ON CERTAIN CLASSES OF ALMOST PRODUCT STRUCTURES

A. Montesinos

1. Introduction. A. M. Naveira [2] gave a classification of Riemannian almost product structures (M, g, P) attending to the invariances of ∇P under the action of $O(p) \times O(q)$. The essential conditions defining the classes are F (foliation), C_1 (Vidal's), C_2 (minimal), C_3 (umbilical). O. Gil-Medrano [1] gave an interpretation of C_i under the general assumption of integrability.

We first show the transversal nature of the conditions C_i when integrability is assumed. Then, we give a geometric interpretation of these conditions without integrability by expressing them in terms of Lie derivatives.

Condition C_2 turns out to depend only on the volume form induced by g on the distribution $\mathcal{H}$. It can be rephrased in terms of the *expansion* of $\mathcal{H}$, which in certain sense is dual to the divergence of the complementary distribution $\mathcal{V}$, and becomes the *complementary form* of Vaisman [5] when $\mathcal{V}$ is integrable.

We see that C_3 can be written as C_1 at each point by a conformal transformation, and give an example. If in addition $\mathcal{V}$ is integrable, we have a conformal foliation.

If $\mathcal{V}$ is a conformal foliation of codimension $q \geqslant 3$, S. Nishikawa and H. Sato [3] have proved that $\mathrm{Pont}^k(\mathcal{H}; \mathbf{R}) = 0$ in cohomology for $k > q$, by using Cartan connections and classifying spaces. In a forthcoming paper on the conformal curvature of a conformal foliation we shall give a differential geometric proof of that result for arbitrary q. Another proof with standard techniques, less conceptual but more direct, could be given from Proposition 5.1.

2. General set-up. Let (M, g, P) be a Riemannian almost-product structure, i.e. g is a Riemannian metric on M and P is an $(1,1)$ tensor field such that $P^2 = 1$, $g(P, P) = g$. Let $\mathcal{V}$ and $\mathcal{H}$ be the *vertical* and *horizontal* distributions, corresponding to the projectors $v = \frac{1}{2}(I+P)$, $h = \frac{1}{2}(I-P)$, and assume $\dim \mathcal{V} = p$, $\dim \mathcal{H} = q \neq 0$. The capitals $A, B, C, \ldots; X, Y, Z, \ldots; Q, S, T, \ldots$ will denote vector fields that are, respectively, vertical, horizontal and unrestricted. All objects are supposed C^∞.

Let ∇ be the Levi-Civita connection and put $\alpha(Q, S, T) = g((\nabla_Q P)S, T)$. Then

$$(1) \qquad \alpha(Q, S, T) = \alpha(Q, T, S) = -\alpha(Q, PS, PT).$$

Let $\{e_u\}$ $(u: p+1, \ldots, p+q)$ denote in the sequel an orthonormal local base of horizontal vector fields. Then the 1-form λ is globally well defined through the local expression $\lambda(Q) = (1/q)\, \Sigma_u\, \alpha(e_u, e_u, Q)$, and it is clear from (1) that $\lambda = \lambda v$.

Received April 22, 1981. Revision received June 31, 1982.
Michigan Math. J. 30 (1983).

We say that $\mathcal{H}$ is:
i) C_1, if $\alpha(X,X,A)=0$ (*Vidal* [6]);
ii) C_2, if $\lambda=0$ (*minimal*);
iii) C_3, if $\alpha(X,X,A)=g(X,X)\lambda(A)$ (*umbilical*).

Apart from *foliation,* whose interpretation is obvious, these are the essential conditions leading to the Naveira classification [2]. Now, we write them in terms of Lie derivatives.

PROPOSITION 2.1. *$\alpha(X,X,A)=(L_A g)(X,X)$, and*

$$\lambda(A)=-\frac{2}{q}(L_{e_u}\theta^u)(A),$$

where $\theta^u=g(e_u,\ )$.

Proof. We have:

$$\alpha(X,X,A)=g((\nabla_X P)X,A)=g(\nabla_X PX,A)-g(P\nabla_X X,A)=-2g(\nabla_X X,A).$$

Since $g(X,A)=0$, we get:

$$-2g(\nabla_X X,A)=2g(X,\nabla_X A)=2g(X,\nabla_A X)-2g(X,L_A X)=(L_A g)(X,X).$$

Now

$$q\lambda(A)=\sum_u \alpha(e_u,e_u,A)=\sum_u (L_A g)(e_u,e_u)=-2\sum_u g(L_A e_u,e_u)$$
$$=2\theta^u(L_{e_u}A)=-2(L_{e_u}\theta^u)(A).$$

COROLLARY 2.2. *Conditions C_i are equivalently written*
C_1: $(L_A g)(X,X)=0$
C_2: $(L_{e_u}\theta^u)v=0$
C_3: $(L_A g)(X,X)=g(X,X)\lambda(A)$.

These conditions refer more to the normal bundle ν of $\mathcal{V}$ than to $\mathcal{H}$. This will be clear when $\mathcal{V}$ is a foliation after the following result.

PROPOSITION 2.3. *Let (M,g,P) be a Riemannian almost-product structure with integrable vertical distribution $\mathcal{V}$, and let $\mathcal{N}$ be a complementary distribution, i.e. $\mathcal{V}\oplus\mathcal{N}=TM$. If $\mathcal{H}$ is respectively C_1,C_2,C_3, then it is possible to choose a metric g' such that $(M,g',\mathcal{V}\oplus\mathcal{N})$ is a Riemannian almost-product structure, and that $\mathcal{N}$ is C_1,C_2,C_3.*

Proof. Let v' and h' be the vertical and horizontal projectors corresponding to $\mathcal{V}\oplus\mathcal{N}$. We put $g'(Q,S)=g(v'Q,v'S)+g(hQ,hS)$. Then, if $g'(Q,Q)=0$, we have $hQ=0$, and Q is vertical; hence $v'Q=Q$, and so $Q=0$. Therefore, g' is Riemannian. Also $g'(h'Q,v'S)=g(hh'Q,hv'S)=0$; thus g' is adapted to $\mathcal{V}\oplus\mathcal{N}$. Now, if Z is h'-horizontal and basic (here we need the integrability):

$$(L_A g')(Z,Z)=L_A g'(Z,Z)=L_A g(hZ,hZ)=(L_A g)(hZ,hZ).$$

Then, in the cases C_1 or C_3:

$$(L_A g')(Z,Z)=g(hZ,hZ)\lambda(A)=g'(Z,Z)\lambda(A),$$

and our claim follows. As for the case C_2, if $\{e'_u\}$ is an orthonormal base of h'-horizontal vectors, we have $\delta_{uv}=g'(e'_u,e'_v)=g(he'_u,he'_v)$. Hence:

$$\sum_u (L_A g')(e'_u,e'_u)=\sum_u (L_A g)(he'_u,he'_u)=0.$$

3. The Vidal condition. The condition C_1 was stated by E. Vidal and E. Vidal-Costa [6] under the form $(D_A g)(X,X)=0$, where D is the Vaisman connection (see also [5]). Its form as $\alpha(X,X,A)=0$ is due to A. M. Naveira [2].

Let us give a geometric interpretation. If $m\in M$, for computing $(L_A X)_m$ it is enough to know the values of X upon the integral curve of A by m. Let ϕ_t be the flow of A and $X_m\in\mathcal{H}_m$. Then, $X_t=\phi_{t*}X_m$ represents the dragging of X_m along the integral curve $\phi_t(m)$. Thus $(L_A X_t)_m=0$ and $(L_A g(X_t,X_t))_m=(L_A g)(X_m,X_m)$. If C_1 holds, this is zero. Hence, C_1 says that *the transport of X_m by means of the flow of A makes the length of X_t stationary at m.*

In pictorial terms, the ribbon X_t, whose sides are the integral curves of A passing by the cue and the tip of X_m, twists but not widens at m. The Vidal condition is a generalization of the Reinhart's [4] in the sense that the former drops the integrability of $\mathcal{V}$. A Reinhart space can be viewed as a foliation whose leaves maintain constant distance. Now we have no leaves, but certainly have curves in $\mathcal{V}$ (1-leaves in $\mathcal{V}$). In this sense, our interpretation generalizes that of the Reinhart structure.

As far as I am aware, there are no examples in the literature of a Riemannian almost-product structure with the condition C_1 only. The following is one. Let $S^3\subset\mathbf{R}^4$ be parametrized by (x,y,z,w), with $x^2+y^2+z^2+w^2=1$, and S^1 parametrized by θ. Let U_1,U_2,U_3 be the parallelization of S^3 given by

$$U_1=(-y,x,-w,z),\qquad U_2=(w,z,-y,-x),\qquad U_3=(-z,w,x,-y).$$

We take for $S^3\times S^1$ the Riemannian structure

$$g=s^1\otimes s^1+s^2\otimes s^2+s^3\otimes s^3+f d\theta\otimes d\theta-(s^1\otimes d\theta+d\theta\otimes s^1),$$

where $f=4+wx-yz$, and $\{s^i,d\theta\}$ is the dual of $\{U_i,\partial/\partial\theta\}$. We put $\mathcal{V}=\{U_3,U_1+\partial/\partial\theta\}$, $\mathcal{H}=\{U_1,U_2\}$.

We have $[U_i,U_j]=-2U_k$, $L_{U_i}s^j=-L_{U_j}s^i=-2s^k$ if i,j,k is a cyclic permutation of $1,2,3$; the remaining Lie derivatives are zero. Hence, neither $\mathcal{V}$ nor $\mathcal{H}$ are foliations. Now, since $U_1(f)=0$, we have:

$L_{U_1}g=0$

$L_{U_2}g=U_2(f)d\theta\otimes d\theta-(s^3\otimes d\theta+d\theta\otimes s^3)$

$L_{U_3}g=U_3(f)d\theta\otimes d\theta+(s^2\otimes d\theta+d\theta\otimes s^2)$

$L_{\partial/\partial\theta}g=0.$

Hence $(L_{U_3}g)(X,X)=(L_{U_1+\partial/\partial\theta}g)(X,X)=0$, for $X\in\mathcal{H}$; thus $\mathcal{H}$ is C_1. Also, $(L_{U_2}g)(U_3,U_3)=0$, $(L_{U_2}g)(U_1+\partial/\partial\theta,U_1+\partial/\partial\theta)=U_2(f)\neq 0$. Therefore $\mathcal{V}$ is not C_1, nor C_2, nor C_3.

4. Minimal distributions. Let $\mathcal{H}$ be a q-dimensional distribution on M and ω a volume form on $\mathcal{H}$, that is a q-form such that $\omega(X_1,\dots,X_q)\neq 0$ if $\{X_1,\dots,X_q\}=\mathcal{H}_m$ for arbitrary m. Let $\mathcal{V}_m=\{Q\in M_m \mid \omega(Q,\ \ )=0\}$. Then, $m\to\mathcal{V}_m$ defines a p-dimensional distribution on M such that $\mathcal{V}\oplus\mathcal{H}=TM$. In other words, the pair $(\mathcal{H},\omega)$ defines an almost-product structure P on M.

Let $\{X_u\}$ $(u\colon 1,\dots,q)$ be a set of vector fields on $U\subset M$ generating $\mathcal{H}$ on U, and such that $\omega(X_1,\dots,X_q)=1$ on U. Then, we define the *expansion* of $\mathcal{H}$ with respect to ω, Ex_ω, as the 1-form given on U by

$$\mathrm{Ex}_\omega(Q)=(L_{vQ}\omega)(X_1,\dots,X_q).$$

It is clear that Ex_ω is globally well defined. Let $\{\theta^u\}$ $(u\colon 1,\dots,q)$ be the dual of $\{X_u\}$, i.e. $\theta^u=-\theta^u P$, $\theta^u(X_v)=\delta^u_v$. Then $\mathrm{Ex}_\omega=\frac{1}{2}\theta^u L_{X_u}P$. In fact, we have

$$\begin{aligned}(L_{vQ}\omega)(X_1,\dots,X_q)&=-\sum_u \omega(X_1,\dots,L_{vQ}X_u,\dots,X_q)\\ &=-\theta^u(L_{vQ}X_u)\omega(X_1,\dots,X_q)=\theta^u(L_{X_u}vQ)\\ &=(\theta^u L_{X_u}v)(Q)=-(L_{X_u}\theta^u)(vQ)=\tfrac{1}{2}(\theta^u L_{X_u}P)(Q).\end{aligned}$$

COROLLARY 4.1. *Let (M,g,P) be a Riemannian almost-product structure. Then $\mathcal{H}$ is C_2 if and only if* $\mathrm{Ex}_\omega=0$, *where ω is the volume form induced by g on $\mathcal{H}$.*

Now, let $\{A_a,X_u\}$ be an adapted frame of P on $U\subset M$ such that

$$\omega(X_1,\dots,X_q)=1$$

on U. Let $\{\alpha^a,\theta^u\}$ be its dual and $\tau=\alpha^1\wedge\cdots\wedge\alpha^p\wedge\theta^1\wedge\cdots\wedge\theta^q$. Then:

PROPOSITION 4.2. $(A_1\wedge\cdots\wedge A_p)(\mathrm{Ex}_\omega,\ \ )=v(\mathrm{div}_\tau(A_1\wedge\cdots\wedge A_p))$.

Proof. We have $L_{A_a}\tau=(\alpha^b([A_b,A_a])+\theta^u([X_u,A_a]))\tau$, whence

$$\begin{aligned}&\mathrm{div}_\tau(A_1\wedge\cdots\wedge A_p)\\ &\quad=\sum_a(-1)^{a+1}\theta^u([X_u,A_a])A_1\wedge\cdots\wedge\hat{A}_a\wedge\cdots\wedge A_p\\ &\qquad-\sum_{a<b}(-1)^{a+b}\theta^u([A_a,A_b])X_u\wedge A_1\wedge\cdots\wedge\hat{A}_a\wedge\cdots\wedge\hat{A}_b\wedge\cdots\wedge A_p\end{aligned}$$

Therefore

$$v(\mathrm{div}_\tau(A_1\wedge\cdots\wedge A_p))=\sum_a(-1)^{a+1}\theta^u([X_u,A_a])A_1\wedge\cdots\wedge\hat{A}_a\wedge\cdots\wedge A_p.$$

Now

$$(A_1\wedge\cdots\wedge A_p)(\mathrm{Ex}_\omega,\ \ )=\sum_a(-1)^{a+1}\,\mathrm{Ex}_\omega(A_a)A_1\wedge\cdots\wedge\hat{A}_a\wedge\cdots\wedge A_p,$$

and $\mathrm{Ex}_\omega(A_a)=\theta^u([X_u,A_a])$, whence our claim follows. □

In this sense, Ex_ω is dual to the divergence of $\mathcal{V}$.

The geometric meaning of Ex_ω is clear. Let $m \in M$ and $X_u \in \mathcal{H}_m$ be such that $\omega_m(X_1,\ldots,X_q)=1$; thus we have at m a horizontal parallelepiped of unit volume. Take a vertical field A, that is a field transversal to $\mathcal{H}$ with respect to ω. Drag the parallelepiped along the flow of A, and compute at m the rate of growth of its volume; the result is $\mathrm{Ex}_\omega(A)_m$. Thus $\mathcal{H}$ is minimal, in the sense of stationary volume along vertical directions, if $\mathrm{Ex}_\omega=0$.

REMARK. ω is a volume form on the normal bundle ν of $\mathcal{V}$; if $\mathcal{V}$ happens to be a foliation, one can do all this after replacing $\mathcal{H}$ by ν, cf. 2.3. Then, Ex_ω becomes the *complementary form* of Vaisman [5].

5. Conformal foliations. Let $\mathcal{H}$ be C_3. Then $(L_A g)(X,X)=g(X,X)\lambda(A)$. If $m \in M$, there is some function f on M such that $2(df)_m=-\lambda_m$. Therefore $(L_A e^{2f}g)(X,X)_m=0$. In other words, C_1 can be realized at m by a conformal change of g. Hence, the condition C_3 is a conformal invariant (cf. [1]). Thus, the geometric interpretation of C_3 reduces to that of C_1.

An interesting case arises when $\mathcal{V}$ is a foliation.

PROPOSITION 5.1. *Let $\mathcal{H}$ be C_3 and $\mathcal{V}$ a foliation. Then, for each $m \in M$ there is some open neighborhood U of m on which the given Riemannian almost-product structure is conformally Reinhart.*

Proof. Let $\{dx^u\}$ be a coordinate base of horizontal 1-forms on U and $\{X_u\}$ its dual base of horizontal vector fields; let ω be the volume form on $\mathcal{H}$ and $2qf=\ln\omega(X_1,\ldots,X_q)^2$. We have

$$\lambda(A)=-\frac{2}{q}(L_{e_u}\theta^u)(A),$$

when $\{e_u\}$ is orthonormal; if $e_u=B_u^v X_v$ and $\theta^u=\underline{B}_v^u dx^v$, where the matrix $\underline{B}_v^u$ is the inverse of B_v^u, then we obtain by substitution:

$$q\lambda(A)=-2(\underline{B}_w^u dB_u^w)(A)=-A(\ln(\det B_u^w)^2)=-A(2qf).$$

Hence $(L_A e^{2f}g)(X,X)=0$ on U. □

Then, we have proved that $\mathcal{V}$ is a conformal foliation (cf. [3], [5]) if and only if $\mathcal{H}$ is C_3. Not every conformal foliation admits a global conformal transformation making it a Reinhart structure, as it is known [3]. The following is another example; it allows to visualize clearly the global obstruction. Let $M=S^1\times\mathbf{R}$ be parametrized by (θ,x). Take $g=d\theta\otimes d\theta+dx\otimes dx$, $\mathcal{V}=\{\partial/\partial\theta+x\partial/\partial x\}$, $\mathcal{H}=\{-x\partial/\partial\theta+\partial/\partial x\}$. Then $\mathcal{H}$ is trivially C_3. However, since the leaf l_0 of $\mathcal{V}$ at $x=0$ is a circle and the nearby leaves approach more and more that circle after whole turns, it is impossible to take a global metric making constant the distance from l_0 to a nearby leaf. In other words, that structure is not Reinhart whatever may be g.

REFERENCES

1. O. Gil-Medrano, *On the geometric properties of some classes of almost product structures,* preprint.
2. A. M. Naveira, *A classification of Riemannian almost product manifolds,* Rend. Mat., to appear.
3. S. Nishikawa and H. Sato, *On characteristic classes of Riemannian, conformal, and projective foliations,* J. Math. Soc. Japan 28 (1976), 223-241.
4. B. Reinhart, *Foliated manifolds with bundle-like metrics,* Ann. of Math. (2) 69 (1959), 119-132.
5. I. Vaisman, *Conformal foliations,* Kodai Math. J. 2 (1979), 26-37.
6. E. Vidal and E. Vidal-Costa, *Special connections and almost foliated metrics,* J. Differential Geometry 8 (1973), 297-304.

Departamento de Geometría y Topología
Facultad de Matemáticas
Burjasot (Valencia), Spain