. It follows, in particular, that E is not Hilbert. □

For R not satisfying d.c.c. on prime ideals, it can happen that the polynomial ring $R[\{X_i\}_{i=1}^{\infty}]$ is Hilbert and yet R does not satisfy conditions (1) and (2). As indicated by the argument given above, the existence of an R with this property is related to the existence of an integral domain D properly contained in its quotient field K such that K is countably generated as a D-algebra, but such that each ring between D and K that is countably generated as a D-algebra is Hilbert. Before presenting an example of an integral domain D having this property, we state and complete the proof for our main positive result.

THEOREM. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be an infinite set of indeterminates. If R is a Noetherian ring (or, more generally, if* Spec R *is Noetherian and R satisfies d.c.c. on prime ideals), then the following are equivalent:*
(i) *the polynomial ring $R[\{X_\lambda\}]$ is a Hilbert ring;*
(ii) *every $|\Lambda|$-generated R-algebra that is a field is an integral R-algebra; and*
(iii) *for each maximal ideal m of R, the field R/m has cardinality $>|\Lambda|$, and for each nonmaximal prime p of R, the set of primes q of R such that $p \subset q$ and* ht $q/p = 1$ *has cardinality $>|\Lambda|$.*

Proof. Condition (ii) is equivalent to conditions (1) and (2) of Theorem 3.5 of [2] listed above. Hence, from what we have shown above, (ii) implies (i) for any commutative ring R, and (i) implies (ii) for any R that satisfies d.c.c. on prime ideals. Also we have observed above that (i) implies R/m has cardinality $>|\Lambda|$ for each maximal ideal m of R. Let p be a nonmaximal prime of R and let $D = R/p$. Since R has d.c.c. on prime ideals, each nonzero prime ideal of D contains a prime ideal of D of height one. Hence if $\{\bar{q}_\alpha\}$ is the set of prime ideals of D of height one and $t_\alpha \in \bar{q}_\alpha$, $t_\alpha \neq 0$, then $D[\{t_\alpha^{-1}\}]$ is the quotient field of D. Condition (ii) implies that $|\{\bar{q}_\alpha\}| > |\Lambda|$. Therefore (i) and (ii) imply (iii) for any ring R having d.c.c. on prime ideals.

To complete the proof of the theorem, we show that (iii) implies (ii). Let L be a field that is $|\Lambda|$-generated as an R-algebra, and let D be the canonical homomorphic image of R in L. Condition (iii) implies that D has cardinality $>|\Lambda|$. Hence by [2, Theorem 2.5], L is algebraic over the quotient field K of D. As we have observed above, K is $|\Lambda|$-generated as a D-algebra since L is $|\Lambda|$-generated as a D-algebra. But if $D \neq K$, then (iii) implies that the set $\{q_\alpha\}$ of height-one prime ideals of D has cardinality $>|\Lambda|$. Since Spec D is Noetherian, a nonzero element t of D is contained in only a finite number of the $\bar{q}_\alpha$. Therefore, $D \neq K$ implies that K is not $|\Lambda|$-generated as a D-algebra. We conclude that $D = K$, and therefore that (iii) implies (ii). This completes the proof of the theorem. □

To obtain an example showing the necessity of the d.c.c. hypothesis in the above theorem, we prove the existence of an integral domain R with the following properties:

(a) R contains a field of uncountable cardinality.
(b) Spec R is Noetherian.
(c) For each nonzero prime ideal p of R, the residue class ring R/p is finite-dimensional, and the local ring R_p is an infinite dimensional valuation ring.
(d) For each nonzero nonmaximal prime p of R, there exists an uncountable number of prime ideals q of R such that $p \subset q$ and ht $q/p = 1$.
(e) There exists a set $\{t_i\}_{i=1}^{\infty}$ of nonzero nonunits of R such that each nonzero prime ideal of R contains t_i for some i.

Condition (e) implies that $R[\{t_i^{-1}\}_{i=1}^{\infty}]$ is the quotient field K of R, and hence that K is a homomorphic image of the polynomial ring $S = R[\{X_i\}_{i=1}^{\infty}]$. Therefore R and S do not satisfy conditions (1) and (2) of [2, Theorem 3.5], or the equivalent condition (ii) of the above theorem. But conditions (a)–(d) imply that $S = R[\{X_i\}_{i=1}^{\infty}]$ is a Hilbert ring. To show that S is Hilbert, it suffices to show that every nonmaximal prime P of S is an intersection of prime ideals of S that properly contain it ([4, p. 138] or [1, p. 71]). If $P \cap R = p \neq (0)$, then $S/pS \cong (R/p)[\{X_i\}]$ is Hilbert by the Theorem, so that P is an intersection of maximal ideals of S in this case. Suppose that $P \cap R = (0)$, and let $T = S/P$. If P is not equal to the intersection of the prime ideals of S that properly contain P, then the quotient field L of T is a simple ring extension of T. Hence L is countably generated as an R-algebra. Condition (a) implies that L is algebraic over K. The fact that L is a simple ring extension of T implies that any valuation ring W such that $T \subset W < L$ is contained in a rank one valuation ring U such that $T \subset U < L$. Since L/K is algebraic, $U \cap K = V$ is a rank one valuation ring on K. But condition (c) implies that $R_q = V$, where q is the center of V on R. This contradicts the fact that R_q is an infinite dimensional valuation ring. We conclude that conditions (a)–(d) imply that $S = R[\{X_i\}_{i=1}^{\infty}]$ is a Hilbert ring.

EXAMPLE. It remains to prove the existence of an integral domain R that satisfies conditions (a)–(e). For this purpose we use an existence theorem due to Jaffard [7, p. 78]. Jaffard has shown that if G is a lattice-ordered abelian group, then there is an integral domain R which has G as its group of divisibility. The construction is carried out using the group ring $B(G)$ of G with respect to an arbitrary field F. R has the same quotient field as $B(G)$ and F is contained in R. By taking the field F to be of uncountable cardinality, we insure that condition (a) is satisfied. An argument due to J. Ohm presented in [5, p. 1380] shows that the Jaffard construction actually yields a Bezout domain R. This implies that R_p is a valuation ring for each prime ideal p of R. Moreover, as observed in [5], there is a one-to-one inclusion preserving correspondence between the nonzero prime ideals of R and nonempty subsets J of the set G^+ of positive elements of G that have the following properties: $0 \notin J$; $a \in J$ and $b > a$ implies $b \in J$; $a, b \in J$ implies $\inf\{a, b\} \in J$, and $G^+ \backslash J$ is closed under addition. To obtain R satisfying conditions (b)–(e), we construct G as follows. Let $\mathbf{Z}$ denote the group of integers, and let Λ be an uncountable set. Let $A_1 = \mathbf{Z}^{(\Lambda)} = \{f : f$ is a finitely nonzero function from Λ to $\mathbf{Z}\}$. Then A_1 with componentwise ordering is a lattice-ordered abelian group. Let $B_1 = \mathbf{Z} \oplus A_1$, where B_1 is partially ordered by defining $(b, a) > (b', a')$ in $\mathbf{Z} \oplus A_1$ if $b > b'$ in $\mathbf{Z}$, or $b = b'$ and $a > a'$ in A_1. We identify A_1 with $(0, A_1)$ in B_1. Let $A_2 = B_1^{(\Lambda)} = \{f : f$ is a finitely nonzero function from Λ to $B_1\}$. Then A_2 with componentwise ordering is a lattice-ordered abelian group. Let $B_2 = \mathbf{Z} \oplus A_2$, where B_2 is partially ordered by defining $(b, a) > (b', a')$ if $b > b'$, or $b = b'$ and $a > a'$ in A_2. Similarly, we define $A_n = B_{n-1}^{(\Lambda)}$, and $B_n = \mathbf{Z} \oplus A_n$ for each positive integer n. If R_n is a domain obtained using the Jaffard construction with group of divisibility B_n, then $\dim R_n = n + 1$, $\operatorname{Spec} R_n$ is Noetherian, and R_n has a unique minimal nonzero prime ideal that corresponds to the set of elements (b, a) in $\mathbf{Z} \oplus A_n = B_n$ such that $b > 0$ in $\mathbf{Z}$. Moreover, for each nonzero nonmaximal prime ideal p of R_n, there exist uncountably many prime ideals q of R_n such that $p \subset q$ and $\operatorname{ht} q/p = 1$. For a fixed $\lambda \in \Lambda$ we identify B_{n-1} with the λ-component of $A_n = B_{n-1}^{(\Lambda)}$. Then B_{n-1} is a lattice-ordered subgroup of A_n. Also we identify A_n with the subgroup $0 \oplus A_n$ of $B_n = \mathbf{Z} \oplus A_n$ so that A_n is a lattice-ordered subgroup of B_n. Let G be the group obtained by taking the directed union of the B_n. Using the Jaffard construction over an uncountable field, we construct R with group of divisibility G. Then $R = \bigcup_{n=1}^{\infty} R_n$, where R_n has group of divisibility $B_n \subset G$. Let

$$J_n = \{g \in G : \text{for some } a_n \in A_n,\ g \geqslant (1, a_n) \in \mathbf{Z} \oplus A_n \subset G\}$$

and let P_n denote the prime ideal of R corresponding to J_n. Then $P_n = (P_n \cap R_n)R$ and $R/P_n \cong R_n/(P_n \cap R)$. Hence R/P_n is finite dimensional and $\operatorname{Spec}(R/P_n)$ is Noetherian. If t is any nonzero element of R and g is the image of t in G, then $g \in A_n$ for some n. It follows that $P_n \subset tR$. In particular, if Q is any nonzero prime ideal of R, then $P_n \subset Q$ for some n. Therefore R satisfies conditions (b), (d) and (e). Since $P_{n+1} < P_n$ for each n, R also satisfies condition (c), and hence provides the desired example.

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