A LATTICE POINT PROBLEM IN HYPERBOLIC SPACE

Peter Nicholls

1. Introduction. We consider the model of n-dimensional hyperbolic space which is given by the interior of the unit ball, $B=\{x: |x|<1\}$, where $x=(x_1,x_2,\dots,x_n)\in\mathbf{R}^n$ and $|x|=(\Sigma x_i^2)^{1/2}$. Lines in the space are arcs of circles orthogonal to the unit sphere, $S=\{x: |x|=1\}$, and angle is Euclidean angle. The hyperbolic metric ρ is derived from the differential

$$d\rho=\frac{2|dx|}{1-|x|^2}$$

and the hyperbolic lines are geodesics for this metric.

A Moebius transform preserving B is a product of an even number of inversions in spheres orthogonal to S and such transforms preserve the hyperbolic metric ρ. We denote by M the full group of all Moebius transforms preserving B. If G is a discrete subgroup of M and we select a point $x\in B$, then the collection of G-equivalents of x form a lattice of points in B. We shall be concerned in this paper with the way in which such a lattice is distributed in B.

Suppose x_1, x_2 are two points of B and s is a positive real number. For the discrete group G we define the counting function $N(s,x_1,x_2)$ to be the number of transforms $\gamma\in G$ such that $\rho(x_1,\gamma(x_2))<s$. We are concerned with the asymptotic behavior of $N(s,x_1,x_2)$ as s approaches infinity - this can be viewed as the hyperbolic analog of the Gauss circle problem.

The Dirichlet region D for the group G is defined by

$$D=\{x\in B: \rho(x,0)<\rho(\gamma(x),0) \text{ all } \gamma\in G, \gamma\neq I\}.$$

Now if γ is a Moebius transformation we denote by $\gamma'(x)$ the Jacobian matrix of γ at x and by $|\gamma'(x)|$ the positive number such that $\gamma'(x)/|\gamma'(x)|$ is orthogonal. In other words $|\gamma'(x)|$ is the linear change of scale at x, the same in all directions. Since $|\gamma'(x)|=(1-|\gamma(x)|^2)(1-|x|^2)^{-1}$ (see [1: ch. II]) then $|\gamma'(x)|<1$ if and only if $|\gamma(x)|>|x|$ and we see that

$$D=\{x\in B: |\gamma'(x)|<1 \text{ all } \gamma\in G, \gamma\neq I\}.$$

Hyperbolic volume V in B is derived from the differential

$$dV=\frac{2^n dx_1\, dx_2 \dots dx_n}{(1-|x|^2)^n},$$

where $x=(x_1,\dots,x_n)$. We denote by $V(G)$ the hyperbolic volume of D. In this paper we are concerned solely with the situation when $V(G)<\infty$ (the infinite volume case is discussed in an earlier paper of the author [11]).

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A few years ago S. J. Patterson [12] showed that, in dimension two, for a group G of finite volume

$$N(s,x_1,x_2) \sim V\{x: \rho(x,0)<s\}/V(G) \quad \text{as } s \to \infty$$

—a result that accords exactly with intuition. His proof makes essential use of the Selberg theory concerning the spectral decomposition of the Laplace operator on the quotient space and yields also the second order term. A weaker, but analogous result in dimension three has been proved by F. Fricker [4].

Similar analytic methods have been used very recently by Lax and Phillips [7] to obtain asymptotic estimates for the counting function for a wide class of discrete groups of Euclidean and non-Euclidean motions. In particular, their results contain Theorem A below.

Quite recently Aaronson and Sullivan (conversation with J. Aaronson) have proposed a new method for deriving the asymptotic formula above. This method, using an ergodic result of E. Hopf [5] concerning the geodesic flow on the quotient space, is valid in all dimensions but yields only the first order term. The asymptotic result is as follows.

THEOREM A. *Let G be a discrete group of finite volume acting in the unit ball of Euclidean n-space. If $x_1, x_2 \in B$ then*

$$N(s,x_1,x_2) \sim V\{x: \rho(x,0)<s\}/V(G) \quad \textit{as } s \to \infty.$$

In the case where G not only has finite volume but has a Dirichlet region D whose closure is compact in B we can do better.

THEOREM 1. *Let G be a discrete group acting in the unit ball of Euclidean n-space, if $\bar{D} \subset B$ then*

$$N(s,x_1,x_2) \sim V\{x: \rho(x,0)<s\}/V(G) \quad \textit{as } s \to \infty.$$

uniformly for all x_1, x_2 in B.

We remark that Theorem 1 is false without the assumption that $\bar{D} \subset B$. To see this, we consider a sequence $\{a_n\}$ of points of D with $|a_n|$ converging to one. Define $s_n = \rho(0, a_n)$ which is unbounded as n tends to infinity and note that, for all n, $N(s_n, 0, a_n) = 0$.

Since this proof of Theorem A has not appeared in the literature we will prove it, and Theorem 1, in Section 3. Our main purpose however, is to show how the ergodic method can be refined to obtain sharper orbital distribution results. We will show that each orbit under a discrete group of finite volume is uniformly distributed in all directions.

Let θ be a region on S which is obtained as the intersection of S with the interior of a ball. It is possible to consider more general regions than this—the crucial properties are that θ should be open, connected and such that the part of θ within ϵ of its boundary should have Euclidean $(n-1)$ dimensional area which goes to zero with ϵ. The region θ subtends at the origin a solid angle which we denote by Θ and, for s positive, we define

$$\Theta(s)=\Theta\cap\{x: \rho(x,0)<s\}.$$

For the discrete group G we wish to count orbit points which lie in $\Theta(s)$ and accordingly define, for s positive, $a\in B$ and Θ as above, $N(\Theta,a,s)$ to be the number of $\gamma\in G$ with $\gamma(a)\in\Theta(s)$. The measure of solid angles subtended at the origin is denoted by w and we state our main result.

THEOREM 2. *Let G be a discrete group of finite volume acting in the unit ball of Euclidean n-space. If $a\in B$ and θ is the intersection of S with the interior of a ball then*

$$N(\Theta,a,s)\sim w(\theta)V\{x: \rho(x,0)<s\}/V(G)w(S) \quad \textit{as } s\to\infty.$$

We note that weaker, integrated, versions of this result have been obtained by Tsuji [14: p. 557] in dimension two and the author [9] in dimension three.

We give two applications of these asymptotic results. The first is to the convergence of certain classical series associated with a discrete group.

THEOREM 3. *Let G be a discrete group of finite volume acting in the unit ball of Euclidean n-space and let t be a positive real number. The series*

$$\sum_{\gamma:|\gamma(0)|<s}(1-|\gamma(0)|)^t$$

converges, as $s\to 1$, if and only if $t>n-1$. Further,

$$\sum_{\gamma:|\gamma(0)|<s}(1-|\gamma(0)|)^{n-1}\sim\frac{w(S)}{V(G)}\log\left(\frac{1}{1-s}\right) \quad \textit{as } s\to 1,$$

and, if $t<n-1$,

$$\sum_{\gamma:|\gamma(0)|<s}(1-|\gamma(0)|)^t\sim\frac{w(S)}{V(G)}(n-1-t)^{-1}(1-s)^{t-n+1} \quad \textit{as } s\to 1.$$

Certain estimates on the partial sums of such series have been obtained, in the case $n=2$, by Lehner [8] and, in the case $n=3$, by Beardon and Nicholls [3].

The second application yields some quantitative results connected with a class of limit points called *points of approximation* [2], or *conical limit points* [13]. We first define a Stolz cone. Choose $\xi\in S$ and let $L(\xi)$ denote the radius to ξ. Now for $s>0$ we define the Stolz cone of opening s at ξ, denoted $S(\xi,s)$, by

$$S(\xi,s)=\{x\in B: |x|>\tfrac{1}{2} \text{ and } \rho(x,L(\xi))\leqslant s\}.$$

The point ξ is said to be a conical limit point for the discrete group G if one (and hence every) G orbit meets some Stolz cone at ξ in infinitely many points. This class of limit points is of great importance in the theory of discrete groups and has been studied extensively [2, 10, 13]. Suppose G is a discrete group with $\bar{D}\subset B$, then it is known that every point of S is a conical limit point. We can obtain estimates of the number of orbit points which approach a given conical limit point in a Stolz cone. Defining then, for $t,s>0$ and $\xi\in S, C(t,\xi,s)$ to be the number of γ in G with $\rho(0,\gamma(0))<t$ and $\gamma(0)\in S(\xi,s)$ we have the following result.

THEOREM 4. *Let G be a discrete group acting in the unit ball of Euclidean n-space such that $\bar{D}\subset B$. There exist real positive constants t_0, s_0, a, b, such that if $t>t_0$, $s>s_0$ then*

$$at \leqslant C(t,\xi,s) \leqslant bt$$

for any $\xi \in S$.

Theorem 4 is motivated by a classical result in the theory of Diophantine approximation and we conclude the introduction by indicating the connection.

Consider the modular group Γ,

$$\Gamma=\{\gamma : \gamma(z)=(az+b)(cz+d)^{-1};\ a,b,c,d\in\mathbf{Z},\ ad-bc=1\},$$

which is a Fuchsian group acting in the upper half of the complex plane. The set of limit points for this group comprises the extended real line and the conical limit points are precisely the finite irrationals.

Suppose $\gamma\in\Gamma$ and $\alpha\in\mathbf{R}$, denote by θ the angle between the (Euclidean) join of α and $\gamma(i)$ and the line $\{\operatorname{Re} z=\alpha,\ \operatorname{Im} z>0\}$. Some easy trigonometry and algebra lead to:

$$\operatorname{Cos}\theta=\frac{\operatorname{Im}\gamma(i)}{|\gamma(i)-\alpha|}=\{1+[ac+bd-\alpha(c^2+d^2)]^2\}^{-1/2},$$

where $\gamma(z)=(az+b)(cz+d)^{-1}$. Thus $|\theta|<\beta$ if and only if $|ac+bd-\alpha(c^2+d^2)|<M$ where β and M are related by a (complicated) formula. Now let $\{\gamma_n\}$ ($\gamma_n=(a_n z+b_n)(c_n z+d_n)^{-1}$) be a sequence in Γ with $\{\gamma_n(i)\}$ approaching α in a Stolz angle. We see, from the above, that for each such n, $|a_n c_n+b_n d_n-\alpha(c_n^2+d_n^2)|<M$. Now, consideration of isometric circles or some elementary algebra shows that the sequence $\{-d_n/c_n\}$ also converges to α and thus, for n large enough, $|d_n/c_n|<|\alpha|+1$. For such n,

$$|\alpha-a_n/c_n|<\left|\alpha-\frac{a_n c_n+b_n d_n}{c_n^2+d_n^2}\right|+\left|\frac{a_n}{c_n}-\frac{a_n c_n+b_n d_n}{c_n^2+d_n^2}\right|<\frac{M+|\alpha|+1}{c_n^2+d_n^2}<\frac{k}{c_n^2},$$

where k depends on the opening of the Stolz angle. Estimates such as this serve to show that counting solutions of the inequality $|\alpha-a/c|<k|c|^{-2}$ is essentially the same as counting group images of i approaching α in a Stolz angle.

A result in Diophantine approximation [6: p. 27] states that for almost all real α the number of integer solutions p,q of the inequalities $0<q\alpha-p<1/q$ and $1\leqslant q<N$ is asymptotic to $k\log N$ as $N\to\infty$. Theorem 4 gives estimates of this type (weaker, since they are only bounds and not asymptotic results) for more general discrete groups in all dimensions.

In Section 2 we give some details concerning the geodesic flow and the ergodic result we need. The proofs of Theorem A and Theorem 1 are given in Section 3 and Section 4 comprises the proof of Theorem 2. Finally, in Section 5, we give the proofs of Theorem 3 and 4.

2. The geodesic flow. In this section we define the geodesic flow and state some of its properties. The geodesic flow is first defined on the unit tangent space of the ball (we follow the treatment in Ahlfors [1] for this). We then show how to define the flow on the quotient of the tangent space by a discrete group G. Finally, we give the crucial result (due to E. Hopf [5: p. 291]) that when G is of finite volume then the geodesic flow on the quotient space is mixing.

Denote by $T(B)$ the unit tangent space of B. Thus a point of $T(B)$ consists of a *point* $x \in B$ and a direction at that point. The direction will be given by a unit vector $\xi \in S$. Thus $T(B)$ is the space of directed line elements (x, ξ). The Moebius group M acts in an obvious way on $T(B)$. If $\gamma \in M$ then x is mapped to $\gamma(x)$ and at the same time ξ is transformed by the matrix $\gamma'(x)$ to give the new direction $\gamma'(x)\cdot\xi$ for the line element at $\gamma(x)$—but in order to obtain a unit vector we must divide by $|\gamma'(x)|$.

Thus the action of γ on $T(B)$ is defined by

$$\gamma(x, \xi) = \left(\gamma(x), \frac{\gamma'(x)}{|\gamma'(x)|}\xi\right).$$

There is an invariant volume m on $T(B)$ derived from the element $dm = dV(x)\,dw(\xi)$ where $w(\xi)$ denotes the solid angle. The volume V is invariant under a Moebius γ, and ξ undergoes a rotation which keeps the spherical measure invariant.

We now define the *geodesic flow*, which is a one parameter group of diffeomorphisms g_t of $T(B)$ which satisfy $g_t \circ g_s = g_{t+s}$.

Every line element (x, ξ) determines a geodesic ray which starts from x in the direction ξ. Fix a real number t. Let x move along the geodesic from x to a point x' at a directed hyperbolic distance t from x. At the same time let the vector ξ slide to the positive tangent vector ξ' at x'. We define $g_t(x, \xi) = (x', \xi')$.

It is quite evident that $g_t \circ g_s = g_{t+s}$ and that $g_t^{-1} = g_{-t}$. It is also clear that the flow is invariant under Moebius transforms in the sense that $g_t \circ \gamma = \gamma \circ g_t$ for any Moebius γ preserving B. It is known that g_t is a flow in the sense that each g_t leaves the volume element dm invariant (see [1: p. 76] for a proof).

Now suppose G is a discrete subgroup of M with Dirichlet region D as defined in Section 1. Consider the quotient space $\Omega = T(B)/G$ with the projection map $\pi: T(B) \to T(B)/G$. A subset A of Ω is said to be *measurable* if $\pi^{-1}(A)$ is m-measurable in $T(B)$ and in that case we define the m-measure of A: $m(A) = m\{(x, \xi) \in \pi^{-1}(A) : x \in D\}$. Note that Ω is measurable and that $m(\Omega)$ is finite if and only if G is a group of finite volume. In that case we have $m(\Omega) = V(D)\,w(s)$.

Since g_t is invariant under Moebius transforms and preserves the measure m on $T(B)$, we see that it acts as a flow on Ω. This action, of g_t on the quotient space Ω, has been studied extensively for many years and much is known. From our point of view the most important property of g_t is that it is *mixing* for groups of finite volume. This result, Lemma 2.1 below, is due to E. Hopf [5, p. 291].

LEMMA 2.1. *If G is a group of finite volume then the flow g_t on Ω is mixing. In other words, if A_1 and A_2 are measurable subsets of Ω then*

$$\lim_{t\to\infty} m\{A_1 \cap g_t(A_2)\} = \frac{m(A_1)\cdot m(A_2)}{m(\Omega)}.$$

For our purposes it is more useful to have a formulation of this result in the line element space $T(B)$. Considering one sheet of $\pi^{-1}(T(B)/G)$ we derive the following from Lemma 2.1.

LEMMA 2.2. *If G is a group of finite volume and if A_1, A_2 are measurable subsets of $D\times S$ (itself a subset of $T(B)$) then*

$$\lim_{t\to\infty} m\{G(A_1)\cap g_t(A_2)\} = \frac{m(A_1)m(A_2)}{V(D)\cdot w(S)}.$$

3. Proofs of Theorem A and Theorem 1. Let G be a discrete group of finite volume acting in B with Dirichlet region D centered at the origin. Let $\Delta\subset D$ be a ball centered at $a\in D$. We will use the ergodic result, Lemma 2.2, to prove the following asymptotic formula.

$$\lim_{t\to\infty}\frac{w[\{x:\rho(x,0)=t\}\cap G(\Delta)]}{w(S)} = \frac{V(\Delta)}{V(D)}, \tag{3.1}$$

where w, as before, denotes solid angle subtended at the origin. Thus (3.1) says that the group images of Δ ultimately cover their fair share of large enough hyperbolic spheres centered at the origin.

We show first how (3.1) leads to the desired asymptotic formula for the counting function.

Upon integration we obtain from (3.1):

$$\lim_{t\to\infty}\frac{V[\{x:\rho(x,0)<t\}\cap G(\Delta)]}{V\{x:\rho(x,0)<t\}} = \frac{V(\Delta)}{V(D)}. \tag{3.2}$$

Now choose $\epsilon>0$ and find $\delta>0$ so small that for $t>t_0$, say,

$$V\{x:\rho(x,0)<t+\delta\}/V\{x:\rho(x,0)<t\}<1+\epsilon,$$

which may be done since, in n-dimensions, $V\{x:\rho(x,0)<t\}\sim k\exp[(n-1)t]$. We apply (3.2) to the ball Δ of radius δ and deduce that, for $t>t_1$, say,

$$V[\{x:\rho(x,0)<t+\delta\}\cap G(\Delta)]/V\{x:\rho(x,0)<t+\delta\}<V(\Delta)(1+\epsilon)/V(D).$$

Now if $\rho(\gamma(a),0)<t$ then $\gamma(\Delta)\subset\{x:\rho(x,0)<t+\delta\}$, and so

$$V(\Delta)N(t,0,a)\leqslant V[\{x:\rho(x,0)<t+\delta\}\cap G(\Delta)].$$

Using the inequalities above we see that, for $t>t_1$,

$$N(t,0,a)/V\{x:\rho(x,0)<t\}\leqslant(1+\epsilon)^2/V(G).$$

A similar lower bound shows that

$$N(t,0,a)\sim V\{x:\rho(x,0)<t\}/V(G) \tag{3.3}$$

as $t\to\infty$.

Now consider two points x_1, x_2 in B and let γ be a Moebius transform with $\gamma(x_1)=0$, $\gamma(x_2)=w$, say. We write $\Gamma=\gamma G\gamma^{-1}$ and note that $V(\Gamma)=V(G)$. Clearly,

$$\rho(x_1, g(x_2)) = \rho(\gamma(x_1), \gamma g(x_2))$$
$$= \rho(0, \gamma g \gamma^{-1}(w))$$

for any $g \in G$. It follows that

$$N_G(s, x_1, x_2) = N_\Gamma(s, 0, w).$$

Theorem A is now an immediate consequence of (3.3).

It remains to prove (3.1). The idea is as follows. Let Δ be the fixed ball with hyperbolic radius δ and hyperbolic center a. Now let C be a ball centered at the origin of hyperbolic radius r. We define two subsets A_1, A_2 of $T(B)$ by

$$A_1 = \Delta \times S, \quad A_2 = C \times S$$

and apply Lemma 2.2. Note that $g_t(A_2)$ is the "annulus"

$$\{x: t - r < \rho(x, 0) < t + r\}$$

together with a set of directions at each point. The lemma now gives information about how much of this "annulus" is covered by group images of A_1. We are able to account for the direction set at each point and, by letting r tend to zero, we will be able to derive (3.1).

To save writing we will denote $g(s) = w[\{x: \rho(x, 0) = s\} \cap G(\Delta)]$ and the following result is needed.

LEMMA 3.1. *The function $g(s)$ is uniformly continuous on* $(0, \infty)$.

Proof. We will show that, in dimension n $(n \geqslant 2)$, $|g(s) - g(s-\epsilon)| = O(\epsilon^{n-1})$ uniformly in s as $\epsilon \to 0$. We assume first that the radius, δ, of Δ is so small that $G(\Delta)$ is a non-overlapping set of balls. We ask how many such balls can intersect the sphere $\{x: \rho(x, 0) < s\}$, and accordingly define $n(s, \Delta)$ to be the number of $\gamma \in G$ with the property that $\gamma(\Delta) \cap \{x: \rho(x, 0) < s\} \neq \varnothing$. We can obtain an upper bound on $n(s, \Delta)$ by a volume argument and, in fact,

$$n(s, \Delta) < kV\{x: \rho(x, 0) < s\} \quad \text{for } s \geqslant s_0, \text{ say,} \tag{3.4}$$

where k is an absolute constant. The proof of (3.4) is exactly analogous to that given by Tsuji in dimension two [14: p. 516] and by the author in dimension three [9].

Now consider a single image Δ' of Δ and we claim that the difference

$$|w(\Delta' \cap \{x: \rho(x, 0) = s - \epsilon\}) - w(\Delta' \cap \{x: \rho(x, 0) = s\})|$$

is maximized if Δ' is internally tangent to the sphere $\{x: \rho(x, 0) = s\}$. To see this, consider the two-dimensional case with s large and ϵ small (compared to δ). The intersection $\Delta' \cap \{x: \rho(x, 0) = s\}$ is essentially a chord of Δ' and the difference given above reduces to a multiple (depending on s and ϵ) of the difference in angle subtended at the Euclidean center of Δ' by two parallel chords. Our claim follows easily from this. □

Now the set $\Delta' \cap \{x: \rho(x, 0) = s - \epsilon\}$ is almost (for s large and ϵ small) an $(n-1)$ dimensional ball, situated in the hyperplane normal to the radius of B which

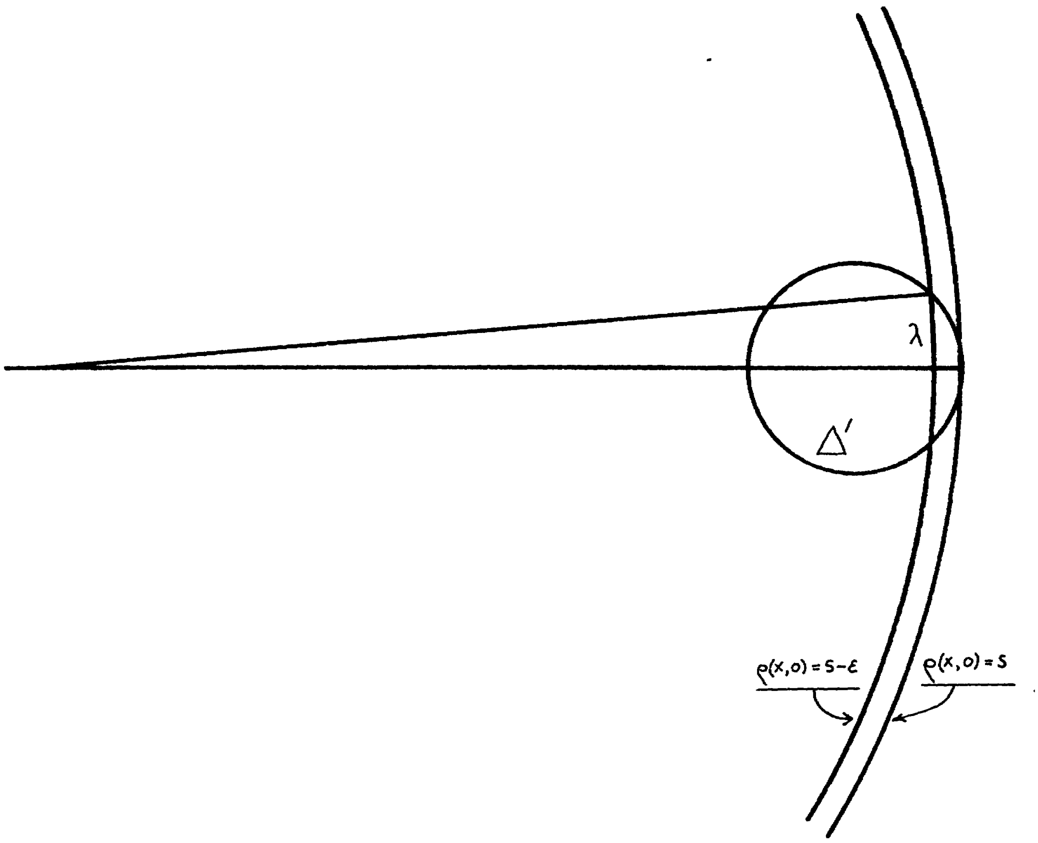

Figure 1

passes through the center of Δ', and whose hyperbolic distance from the origin is $s-\epsilon$. See Figure 1, which illlustrates the situation in dimension two.

The solid angle subtended at the origin by this $(n-1)$ dimensional ball is asymptotic to a constant times λ^{n-1} where λ is its Euclidean radius.

Calculation shows that λ is asymptotic (as $s\to\infty$) to $\epsilon\delta\exp(-s)$. As can be seen from Figure 1, this calculation requires only Pythagoras' theorem but is fairly tedious as it involves the *Euclidean* radius of Δ' which is given by the formula

$$r=\frac{1}{2}\left[\frac{e^t-1}{e^t+1}-\frac{e^{t-2\delta}-1}{e^{t-2\delta}+1}\right].$$

Thus the difference

$$|w(\Delta'\cap\{x:\rho(x,0)=s-\epsilon\})-w(\Delta'\cap\{x:\rho(x,0)=s\})|$$

is bounded by a quantity which, as $s\to\infty$, is asymptotic to a constant times $(\delta\epsilon)^{n-1}\exp[-(n-1)s]$. If we assume that this maximum difference is attained for every single image Δ' of Δ which intersects the ball $\{x:\rho(x,0)\leqslant s\}$, we obtain from the above and (3.4) that $|g(s)-g(s-\epsilon)|$ is bounded above by a quantity which is asymptotic (as $s\to\infty$) to

$$k(\delta\epsilon)^{n-1}\exp[-(n-1)s]V\{x:\rho(x,0)\leqslant s\}.$$

The proof of Lemma 3.1 is complete when we note that $V\{x:\rho(x,0)\leqslant s\}$ is asymptotic to a constant times $\exp[(n-1)s]$ (this is obtained by integrating the volume element dV over the ball).

We now prove (3.1). Let $a\in D$ be fixed and let $\Delta\subset D$ be a fixed hyperbolic disc centered at a. Define A_1 and A_2 as before. It becomes necessary at this point to consider both Euclidean and non-Euclidean radii of spheres centered at the origin. For the remainder of this section, whenever s is a positive real we define

$$s_1=\frac{\sinh(\frac{1}{2}s)}{[1+\sinh^2(\frac{1}{2}s)]^{1/2}}$$

and note that $\rho(x,0)=s$ if and only if $|x|=s_1$.

We make two further definitions. If $0<t<\infty$ then set

$$X(t)=\{x\colon (x,\xi)\in g_t(A_2) \text{ for some } \xi\in S\}$$

and for any $x\in B$ set $I(x,t)=\{\xi\in S\colon (x,\xi)\in g_t(A_2)\}$. Trivially $I(x,t)\neq\phi$ if and only if $x\in X(t)$. It follows from the symmetry of the situation that if $\rho(x_1,0)=\rho(x_2,0)=s$ then

$$\begin{aligned} w[I(x_1,t)]&=w[I(x_2,t)]\\ &=L(s,t), \text{ say.}\end{aligned}$$

We are now in a position to use Lemma 2.2, and we observe that

$$\begin{aligned} m[G(A_1)\cap g_t(A_2)]&=\int_{X(t)} w[I(x,t)]\chi_{G(\Delta)}(x)\,dV(x)\\ &=\int_{t-r}^{t+r}\frac{L(s,t)g(s)s_1^{n-1}}{(1-s_1^2)^n}\frac{ds_1}{ds}\,ds\end{aligned}$$

where we recall that $X(t)$ is the annulus $\{x\colon t-r<\rho(x,0)<t+r\}$. Now observe that $m(A_2)=m(g_t(A_2))$ and

$$\begin{aligned} m(g_t(A_2))&=\int_{X(t)} w[I(x,t)]\,dV(x)\\ &=\int_{t-r}^{t+r}\frac{L(s,t)w(S)s_1^{n-1}}{(1-s_1^2)^n}\frac{ds_1}{ds}\,ds.\end{aligned}$$

We thus write $m[G(A_1)\cap g_t(A_2)]/m(A_2)$ as the quotient of two integrals and, from the continuity of the integrands, we see that for some s satisfying $t-r<s<t+r$ we have $m[G(A_1)\cap g_t(A_2)]/m(A_2)=g(s)/w(S)$. However, by Lemma 3.1, g is uniformly continuous and so, given $\epsilon>0$, we find $r>0$ so small that $|m[G(A_1)\cap g_t(A_2)]/m(A_2)-g(t)/w(S)|<\epsilon$. From Lemma 2.2,

$$m[G(A_1)\cap g_t(A_2)]/m(A_2)\to m(A_1)/V(D)w(S)$$

as $t\to\infty$. We note that $m(A_1)=V(\Delta)w(S)$ and deduce that

$$\lim_{t\to\infty} g(t)/w(S)=V(\Delta)/V(D).$$

This is (3.1) and the proof of Theorem A is complete. □

We now prove Theorem 1. Suppose G is a discrete group with $\bar{D}\subset B$, suppose further that there exist sequences $\{a_n\}\subset B$ and $\{s_n\}$, with $s_n\to\infty$ and that, for some $\epsilon>0$,

$$|N(s_n,0,a_n)/V\{x:\rho(x,0)<s_n\}-1/V(G)|>\epsilon \tag{3.5}$$

for all n. We will derive a contradiction. It is clear that for all $\gamma\in G$ and any $w\in B$, $N(s,0,w)=N(s,0,\gamma(w))$, and so we may assume that for each n, $a_n\in\bar{D}$. Thus, passing to a subsequence if necessary, we assume $a_n\to a$ in B. From (3.5) we verify that $|N(s_n,0,a)/V\{x:\rho(x,0)<s_n\}-1/V(G)|>\epsilon$ for n large enough. This contradiction with Theorem A completes the proof of Theorem 1. □

4. Proof of Theorem 2. Let G be a discrete group of finite volume acting in B with Dirichlet region D centered at the origin. Let $\Delta\subset D$ be a disc centered at the origin. Let $\theta\subset S$ be the intersection of S with the interior of a ball, and denote by Θ the solid angle subtended at the origin by θ. For $s>0$ set $\Theta(s)=\Theta\cap\{x:\rho(x,0)<s\}$. We will use the ergodic result, Lemma 2.2, to prove the following asymptotic formula:

$$\lim_{t\to\infty}\frac{w[\{x:\rho(x,0)=t\}\cap\Theta\cap G(\Delta)]}{w(\theta)}=\frac{V(\Delta)}{V(D)}. \tag{4.1}$$

Note that this formula generalises (3.1) and shows that the group images of Δ ultimately cover their fair share of all conical sections of large enough hyperbolic spheres centered at the origin.

Before deriving (4.1) we show how it leads to the desired asymptotic formula for the counting function.

Upon integration we obtain from (4.1):

$$\lim_{t\to\infty}\frac{V[\Theta(t)\cap G(\Delta)]}{V[\Theta(t)]}=\frac{V(\Delta)}{V(D)}. \tag{4.2}$$

We recall that $N(t,\Theta)$ is the number of $\gamma\in G$ with $\gamma(0)\in\Theta(t)$ and, using estimates similar to those given in Section 3, we see that

$$\lim_{t\to\infty}\frac{V(\Delta)\cdot N(t,\Theta)}{V[\Theta(t)\cap G(\Delta)]}=1.$$

Combining this with (4.2) yields:

$$\lim_{t\to\infty}\frac{N(t,\Theta)}{V[\Theta(t)]}=\frac{1}{V(D)},$$

which is the required asymptotic estimate for $N(s,\Theta)$.

It remains now to prove (4.1). The proof proceeds along similar lines to the proof of (3.1). Accordingly we fix Δ and define $A_1=\Delta\times S$; now let C be a disc centered at the origin and of hyperbolic radius r. We define $A_2=C\times\theta$ and note that A_2 is different from the A_2 defined in Section 3—in the present case we assign to each point in C only those directions in θ. It is useful to observe that $\Theta(t)=\bigcup_{0<s<t}g_t(\{0\}\times\theta)$ and thus $g_t(A_2)$ is (for small r) a thin shell whose

cross-section is approximately $g_t(\{0\}\times\theta)$ (i.e., $\Theta\cap\{x:\rho(x,0)=t\}$), together with a set of directions at each point.

Explicitly, this shell is given by $X(t)=\{x: (x,\xi)\in g_t(A_2)$ for some $\xi\in S\}$ and the set of directions at each point x of $X(t)$ is given by

$$I(x,t)=\{\xi\in S: (x,\xi)\in g_t(A_2)\}.$$

For economy of notation we have used the same symbols, $X(t)$ and $I(x,t)$, as were used in Section 3—their meaning is of course different in this section. The shell $X(t)$ is now a subset of the "annular region" $\{x: t-r<\rho(x,0)<t+r\}$. Of great importance is the fact that, contrary to the situation in Section 3, the values of the angular measure w of two direction sets $I(x_1,t), I(x_2,t)$ associated with points x_1, x_2 of equal modulus in $X(t)$ are not necessarily equal. This fact gives rise to an added difficulty in computing $m[G(A_1)\cap g_t(A_2)]/m(A_2)$, which is required for the application of Lemma 2.2.

We will show that given t and s satisfying $t-r<s<t+r$, then

$$X(t)\cap\{x:\rho(x,0)=s\}$$

comprises an *admissible part,* any two points of which have direction sets of the same angular magnitude, and an *inadmissible part* which, for r close enough to zero, is so small as to make no difference in our asymptotic estimates.

Accordingly, suppose $x\in X(t)$. Then, for some $\xi\in S$, $(x,\xi)\in g_t(A_2)$ and so the sphere $\{y:\rho(x,y)=t\}$ intersects C. Join each point z of this intersection to x by a geodesic ray and let ξ_z be the direction at z which determines this geodesic. We say that x is *admissible* if each such $\xi_z\in\theta$. To put it another way: If $x\in X(t)$ then x is obtained by moving a distance t from a point of C along a geodesic in a direction belonging to θ; x is *admissible* if it can be so obtained from *any* point of C which is distant t from x. Figure 2 illustrates an admissible point x and the set of directions $I(x,t)$. It will be seen that if $x_1,x_2\in X(t)$ are both admissible and if $|x_1|=|x_2|$, then $I(x_2,t)$ is merely a rotation of $I(x_1,t)$ and consequently $w[I(x_2,t)]=w[I(x_1,t)]$.

As regards the inadmissible set we have the following.

LEMMA 4.1. *Given* $\epsilon>0$ *there exists* $r_0>0$ *such that if* $r<r_0$ *and* s,t *satisfy* $1<t-r<s<t+r$ *then the inadmissible part of* $X(t)\cap\{x:\rho(x,0)=s\}$ *has angular measure* (w) *less than* ϵ.

Proof. Suppose $x\in X(t)$ is inadmissible. Then there exists a point $z\in C$ with $\rho(z,x)=t$ and such that the geodesic connecting z to x determines a direction at z which does not belong to θ. On the other hand, x *is* obtained by moving a distance t from a point of C along a geodesic in a direction belonging to θ. We may as well suppose that this latter point is on the radius joining 0 to x. Figure 3 illustrates the situation.

A straightforward calculation shows that the Euclidean separation of the two points ξ_1,ξ_2 of Figure 3 is $O(r)$ as $r\to 0$ provided that $\rho(x,0)>1$, say. It follows then that the radial projection of x onto S has a separation from the boundary of

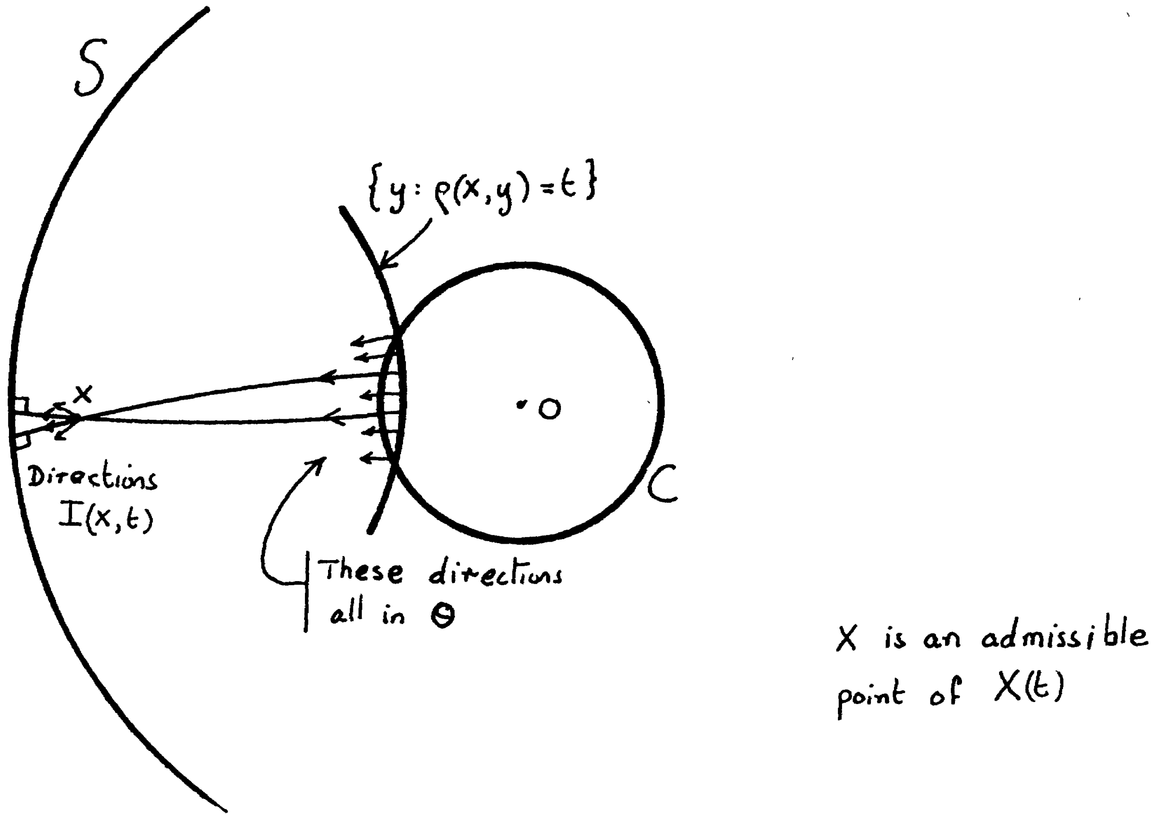

Figure 2

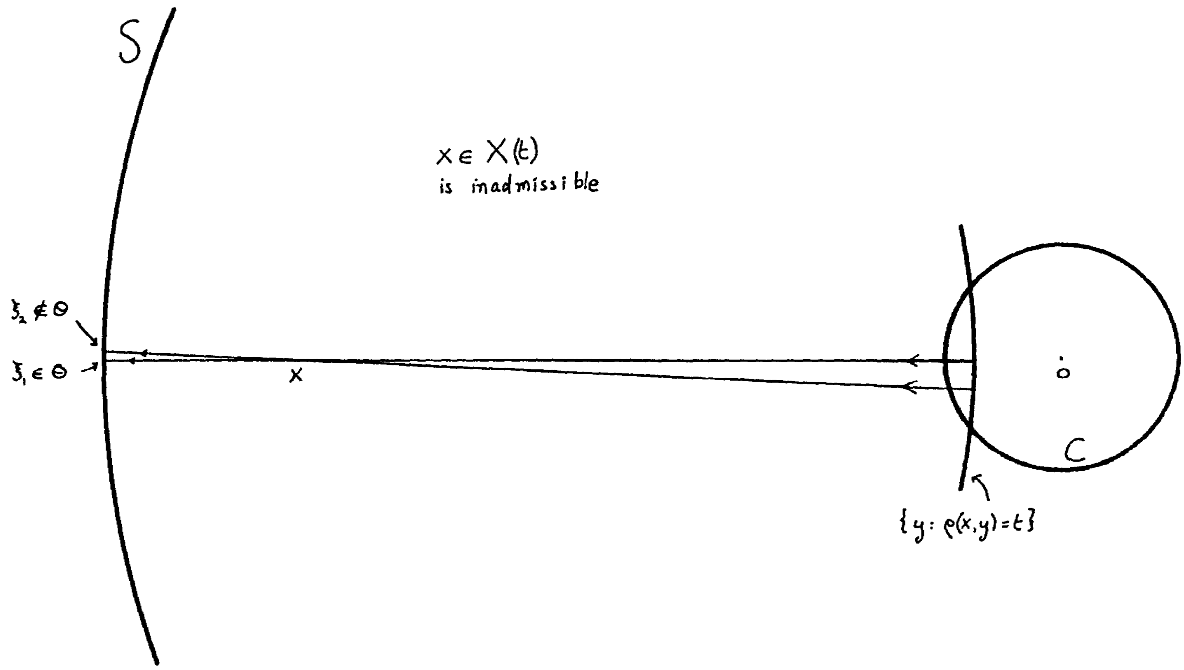

Figure 3

θ which is of the order $O(r)$. Thus the radial projection of the inadmissible set onto S is contained in a band of width $O(r)$ around the edge of θ. Since θ is the intersection of S with the interior of a ball, this band has an area which approaches zero with r. Since the angular measure of our inadmissible set is the area of its projection onto S, we see that the lemma is proved. □

We return to the proof of (4.1) and note that

$$m[G(A_1)\cap g_t(A_2)] = \int_{X(t)} w[I(x,t)]\chi_{G(\Delta)}(x)\,dV(x). \tag{4.3}$$

The right side of (4.3) is computed as a double integral—first over the sphere $\{x: \rho(x,0)=s\}$ and then radially, letting s vary from $t-r$ to $t+r$. We approximate this first integral, replacing $X(t)\cap\{x: \rho(x,0)=s\}$ by the slightly smaller set $\Theta\cap\{x: \rho(x,0)=s\}$ and then assuming that every point in the latter set is admissible. The error obtained may be estimated using Lemma 4.1. For admissible x in $X(t)$ satisfying $\rho(x,0)=s$, we denote by $L(s,t)$ the angular measure $w[I(x,t)]$.

Given $\epsilon>0$ we find r_0 so small that if $r<r_0$ and $t>1$, say, then

$$m[G(A_1)\cap g_t(A_2)] \Big/ \int_{t-r}^{t+r} \frac{L(s,t)s_1^{n-1}}{(1-s_1^2)^n} w[\{x: \rho(x,0)=s\}\cap\Theta\cap G(\Delta)]\frac{ds_1}{ds}\,ds$$

differs from 1 by at most ϵ.

Similarly, for the same values of r and t,

$$m[g_t(A_2)] \Big/ \int_{t-r}^{t+r} \frac{L(s,t)s_1^{n-1}}{(1-s_1^2)^n} w(\theta)\frac{ds_1}{ds}\,ds$$

differs from 1 by at most ϵ.

We proceed as in the proof of (3.1), using the uniform continuity of $w[\{x: \rho(x,0)=s\}\cap\Theta\cap G(\Delta)]$ (which follows from the proof of Lemma 3.1), to deduce that

$$\lim_{t\to\infty} w[\{x: \rho(x,0)=t\}\cap\Theta\cap G(\Delta)]/w(\theta) = \lim_{t\to\infty} m[G(A_1)\cap g_t(A_2)]/m[A_2].$$

By Lemma 2.2 this latter limit is equal to $m(A_1)/V(D)\cdot w(S)$ and, since $m(A_1)=V(\Delta)\cdot w(S)$, we see that

$$\lim_{t\to\infty} w[\{x: \rho(x,0)=t\}\cap\Theta\cap G(\Delta)]/w(\theta) = V(\Delta)/V(D).$$

This is (4.1) and the proof of Theorem 2 is complete. □

5. Proofs of Theorem 3 and 4. Let G satisfy the hypotheses of Theorem 3 and define, for $0<r<1$, $n(r)$ to be the number of $\gamma\in G$ with $|\gamma(0)|<r$. Clearly $n(r)=N(\log((1+r)/(1-r)),0,0)$ and so, by Theorem A, $n(r)\sim V\{|x|<r\}/V(G)$ as $r\to 1$. From the definition of solid angle measure,

$$V\{|x|<r\} = \int_0^r \frac{2^n w(S)t^{n-1}}{(1-t^2)^n}\,dt,$$

and we see that

$$n(r) \sim \frac{w(S)}{(n-1)V(G)(1-r)^{n-1}}. \tag{5.1}$$

Theorem 3 now follows from (5.1) and the fact that, for positive t,

$$\sum_{|\gamma(0)|<s} (1-|\gamma(0)|)^t = \int_0^s (1-r)^t \, dn(r).$$

Now let G satisfy the hypotheses of Theorem 4 and, from Theorem 1, we find s_0 such that if $s \geqslant s_0$

$$\tfrac{1}{2}V\{x: \rho(x,0)<s\}/V(G) < N(s,x_1,x_2) < 2V\{x: \rho(x,0)<s\}/V(G) \tag{5.2}$$

for all x_1, x_2 in B. Now choose $\xi \in S$ and form a sequence of discs D_n, each of hyperbolic radius s_0 and whose centers a_n lie on the radius to ξ and satisfy $\rho(a_n, 0) = 2ns_0$.

Those open discs are mutually exterior and are all contained in the Stolz cone $S(\xi, s_0)$. From (5.2) we see that the number of images of 0 contained in each D_n is at least $\frac{1}{2}V\{x: \rho(x,0)<s\}/V(G)$. The lower bound of Theorem 4 now follows when we observe that the number of complete discs D_n contained in $\{x: \rho(x,0)<t\}$ is, for t large enough, the integer part of $t/2s_0$ plus one.

To obtain the upper bound of Theorem 4 we proceed as follows. Any point of $S(\xi, s_0)$ is at most a hyperbolic distance s_0 from the radius to ξ and therefore at most a hyperbolic distance $2s_0$ from one of the points a_n previously defined. Thus the cone $S(\xi, s_0)$ is contained in the union of discs D_n' which are each of radius $2s_0$ and centered at the points a_n. The required result follows from (5.2) when we observe that $S(\xi, s_0) \cap \{x: \rho(x,0)<t\}$ is contained in $\bigcup_{n=0}^{N} D_n'$, where N is the integer part of $t/4s_0$ plus one.

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Department of Mathematics
Northern Illinois University
DeKalb, Illinois 60115