STABLE BASE LOCI OF REPRESENTATIONS OF ALGEBRAIC GROUPS

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Introduction. It is well known that the ring of invariant functions under the action of a group G on an affine k-algebra R need not again be an affine k-algebra. A counterexample to the original problem posed by Hilbert (the 14th problem) was first given by Nagata [7] and was suggested by earlier work of Zariski [12] and Rees [8]. This paper investigates the connections between the ideas discussed in [7] and [12] from the point of view of quotient spaces. Zariski's reformulation of the original 14th problem showed that the matter rests with the behavior of certain linear systems on "almost canonical" projective varieties associated to the pair R and G. We describe these linear systems here as the base loci of the canonical rational maps determined by invariant functions of a given degree (cf. Section 2). The stable behavior of these linear systems plays a key role in the problem of finite generation (Proposition 2.2).

The rational maps determined by these linear systems are regular on certain open sets and on suitable domains, called quotient domains, actually determine an orbit map. The existence of a sufficiently large quotient domain also plays a role in our main result (4.4) which asserts that stable base loci (cf. Section 2) and sufficiently large quotient domains give finite generation. We give interpretations of these results in the case of Nagata's counterexample in Examples 2.1 and 5.3.

We now fix our terminology. All schemes will be reduced algebraic k-schemes, with k a fixed algebraically closed field. A variety is a separated integral scheme. Almost all schemes appearing after Section 1 will be varieties. All algebraic groups are assumed to be affine algebraic varieties. For any irreducible scheme X we identify $\Gamma(X, O_X)$ with the subring of everywhere defined rational functions in $k(X)$ - the function field of X. Unless otherwise stated, "points" will mean closed points.

1. Generalities on group actions and linear systems. This section gives a brief summary of the results on actions of algebraic groups on varieties and the theory of linear systems which will be used in the following sections. They are given here essentially for convenience of reference.

1.1. Let G be an algebraic group acting rationally on a scheme X. A pair (Y, q) consisting of a scheme Y and a morphism $q: X \to Y$ is a geometric quotient of X by G denoted $X \bmod G$ if the following conditions hold:

(i) q is open and surjective

(ii) $q_*(O_X)^G = O_Y$

(iii) q is an orbit map; i.e., the fibers of closed points are orbits.

Received February 6, 1981. Revision received February 28, 1983.
Michigan Math. J. 30 (1983).

The action of G on X is said to be locally trivial if each point $x \in X$ is contained in a G-stable open subset U of X which is equivariantly isomorphic to $G \times S$ for some scheme S.

THEOREM 1.1 (Generic Quotient Theorem [10]). *Let G act on an algebraic scheme X. Then there exists a G-stable open subset U of X such that $Y = U \bmod G$ exists and Y is a quasi-projective variety.*

In general the determination of the open set described in 1.1 is a non-trivial task. However, for reductive algebraic groups somewhat more can be said. Let G be reductive and V a finite dimensional rational G-module. Let $P(V)$ denote the associated projective space consisting of lines through the origin in V and let R be the ring of polynomial functions on V (with respect to some basis). A point $v \in P(V)$ is called semi-stable if there exists an invariant nonconstant homogeneous element $f \in R$ with $f(v) \neq 0$. A point $v \in P(V)$ is stable if it is semi-stable and the orbit $G \cdot v$ is closed.

THEOREM 1.2 (Mumford; [6: 1.10]). *Let G and V be as above and let X be the set of stable points of $P(V)$. Then X is open and $Y = X \bmod G$ exists.*

The only other result of a general nature aside from Mumford's theorem is a result due to Seshadri which we now describe. If G is a connected algebraic group acting on a scheme X, then the action is said to be proper if the map $G \times X \to X \times X$ given by $(g, x) \to (g \cdot x, x)$ is proper.

THEOREM 1.3 (Seshadri [11]). *Let G be a connected algebraic group acting on a variety X such that for each point x in X the isotropy subgroup of G at x is finite. Then there exists a morphism $p: Z \to X$ such that*

(i) *Z is a normal variety, G operates on Z and p is a finite surjective G-morphism.*

(ii) *G operates freely on Z, the geometric quotient $W = Z \bmod G$ exists, and the quotient map $q: Z \to W$ is a locally trivial principal fibre space with structure group G.*

(iii) *If the action of G on X is proper then the action of G on Z is proper and W is separated.*

(iv) *$k(Z)$ is a finite normal extension of $k(X)$ and the canonical action of $\mathrm{Aut}(k(Z)/k(X))$ on Z commutes with the action of G.*

REMARK 1.3. Note that in both Mumford's theorem and Seshadri's theorem two requirements are crucial for the existence of geometric quotients. First, there must exist (affine) open sets U stable under the action of G such that $\Gamma(U, O_U)^G$ is finitely generated. Second, the orbits of points in U must be closed. The emphasis in this paper is on the first of these conditions.

REMARK 1.4. In the special case where $\dim G = 1$ some additional results are known. For example, if $G = G_m$ and X is affine then a quotient exists when the orbits are closed (cf. [9]). For $G = G_a$ the general case is discussed in [1] and [2].

Note that the geometric structure of $X \bmod G$, when it exists, need not be quasi-projective if X is so, nor even separated when X is a variety. Nevertheless, when X is affine or even quasi-affine there is a G-stable open subvariety U of X such that $U \bmod G$ exists and is quasi-affine. To see this, start with an open G-stable subset X_0 such that $Y = X_0 \bmod G$ exists and is quasi-projective. Such an X_0 exists by 1.1. Let V be a maximal quasi-affine open subset of Y and let $U = q^{-1}(V)$ where $q: X_0 \to Y$ is the quotient morphism. Then U is the desired open set. Since $\Gamma(V, O_V) = \Gamma(U, O_U)^G$, V is, in a weak sense, determined by the ring of invariant functions on U.

1.2. *Linear systems and base loci.* Let V be a complete variety and D a Cartier divisor on V with sheaf $L = O_V(D)$. Then the sections of L determine a rational mapping of V into the projective space $P = P(H^0(V, L))$. Suppose $s_0, \ldots, s_r$ form a basis of $H^0(V, L)$ over k. Then the map φ is given by $\varphi(v) = [s_0(v), \ldots, s_r(v)]$. The proof that this is well defined as long as some $s_i(v) \neq 0$ is given in [4].

If $v \in V$ then $L_v \xrightarrow{\sim} O_{V,v}$ and the images of the s_i in $O_{V,v}$, generate an ideal of $O_{V,v}$. These local ideals determine a well defined sheaf of ideals $I(\varphi)$ of O_V called the base locus ideal of the rational mapping φ. If instead of the complete linear system we choose any finite set of elements $s_0, \ldots, s_m$ in $H^0(V, L)$, then we still obtain a rational mapping into $\mathbf{P}^m$ and the base locus of this mapping is determined as above.

THEOREM 1.5 [4: 7.17.3.]. *Let V be a complete variety, D a Cartier divisor on V and $s_0, \ldots, s_n$ elements of $H^0(V, O_V(D))$. Let φ be the rational mapping determined by $s_0, \ldots, s_n$ and $I(\varphi)$ the base locus ideal of φ. If Z is the blow-up of V along the sheaf of ideals $I(\varphi)$, then the rational mapping φ extends to a morphism from Z to $\mathbf{P}^m$.*

The most important case of 1.5 for our purposes is when $V = \mathbf{P}^n$ and $s_0, \ldots, s_m$ are forms of degree K in the homogeneous coordinate ring of $\mathbf{P}^n$ identified with elements of $H^0(\mathbf{P}^n, O(K))$. In this case it is easy to see that Z is just the closure in $\mathbf{P}^n \times \mathbf{P}^m$ of the graph of the rational map φ. If the sections s_i are invariants with respect to some linear action of G on $\mathbf{P}^n$ then φ becomes a candidate for a quotient map on some open subset of $\mathbf{P}^n$.

EXAMPLE. Let $V = \mathbf{P}^2$ with homogeneous coordinates u, v and w. Let $G = G_a$ act on V by

$$t(u, v, w) = (u + tw, v + tw, w).$$

Let φ be the rational mapping determined by $u - v$, w. Then φ is regular at every point except $[1, 1, 0]$. The closure Z of the graph of φ in $\mathbf{P}^2 \times \mathbf{P}^1$ is given by $x_1(u - v) = wx_0$ where x_0, x_1 are homogeneous coordinates on $\mathbf{P}^1$. The extension of φ is just the restriction of the projection from the second factor to Z. The affine subvariety $\mathbf{P}^2_w \cong \mathbf{A}^2$ is G-stable and has a quotient isomorphic to $\mathbf{A}^1$. The restriction of φ to $\mathbf{P}^2_w$ is given by $[u/w, v/w, 1] \to [u/w - v/w, 1]$ and this is the quotient map. Note that if $w(p) = 0$ then p is a fixed point. For G_a this means no open neighborhood U of v exists for which $U \bmod G_a$ exists. It is easy to see also that $k[\mathbf{P}^2_w]^G = k[(u - v)/w]$.

The above example, though certainly trivial, illustrates the connections between base loci, rings of invariants and geometric quotients which are of primary concern in this paper.

2. The base locus of a rational representation. Let H be a connected algebraic group defined over k and $\rho: H \to GL(V)$ a finite dimensional rational representation of H. We denote by R the ring of k-valued polynomial functions on V (with respect to a fixed basis of V) and by A the subring of H-invariant functions. The ring A is positively graded with $A_0 = k$. The ideal $A_+ \cdot R$ of R is called the *base locus ideal* of ρ and denoted $I(\rho)$. It is a graded ideal in R. For each positive integer l we denote by $I(\rho, l)$ the ideal $A_l \cdot R$ where A_l denotes the lth graded piece of A.

We say the base locus of ρ is *stable* if there exists an integer $e > 0$ such that $I(\rho, le)/I(\rho, e)^l$ has finite length as an R-module for all l sufficiently large. Recall that this is equivalent to saying that $I(\rho, le)$ and $I(\rho, e)^l$ define the same sheaf of ideals on the projective space $P(V)$ of lines through the origin in V. We denote by $B(\rho)$ the subscheme of $P(V)$ defined by $\sum_{l=1}^{\infty} I(\rho, l)$ and call $B(\rho)$ the base locus of the representation ρ.

REMARK. Recall that when H is reductive a point $v \in V$ is called unstable if every homogeneous invariant function vanishes at v. The base locus $B(\rho)$ defined here is just the subscheme of unstable points of $P(V)$.

For a fixed representation ρ let $V' = V \oplus k$. Then $G = GL(V)$ is canonically embedded in $GL(V')$ and $\rho' = \rho \oplus 1$ extends the action of H on V to an action on V'. Let B be the ring of polynomial functions on V'. Then $B = R[x_0]$ where x_0 is an H-invariant function and $P(V') - x_0^{-1}(0)$ is canonically isomorphic to V. For each integer K put $J(K) = I(\rho, K)B + x_0^K B$. We say that $B(\rho)$ is σ-stable if for all K, K' sufficiently large the blow-ups of $P(V')$ along $J(K)$ and $J(K')$ are isomorphic via some H-equivariant isomorphism.

Now let X be a complete variety on which H acts rationally. Let L be a very ample H-linearized invertible sheaf on X (cf. [6]); that is, there exists an H-equivariant embedding $\tau: X \to \mathbf{P}^N$ for some N such that H acts linearly on $\mathbf{P}^N$ and $L = \tau^*(O(1))$. In fact, since L is very ample we may take $\mathbf{P}^N = P(H^0(X, L)$. We define the base locus ideals $I(L, e)$ to be the trace on X of $I(\rho_L, e)$ where $\rho_L: H \to GL(H^0(X, L))$ gives the H-linearization of L. As above, define the base locus $B(X, L)$ to be the subscheme defined by the ideal $\sum_{n=1}^{\infty} I(L, ne)$. We say $B(X, L)$ is stable if there exists an integer e such that for all n sufficiently large $I(L, ne) = I(L, e)^n$. We define σ-stable in a manner analogous to the case of rational representations.

Now let Y be an H-stable subvariety of a complete H-variety X. We say Y is stably embedded in X if there exists an H-linearized very ample invertible sheaf L on X such that $B(X, L)$ is σ-stable. If there exists such an X we say Y is *stably embeddable.*

EXAMPLE: Let Y be an affine variety on which H operates. Then it is well known that Y can be embedded in an affine space $V \cong \operatorname{Spec} k[u_1, \ldots, u_n]$ on which

H acts linearly in such a way that Y is H-stable. Let $u_i = x_i/x_0$, $1 \leqslant i \leqslant n$ and identify V with the complement in $\mathbf{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$ of $L: x_0 = 0$. Then there exists a unique extension of the action of H on V to a linear action of H on $\mathbf{P}^n$ so that x_0 is invariant. If X is the closure of Y in $\mathbf{P}^n$ then $L \cap X_-$ defines an H-linearized very ample sheaf on X.

THEOREM 2.1. *Let Y be an affine variety on which H operates. Let A denote the coordinate ring of Y. If A^H is finitely generated over k, then Y is stably embeddable. Moreover, such a stable embedding $Y \subset X$ may be chosen so that the elements of A^H are restrictions of sections in $\bigoplus_n H^0(X, L^n)$ with L a fixed very ample G-linearized invertible sheaf.*

Proof. Let $R = A^H = k[t_1, \ldots, t_m]$ and write $A = R[u_1, \ldots, u_s]$ so that for $1 \leqslant i \leqslant s$

$$h \cdot u_i = \sum_{K=1}^{s} a_{iK}(h) u_K.$$

Put $B = k[x_0, \ldots, x_{m+s}]$ and let H act on B by $h \cdot x_i = x_i$, $0 \leqslant i \leqslant m$ and $h \cdot x_{m+j} = \sum_K a_{jK}(h) x_{m+K}$, $1 \leqslant j \leqslant s$. Then $B_{(x_0)} = k[y_1, \ldots, y_{m+s}]$, where $y_j = x_j/x_0$ maps H-equivariantly onto A.

Let $\mathbf{P}^{m+s} = \operatorname{Proj} B$ and let X denote the closure of $Y = \operatorname{Spec} A$ in $\mathbf{P}^{m+s}$. Let D be the hyperplane section of X defined by $x_0 = 0$ and L the corresponding very ample sheaf on X. Then L is H-linearized and we claim $B(X, L)$ is stable.

Let $C = B/J$ be the homogeneous coordinate ring of X. Then $\mathfrak{I}(L, r)$ is the sheaf defined by the image of $I(\rho_L, r)$ in C. Let $f \in I(\rho_L, r)$. It follows from the definitions given above that $f/x_0^r \equiv g(y_1, \ldots, y_m) \bmod J$ for some $f \in k[x_0, \ldots, x_m]$. So $f = x_0^r g(y_1, \ldots, y_m) + g'$ with $g' \in J$. Now $x_0^r g(y_1, \ldots, y_m) = g_0 + g_1 x_0 + \cdots + g_s x_0^s$ with $g_i \in k[x_1, \ldots, x_m]$. Since f has degree r, we can assume $\deg g_i = r - i$ and $g_i \in (x_1, \ldots, x_r)^{r-i}$. It follows that $f = f_1 + g'$ with $f_1 \in (x_0, \ldots, x_m)^r$ and $g' \in J$. It is now immediate that $I(\rho_L, r) \equiv (x_0, \ldots, x_m)^r \bmod J$ for all r so that $\mathfrak{I}(\rho_L, r) = \mathfrak{I}(\rho_L, 1)^r$ and Y is stably embedded in X as claimed. □

EXAMPLE 2.1. Let $X \to \mathbf{P}^n$ be a stable embedding of the complete variety X via some fixed H-linearized invertible sheaf L. It is not true in general that ρ_L is itself stable. To see this we may use the counterexample to the original 14th problem of Hilbert given by Nagata in [7]. Let $k = \mathbf{C}$ and suppose $a_{ij} \in \mathbf{C}$, $i = 1, 2, 3$ and $1 \leqslant j \leqslant r$ are algebraically independent over $\mathbf{Q}$. Let H be the subgroup of $GL(2, \mathbf{C})^r$ given by elements of the form

$$h = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & \lambda_r \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \sum_{j=1}^{r} a_{ij} \lambda_j = 0, \quad i = 1, 2, 3.$$

Then H acts on $P(\mathbf{C}^{2r})$ in the evident manner. Let $k[x_1, y_1, \ldots, x_r, y_r]$ be the homogeneous coordinate ring with $h \cdot x_i = x_i + \lambda_i y_i$. The ring of invariant polynomial functions is not finitely generated so $B(P(\mathbf{C}^{2r}), O(1))$ is not stable (cf. 5.2, 5.3). Let X be the linear subspace defined by $y_1 = \cdots = y_{r-1} = 0$. Then with respect to the hyperplane section $L: y_r = 0$, $Y = X_{y_r}$ is stably embedded in X. In fact $X \simeq P^r$ and $k[x_1, \ldots, x_r, y_r]$ is its homogeneous coordinate ring. The ring of invariants is

$k[x_1, \ldots, x_{r-1}, y_r]$ and the invariant sections $x_i y_r - x_r y_i$, $i = 1, \ldots, r-1$ restricted to Y generate $k[Y]^H$.

PROPOSITION 2.2. *Let X be a complete variety on which H acts rationally and let L be an H-linearized very ample invertible sheaf. Suppose that $B(X, L)$ is σ-stable. For each integer n let φ_n be the rational map determined by the subspace of H-invariant sections of $H^0(X, L^n)$. Then there exist an integer e and a monoidal transform $\tau: Z \to X$ such that for all $n \gg 0$ the rational mapping $\varphi_{ne} \circ \tau$ on Z is regular.*

Proof. The base locus ideal of φ_n is just the locus defined by the trace on X of the sections in $H^0(X, L^n)^H$. But L is ample so this ideal is the trace on X of the ideal $I(\rho_L, n)$. Since $B(X, L)$ is monoidally stable the blow-ups $Z_{ne} \to X$ of X along $\mathcal{I}(\rho_L, ne)$ are all isomorphic for n sufficiently large. By [4: 7.17.3] $\varphi_{ne} \circ \tau$ is regular on $Z_{n_0 e} = Z$ for some fixed n_0 and the proposition is proved. □

For rational H-modules V we will need a slightly different version of this result. Let R denote the ring of invariant polynomial functions on V and $R(n)$ the space of invariant forms of degree n. Let $P(V')$ be the canonical completion of V and x_0 the irrelevant coordinate so that $V \simeq P(V') - (x_0 = 0)$. Then $R(n)$ and x_0^n are canonically identified with sections in $H^0(P(V'), O(n))$.

PROPOSITION 2.3. *Let $\rho: H \to GL(V)$ be a finite dimensional rational representation of H and $P(V')$ the canonical completion of V. Let φ_n denote the rational mapping of $P(V')$ determined by the subspace $R(n) + X_0^n \cdot k$ of $H^0(P(V'), O(n))$. If $B(\rho)$ is σ-stable there exist an integer e and a monoidal transformation, $\tau: Z \to P(V')$, of $P(V')$ unique up to H-equivariant isomorphism such that the rational mapping $\varphi_{ne} \circ \tau$ is regular on Z for all n.*

Proof. Since ρ is σ-stable there is an integer $e > 0$ such that blow-ups of $P(V')$ along $J(ne)$ are isomorphic for all n. But the sheaf of ideals determined by $J(ne)$ is precisely the base locus of the rational mapping φ_{ne}. It follows that if $\tau: Z \to P(V')$ is the blow-up of $P(V')$ along $J(e)$ then τ satisfies the conditions stated in the conclusion of the proposition. □

3. k-Noetherian quasi-affine varieties. Let X be a quasi-affine variety, i.e. an open subvariety of some affine variety. We say X is *k-Noetherian* if $A = \Gamma(X, O_X)$ is finitely generated over k. We assume throughout that X is normal. If V is a normal affine variety containing X as an open subset we will call V a *quasi-associated variety of X*. Since the canonical map from $k[V]$ to A is injective, there is a one-to-one correspondence between quasi-associated varieties of X and finitely generated integrally closed k-subalgebras R of A such that the canonical map $X \to \operatorname{Spec} R$ is an open immersion of schemes over k.

If V and W are quasi-associated varieties of X we say that W dominates V, written $W > V$ if $k[V] \subset k[W]$. If V is a fixed quasi-associated variety of X then a *dominating chain* or *d-chain* over V is a sequence

$$\cdots \to V_n \to V_{n-1} \to \cdots \to V_1 \to V_0 = V$$

of morphisms between affine varieties such that

(i) each V_n is quasi-associated to X,

(ii) V_n dominates V_{n-1}.

Let O be a discrete valuation ring of $k(X)$. We say O is an *associated divisor* of the pair (V,X) where V is a quasi-associated variety of X if there exists a quasi-associated variety W dominating V such that the valuation determined by O is of the first kind with respect to $k[W]$. In this case we say O belongs to W.

Now fix a quasi-associated variety V of X. A dominating chain $\{V_n: n=0,1,2,\dots\}$ over $V=V_0$ will be called *irredundant* if the following condition holds: Given any associated divisor O of X belonging to V_n, there exists an integer $m>n$ such that O does not belong to V_m.

The following lemma may be found in [7: pp. 44, 50].

LEMMA 3.1. *Let X be a normal quasi-affine variety and V a quasi-associated variety of X. Let I be an ideal in $R=k[V]$ defining $V-X$ and let $A=\Gamma(X,O_X)$. Then A is integrally closed in its quotient field and IA has height at least two.*

LEMMA 3.2. *Let X and V be as above. Then there exists an irredundant d-chain over V.*

Proof. There clearly exist d-chains so it sufficies to exhibit an irredundant d-chain over V. Assume, inductively, that we have constructed a sequence

$$V_n > V_{n-1} > \cdots > V_1 > V_0 = V$$

such that if $j<n$ and O is an associated divisor of X belonging to V_j, then there exists an r with $j<r\leqslant n$ such that O does not belong to V_r. Now let O be an associated divisor of V_n. Let $B=k[V_n]$ and $p\subset B$ the height one prime defining the center of O on V_n. Then $O=B_p$ is an essential valuation ring of B but is not an essential valuation ring of the Krull domain $A=\Gamma(X,O_X)$. Hence there is an element $a\in A-B$ which is not in O. Now there are only finitely many associated divisors O_i belonging to V_n. Let m be the number of such divisors. Choose an $a_i\in A$ for each of them and let C be the integral closure of the ring $B[a_1,\dots,a_m]$. Then $V_{n+1}=\operatorname{Spec} C$ extends the chain. □

If W is an affine variety quasi-associated to X we denote by $\Theta(W,X)$ the set of associated divisors of X belonging to W. The cardinality of $\Theta(W,X)$ is finite and will be denoted by $\theta(W)$. If ϑ is an irredundant d-chain over the quasi-associated variety V denote by $\theta(V,\vartheta)$ the cardinality of the set $\bigcup_{n=0}^{\infty}\Theta(V_n,X)$. Let $\theta(V)$ denote the infimum over all d-chains ϑ over V of $\Theta(V,\vartheta)$.

THEOREM 3.4. *A normal quasi-affine variety X is k-Noetherian if and only if given any quasi-associated variety V of X, $\theta(V)$ is finite.*

Proof. Suppose first that X is k-Noetherian. Let V be a quasi-associated variety of X and put $W=\operatorname{Spec} A$. Then $W>V$ is an irredundant d-chain over V and $\theta(W,X)<\infty$ so $\theta(V)$ is finite.

Conversely, let ϑ be an irredundant d-chain over the quasi-associated variety V of X with $\theta(V;\vartheta)$ finite. Then there exists an index n such that the d-chain

$$V_n > V_{n-1} > \cdots > V_0 = V$$

is irredundant and such that any associated divisor has a center on some V_i, $1 \leqslant i \leqslant n-1$. It follows that the center of an associated divisor O on V_n, if it exists, must have codimension at least two. Now V_n and X being normal we may conclude that $\Gamma(X, O_X) = k[V_n]$ and hence X is k-Noetherian. □

Let R be a domain with quotient field F and I an ideal of R. A discrete valuation ring O will be called positive on I if $R \subset O$ and $IO \subset m$ where m is the maximal ideal of O.

COROLLARY 3.5. *Let X be a normal quasi-affine variety and V a quasi-associated variety of X. Let I be an ideal defining the complement of X in V. If the set of associated divisors of (V, X) which are positive on I is finite, then X is k-Noetherian.*

Proof. Let ϑ be an irredundant d-chain over V. Then for some n, $V_n \in \vartheta$ does not belong to any associated divisor of (V, X) which is positive on I. But every associated divisor O of (V, X) is positive on I since its center is disjoint from X. Thus $\theta(V, \vartheta)$ is finite and the corollary follows from Theorem 3.4. □

4. Stability of base loci and rings of invariants. We turn now to the connection between the stability of the base locus of a rational representation of the connected linear group H and questions of finite generation of the ring of invariants. To begin with there is the trivial case.

THEOREM 4.1. *Let H be a connected linear group and V a finite dimensional rational H-module. Assume that H has no characters and that the base locus ideal $I(\rho, V)$ is principal. Then $B(\rho)$ is stable and $k[V]^H$ is finitely generated.*

Proof. Let $I(\rho) = f \cdot k[V]$ with f homogeneous of degree e. Since H has no characters each prime factor of f is invariant. It follows that f is itself a prime element. If $g \in I(\rho, K)$ then $g = fh$ and since g and f are invariant h is also invariant. From this it follows that $g = f^t$ and $k[V]^H = k[f]$. □

The general case unfortunately cannot be dealt with so easily. We assume throughout this section that the connected group H has no rational characters. The general case can always be reduced to this case in practice. Recall from Section 1 that if X is any variety on which H acts nontrivially then there exists an H-stable open subvariety $Q \subset X$ such that Q/H exists and is quasi-projective. If Q is a maximal open subvariety with this property then we call Q a *quotient domain* for the transformation pair (X, H).

LEMMA 4.2. *Let X be a normal affine variety on which H acts rationally. Let Q be a quotient domain for (X, H) and let $Y = Q/H$. Assume that X is factorial. Then Y is quasi-affine.*

Proof. Let $\{Y_\alpha\}$ be a finite affine open cover of Y and put $Q_\alpha = q^{-1}(Y_\alpha)$ where $q: Q \to Y$ is the quotient map. Then, since X is factorial, $\Gamma(X_\alpha, O_X)$ is either $k[X]$ or $k[X][1/f_\alpha]$ for some invariant function $f_\alpha \in k[X]$ (cf. [5: Corollary 7] and [9]). Since $k[Y_\alpha] = \Gamma(Q_\alpha, O_X)^H$ there exists a finitely generated integrally closed subring R of $k[X]^H$ such that the canonical map $Y_\alpha \to \operatorname{Spec} R$ is an open immersion for each α. It follows that the canonical map $Y \to \operatorname{Spec} R$ is quasi-finite and so by Zariski's main theorem Y is quasi-affine. □

If X is a normal affine variety and $Q \subset X$ an open subvariety we say Q is divisorially dense in X if every cycle on X of codimension one meets Q.

PROPOSITION 4.3. *Let X be a factorial affine variety on which H operates rationally. Let Q be a divisorially dense quotient domain for (X, H). Then $\Gamma(Q/H, O_{Q/H})$ is canonically isomorphic to the ring of H-invariant functions in $k[X]$.*

Proof. Since $k[X]$ is a unique factorization domain and H has no characters, every invariant rational function on X is the quotient of global invariant functions. Now Y is quasi-affine by 4.2 and any global function on Y is in a canonical way a global function on Q. Hence $\Gamma(Y, O_Y) \subset \Gamma(Q, O_Q)^H$. But Q is divisorially dense so $\Gamma(Q, O_Q) = k[X]$. Clearly $k[X]^H \subset \Gamma(Y, O_Y)$; hence $k[X]^H = \Gamma(Q, O_Q)^H \cong \Gamma(Y, O_Y)$. □

In Section 2, Theorem 2, it was shown that if H acts rationally on an affine variety X, and $k[X]^H$ is finitely generated, then X is stably embeddable. The following theorem is a partial converse.

THEOREM 4.4. *Let X be a factorial affine variety on which H acts rationally. Assume that the following conditions hold:*

(i) *(X, H) has a divisorially dense quotient domain.*
(ii) *X can be stably embedded in a projective variety V.*
(iii) *There exists an H-linearized very ample sheaf L on V such that the canonical map from $\bigoplus_{n \geqslant 0} H^0(V, L^n)^H$ to $k[X]^H$ induced by restriction is surjective.*

Then $k[X]^H$ is a finitely generated k-algebra.

Proof. Let Q be a divisorially dense quotient domain for (X, H) and $Y = Q/H$. By 4.2 Y is quasi-affine and by 4.3 $\Gamma(Y, O_Y) \cong k[X]^H$. It suffices then by Section 3 to show Y is k-Noetherian. Let $S_0 \subset \Gamma(Y, O_Y)$ be a finitely generated integrally closed subring such that $Y \to \bar{Y} = \operatorname{Spec} S_0$ is an open immersion. We will show that the set of associated divisors of $(Y, \bar{Y})$ is finite (cf. 3.4).

Let $p \in Q$ be a nonsingular point and let T_0 be an irreducible closed subvariety of V passing through p and transversal to the orbit of H through p. (Note: If the orbit of p is dense there is nothing to prove $k[X]^H = k$.) Since X is stably embedded in V, by 2.2 we can find a birational morphism $\tau: Z \to V$ and an integer $e > 0$ such that the rational maps $\varphi_{ne} \circ \tau$ given by $H^0(V, L^{ne})^H$ are all regular on Z. Let T denote the strict transform of T_0 on Z and denote by ψ_n the restriction of $\varphi_{ne} \circ \tau$ to T. By (iii), choosing n sufficiently large, we may assume that the image of $H^0(V, L^{ne})^H$ in $\Gamma(Y, O_Y)$ generates a subring S_n whose integral closure $\bar{S}_n$ contains S_0. Thus $Y \to \operatorname{Spec} \bar{S}_n$ is an open immersion for all n sufficiently large.

Let D_n be the geometric image of $\varphi_{ne} \circ \tau$ in $P(H^0(V, L^n)^H)$. Then by the choice of T, $\psi_n : T \to D_n$ is generically finite to one so $\dim T = \dim D_n$. Replacing T (D_n) by its normalization $\tilde{T}$ (resp. $\tilde{D}_n$) in $k(T)$ (resp. $k(Y)$) we get a map $\tilde{\psi}_n : \tilde{T} \to \tilde{D}_n$ induced by ψ_n. Now fix n (sufficiently large) and let $I_n \subset \bar{S}_n$ be the ideal defining the complement of Y in $\operatorname{Spec} \bar{S}_n$. If O is an associated divisor of $(Y, \bar{S}_n)$ having a center C of codimension one on $\operatorname{Spec} \bar{S}_n$ then $\tilde{\psi}_n^{-1}(C)$ is a finite union of prime cycles on T each of which is disjoint from $Q \cap T$. It follows that the only associated divisors of $(Y, \bar{S}_n)$ are those which correspond (via $k(Y) \to k(T)$) to the finite set of prime cycles on T with support in $T - Q \cap T$. This is a finite set of discrete valuation rings in $k(Y)$ so by 3.5 Y is k-Noetherian and the theorem is proved. □

COROLLARY 4.5. *Let V be a rational H-module and suppose (V, H) contains a divisorially dense quotient domain. Then if $B(\rho, V)$ is σ-stable, the ring of invariant polynomial functions on V is finitely generated over k.*

Proof. Let $P(V')$ be the canonical completion of V and L the sheaf associated to $x_0 = 0$. Then $\bigoplus_{n \geqslant 0} H^0(P(V'), L^n)^H$ maps onto $k[V]^H$ by definition. The corollary now follows from the same arguments as above using 2.3 instead of 2.2. □

5. Final observations and examples.

5. Final observations and examples. If H is an arbitrary connected group and $\rho : H \to GL(V)$ is a representation of H then $k[V]^H$ will be finitely generated provided $k[V]^U$ is finitely generated where U is the connected unipotent radical of H. This follows from [7] and the fact that H/U is reductive, hence semi-reductive (see [13]). Thus for all practical purposes unipotent actions hold the key to finite generation for arbitrary groups. We assume in this section that H is unipotent.

Let us call the integer e occurring in the definition of stability the index of stability. At first glance one might guess that e is the least common multiple of the degrees of a set of generators of $I(\rho)$. This is in fact false as the following example shows.

EXAMPLE 5.1. Let G_a act on the polynomial ring $R = k[x, y, u, v]$ by $x \to x + ty$, $u \to u + tv$ with y, v invariant. Then $R^{G_a} = k[y, v, vx - uy]$ so $I(\rho)$ is generated by y and v. But $vx - uy$ is not in $(y, v)^2$ so the index of stability here is not 1 but 2.

In a more positive direction we have the following result.

THEOREM 5.2. *Let $\rho : H \to GL(V)$ be a finite dimensional rational representation of the connected unipotent group H. Suppose that there exists a divisorially dense H-stable open set Q of V such that the stability group in H of each point of Q is finite. Then if $B(\rho)$ is σ-stable $k[V]^H$ is finitely generated.*

Proof. Let $\rho: Z_0 \to Q$ be a Seshadri cover of Q (see Theorem 1.3). Let Z be the normalization of $P(V')$ in $k(Z_0)$ and $\varphi: Z \to P(V')$ the canonical map. Then H acts on Z and ν is H-equivariant [8].

Let S be the blow-up of $\mathbf{P}(V')$ along $I(e)$ where e is the index of stability. Let P be a suitable component of $Z \times_{\mathbf{P}(V')} S$ so that the following diagram commutes:

$$\begin{array}{ccc} P & \to & S \\ \sigma \downarrow & & \downarrow \\ Z & \overset{p}{\to} & \mathbf{P}(V'). \end{array}$$

Here all morphisms are surjective and H-equivariant. Moreover, σ is birational (because $S \to \mathbf{P}(V')$ is birational). Let φ_K be the rational map determined by forms of degree K and $\psi_K = \varphi_K \circ p$. Since the base locus is σ-stable it follows that $\psi_{Ke} \circ \sigma$ is regular on P for all K.

Now $p^{-1}(Q)$ is locally trivial so there is an irreducible subvariety T_0 of Z and an H-stable open set $U \subset p^{-1}(Q)$ with $U \simeq H \times (T_0 \cap U)$. Let T denote the strict transform of T_0 in P. Then for all K the mapping $\psi_K = \psi_{Ke} \circ \sigma / T$ is generically finite to one from T to the geometric image D_K of $\psi_{Ke} \circ \sigma$. The rest of the argument parallels the proof of 4.4. □

This theorem allows us to answer another obvious question: If V_1 and V_2 are rational H-modules and $B(\rho_1), B(\rho_2)$ are stable, then is $B(\rho_1 + \rho_2)$ stable? The answer is negative as the following shows.

EXAMPLE 5.3. Consider again Nagata's counterexample to the original 14th problem with $k = \mathbf{C}$. The group H is defined to be a linear subvariety of G_a^r defined by the vanishing of 3 linear forms. If $h = (\lambda_1, \ldots, \lambda_r) \in G_a^r$, then $h \in H$ if and only if $l_i(\lambda_1, \ldots, \lambda_r) = 0$, $i = 1, 2, 3$ where each l_i is a linear form whose coefficients are suitably generic (cf. Example 2.1). The action of H on the polynomial ring $k[x_1, y_1, \ldots, x_r, y_r]$ is via $x_i \to x_i + \lambda_i y_i$, $y_i \to y_i$, $1 \leqslant i \leqslant r$. Let α, β and γ be distinct integers chosen from $\{1, 2, \ldots, r)$ and let $y_{\alpha\beta\gamma} = \{\prod_{i=1}^r y_i\} / y_\alpha y_\beta y_\gamma$. I claim that if $v \in \mathbf{A}^{2r}$ and $y_{\alpha\beta\gamma}(v) \neq 0$ then the stability group of v in H is trivial. If $y_i(v) \neq 0$, $i \neq \alpha, \beta, \gamma$ and $h \in H$ fixes v, then, since $x_i(v) = x_i(v) + \lambda_i y_i(v)$ we must have $\lambda_i = 0$. Thus the only possible nonzero coordinates of $h \in \mathrm{St}_H(v)$ are λ_α, λ_β and λ_γ. Now $l_i(\lambda_\alpha, \lambda_\beta, \lambda_\gamma) = 0$, $i = 1, 2, 3$. Since these forms are generic, the system of 3 equations in the three unknowns $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$ has rank 3 and hence only the trivial solution. Thus $\lambda_\alpha = \lambda_\beta = \lambda_\gamma = 0$ and $\mathrm{St}_H(v) = 0$.

Now let $Q = \bigcup (\mathbf{A}^{2r})_{y_{\alpha\beta\gamma}}$, the union being taken over all $\alpha \leqslant \beta \leqslant \gamma$ say. Then the stability group of each point in Q is trivial. Moreover since $r \geqslant 16$, the forms $y_{\alpha\beta\gamma}$ have no common factor so Q is divisorially dense. Thus by 5.2 if $B(\rho)$ were σ-stable we would have $k[x_1, y_1, \ldots, x_r, y_r]^H$ finitely generated, a contradiction.

On the other hand $V = \mathrm{Spec}\, k[x_1, y_1, \ldots, x_r, y_r] \simeq V_1 \times \cdots \times V_r$ with $V_i = \mathrm{Spec}\, k[x_i, y_i]$. Each V_i is an H-module with stable base locus generated by y_i. Thus $B(\rho_i)$ is stable for $i = 1, \ldots, r$, but $B(\rho_1 + \cdots + \rho_r)$ is not stable. □

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