HAUSDORFF DIMENSION AND INTERPOLATION BY CERTAIN FUNCTION SPACES

Daniel M. Oberlin

1. Introduction. Write C for the space of continuous complex-valued functions f on $[0,2\pi]$ which satisfy $f(0)=f(2\pi)$. For $f\in C$ and an integer n, let $\hat{f}(n)$ be the Fourier coefficient $(2\pi)^{-1}\int_0^{2\pi}f(t)e^{-int}\,dt$. For $0<\alpha<1$, let C_α be the space of all $f\in C$ such that $\sum_{-\infty}^{\infty}|\hat{f}(n)|^2|n|^{1-\alpha}<\infty$. A compact set $K\subseteq[0,2\pi)$ will be called a set of interpolation for C_α if given any continuous function ϕ on K there is some $f\in C_\alpha$ which agrees with ϕ on K. The purpose of this paper is to prove the following theorem.

THEOREM. *Suppose the compact set $K\subseteq[0,2\pi)$ has Hausdorff dimension α_0, where $0<\alpha_0<1$. Then K is a set of interpolation for C_α if $\alpha_0<\alpha<1$ but not if $0<\alpha<\alpha_0$.*

The question of interpolation for C_{α_0} is more subtle, and we do not consider it. The following result on p-Helson sets is an easy consequence of this theorem. The p-Helson sets we consider here are those of [1], not [4]—they are the sets of interpolation for continuous functions whose Fourier coefficients form l^p sequences.

COROLLARY. *If K and α_0 are as in the theorem, then K is p-Helson for $2/(2-\alpha_0)<p<2$.*

The proof of the theorem uses interpolation criteria phrased in terms of a certain space of measures, which we now define. If λ is a (complex Borel) measure on $[0,2\pi)$ and if n is an integer, then $\hat{\lambda}(n)$ is the Fourier-Stieltjes coefficient $\int_0^{2\pi}e^{-int}\,d\lambda(t)$. For $0<\alpha<1$, the measure λ is said to belong to the space M_α if $\sum_{-\infty}^{\infty}|\hat{\lambda}(n)|^2(|n|+1)^{\alpha-1}$ is finite. Our theorem is an immediate consequence of the two lemmas below and of the following fact: if the Hausdorff dimension of K exceeds α, then K supports a probability measure in M_α (see, e.g., [3, p. 40]). In what follows, $|\lambda|$ denotes the total variation measure for λ.

LEMMA 1. *If $\lambda\in M_\alpha$ then $|\lambda|(K)=0$ whenever the compact set K has Hausdorff dimension less than α.*

LEMMA 2. *Suppose that $K\subseteq[0,2\pi)$ is compact.*
(a) *If $|\lambda|(K)=0$ for every $\lambda\in M_\alpha$, then K is a set of interpolation for C_α.*
(b) *If K is a set of interpolation for C_α, then K does not support a positive measure in M_α.*

The proofs of Lemmas 1 and 2 are in Sections 2 and 3. Section 4 contains another remark about p-Helson sets.

Received January 17, 1982. Revision received May 1, 1982.
Partially supported by NSF grant MCS-7827602-01.
Michigan Math. J. 29 (1982).

2. Proof of Lemma 1.

LEMMA 3. *Fix α with $0<\alpha<1$. There is a positive constant $c=c(\alpha)$ such that the following holds: given $0\leqslant a_1<b_1<\cdots<a_K<b_K\leqslant 2\pi$ write χ_j for the characteristic function of (a_j,b_j) and put $f=\sum_1^K\chi_j$. Then*

$$\sum_{-\infty}^{\infty}|\hat{f}(n)|^2(|n|+1)^{1-\alpha}\leqslant c\sum_1^K(b_j-a_j)^\alpha.$$

Proof. The Fourier coefficient sum to be estimated is

$$\left|\frac{1}{2\pi}\sum_1^K(b_j-a_j)\right|^2+\sum_{n\neq 0}\left|\frac{1}{2\pi n}\sum_1^K(e^{-inb_j}-e^{-ina_j})\right|^2(|n|+1)^{1-\alpha}. \tag{1}$$

To bound (1), begin by defining for real t

$$\phi(t)=\begin{cases} t^{(\alpha-1)/2} & 0<t<2\pi\\ 0 & t<0 \quad\text{or}\quad t>2\pi.\end{cases}$$

A change of variable shows that the real part of $\hat{\phi}(n)$ is $[A+o(1)]/(|n|+1)^{(1+\alpha)/2}$, where A is positive. Thus if $g(t)=\sum_1^K(\phi(t-b_j)-\phi(t-a_j))$, then

$$\frac{1}{2\pi}\int_0^{2\pi}|g(t)+g(t+2\pi)|^2\,dt=\sum_{-\infty}^{\infty}\left|\sum_1^K(e^{-inb_j}-e^{-ina_j})\right|^2|\hat{\phi}(n)|^2. \tag{2}$$

Therefore the left-hand side of (2) dominates a constant multiple of the second sum in (1). Writing $\delta_j=b_j-a_j$ and noting that $(\sum_1^K\delta_j)^2$ is dominated by $(2\pi)^{2-\alpha}\sum_1^K\delta_j^\alpha$, we see that it is enough to get an estimate of $\int_0^{4\pi}|g(t)|^2\,dt$ in terms of $\sum_1^K\delta_j^\alpha$. We will actually estimate the sums

$$\sum_1^K\int_0^{4\pi}|\phi(t-b_j)-\phi(t-a_j)|^2\,dt \tag{3}$$

and

$$\sum_{j=2}^K\int_0^{4\pi}|\phi(t-b_j)-\phi(t-a_j)|\left(\sum_{l=1}^{j-1}|\phi(t-b_l)-\phi(t-a_l)|\right)dt. \tag{4}$$

For $a<b<t$ and $b-a=\delta$, calculus gives the inequality

$$(t-b)^{(\alpha-1)/2}-(t-a)^{(\alpha-1)/2}\leqslant\delta(1-\alpha)/2(t-b)^{(3-\alpha)/2}. \tag{5}$$

Consider now

$$\int_0^{4\pi}|\phi(t-b_j)-\phi(t-a_j)|^2\,dt\leqslant\int_{a_j}^{b_j}|\phi(t-a_j)|^2\,dt+\int_{b_j}^{b_j+\delta_j}|\phi(t-b_j)|^2\,dt$$
$$+\int_{b_j+\delta_j}^{2\pi+a_j}|\phi(t-b_j)-\phi(t-a_j)|^2\,dt+\int_{2\pi+a_j}^{2\pi+b_j}|\phi(t-b_j)|^2\,dt.$$

Using the definition of ϕ to estimate the first, second, and fourth integrals on the right-hand side and using (5) to estimate the third, we see that the left-hand side is bounded by a constant times δ_j^α. Summing on j gives the desired estimate of (3).

To estimate (4), observe that for $t > a_j$ and $1 \leqslant l \leqslant j-1$, $\phi(t-b_l) > \phi(t-a_l) > \phi(t-b_{l-1})$. Thus $t > a_j$ implies

$$\sum_{l=1}^{j-1} |\phi(t-b_l) - \phi(t-a_l)| < \phi(t-b_{j-1}) < \phi(t-a_j).$$

Therefore

$$\int_0^{4\pi} |\phi(t-b_j) - \phi(t-a_j)| \left(\sum_{l=1}^{j-1} |\phi(t-b_l) - \phi(t-a_l)| \right) dt$$

$$\leqslant \int_{a_j}^{b_j} |\phi(t-a_j)|^2\, dt + \int_{b_j}^{b_j+\delta_j} |\phi(t-b_j)|^2\, dt \tag{6}$$

$$+ \int_{b_j+\delta_j}^{2\pi+a_j} |\phi(t-b_j) - \phi(t-a_j)| \phi(t-b_j)\, dt + \int_{2\pi+a_j}^{2\pi+b_j} |\phi(t-b_j)|^2\, dt,$$

where the inequality

$$\sum_{l=1}^{j-1} |\phi(t-b_l) - \phi(t-a_l)| < \phi(t-b_j) \quad \text{if} \quad t > b_j$$

led to the second, third, and fourth integrals on the right-hand side of (6). As before, the first, second, and fourth integrals on the right-hand side of (6) are bounded by a constant times δ_j^α. The third integral is bounded, according to (5), by

$$\tfrac{1}{2}(1-\alpha)\delta_j^{(\alpha-1)/2} \int_{b_j+\delta_j}^{4\pi} \phi(t-b_j)\, dt.$$

Again this is less than a constant times δ_j^α, and so the same is true of the left-hand side of (6). Summing on j once more gives an estimate of (4) which completes the proof of this lemma. □

If J is an interval, let $|J|$ denote its length. To prove Lemma 1 we begin by noting that if $\{J_j\}$ is a finite collection of disjoint subintervals of $[0, 2\pi)$ and if λ is a measure in M_α, then there is a positive constant c such that

$$|\sum \lambda(J_j)| \leqslant c(\sum |J_j|^\alpha)^{1/2}. \tag{7}$$

This is a consequence of Lemma 3 and the definition of M_α.

It is enough to prove Lemma 1 for real $\lambda \in M_\alpha$, so fix such a λ and also a compact K which has Hausdorff dimension less than α. Let $\epsilon > 0$ be arbitrary. Let h be a ± 1-valued Borel function on $[0, 2\pi)$ such that $d\lambda = h d|\lambda|$, and let g be a continuous real-valued function on $[0, 2\pi)$ with $\int_0^{2\pi} |h-g| d|\lambda| < \epsilon$. Let $\{J_l'\}_1^\infty$ and $\{J_l''\}_1^\infty$ be the collections of subintervals of $[0, 2\pi)$ on which g is positive and negative, and let N be so large that

$$|\lambda|([0, 2\pi) \sim \bigcup_1^N (J_l' \cup J_l'')) < \epsilon. \tag{8}$$

Put $\delta = \inf\{|J_l'|, |J_l''| : 1 \leqslant l \leqslant N\}$. Since the compact K has Hausdorff dimension less than α, there is a covering of K by a finite collection $\{I_j\}$ of disjoint intervals such that

(9) $$|I_j| < \delta \quad \text{for each} \quad j \quad \text{and} \quad \sum |I_j|^\alpha < \epsilon.$$

Then $|\lambda|(K) \leqslant \Sigma|\lambda|(I_j)$. For each j put $I_j' = I_j \cap (\bigcup_1^N J_l')$, $I_j'' = I_j \cap (\bigcup_1^N J_l'')$. Then

(10) $$|\lambda|(K) \leqslant |\lambda|(\bigcup I_j') + |\lambda|(\bigcup I_j'') + \epsilon$$

by (8). Now

$$||\lambda|(\bigcup I_j') - \lambda(\bigcup I_j')| = 2|\lambda|(\{h = -1\} \cap (\bigcup I_j')) \leqslant 2\int_{\bigcup I_j'} |h - g|\,d|\lambda|,$$

since $g > 0$ on $\bigcup I_j'$. With a similar inequality for $\bigcup I_j''$ and the fact that

$$\int_0^{2\pi} |h - g|\,d|\lambda| < \epsilon,$$

(10) becomes

$$|\lambda|(K) \leqslant |\lambda(\bigcup I_j')| + |\lambda(\bigcup I_j'')| + 3\epsilon.$$

Since (9) implies that $|I_j| < \inf\{|J_l'|, |J_l''| : 1 \leqslant l \leqslant N\}$, each I_j' and each I_j'' is a union of at most two intervals. Thus (7) and (9) combine to show that $|\lambda(\bigcup I_j')| + |\lambda(\bigcup I_j'')| \leqslant 2^{3/2} c \epsilon^{1/2}$. With the preceding inequality this shows that $|\lambda|(K) = 0$ as desired. □

3. Proof of Lemma 2. Some of the arguments in this section are similar to the analogous arguments in [1].

The space C_α is a Banach space under the norm

$$\|f\| = \sup\{|f(t)| : 0 \leqslant t < 2\pi\} + \left(\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 (|n| + 1)^{1-\alpha}\right)^{1/2}.$$

To prove Lemma 2 we need to consider the dual of C_α. Let S_α be the space of complex sequences $s = (s_n)_{-\infty}^{\infty}$ satisfying $\Sigma_{-\infty}^{\infty} |s_n|^2 (|n| + 1)^{\alpha - 1} < \infty$, and write $\|s\|_\alpha$ for the square root of this sum. It is easy to see that Λ is a bounded linear functional on C_α if and only if there exist a complex Borel measure λ on $[0, 2\pi)$ and a sequence $s \in S_\alpha$ such that $\Lambda f = \int_0^{2\pi} f\,d\lambda - \Sigma_{-\infty}^{\infty} \hat{f}(n) s_n$ for $f \in C_\alpha$. The dual space norm of such a Λ is $\inf\{\|\lambda - \nu\|_M + \|s - \hat{\nu}\|_\alpha : \nu \in M_\alpha\}$, where $\|\mu\|_M$ stands for the total variation of the measure μ on $[0, 2\pi)$. The following lemma gives a condition equivalent to K's being a set of interpolation for C_α.

LEMMA 4. *The compact set $K \subseteq [0, 2\pi)$ is a set of interpolation for C_α if and only if there is a positive constant c such that the inequality*

(11) $$\|\lambda\|_M \leqslant c \inf\{\|\lambda - \nu\|_M + \|\hat{\nu}\|_\alpha : \nu \in M_\alpha\}$$

holds for all measures λ on K.

Proof. The proof is standard. Consider the restriction map from C_α into the continuous functions on K. A classical theorem implies that this map is onto precisely when its adjoint has a closed range. The inequality (11) is necessary and sufficient for this adjoint to have a closed range.

Statement (a) of Lemma 2 is an immediate consequence of Lemma 4. To establish (b), assume that K supports a positive measure λ in M_α and we will show that (11) fails by producing measures λ_n on K with $\|\lambda_n\|_M = \|\lambda\|_M$ and $\|\hat{\lambda}_n\|_\alpha \to 0$.

With the positive integer n fixed, partition $[0, 2\pi)$ into disjoint half-open intervals I_j $(1 \leqslant j \leqslant n)$ of equal length and let λ^j be the restriction of λ to I_j. For a choice $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ of signs ± 1, let λ_n^ϵ be the measure $\sum_1^n \epsilon_j \lambda^j$. Then $\|\lambda_n^\epsilon\|_M = \|\lambda\|_M$. The measure λ_n will be λ_n^ϵ for some ϵ which we now prepare to choose.

Write $\phi(t)$ for the kernel $|\sin(t/2)|^{-\alpha}$. Then ([3, pages 35 and 40]) there is a sequence $\{\gamma_l\}_{-\infty}^{\infty}$ of positive numbers with

$$\gamma_l = (A + o(1))(|l| + 1)^{\alpha - 1}, \qquad A > 0, \tag{12}$$

such that $\int_0^{2\pi}\int_0^{2\pi} \phi(t-u)\, d\mu(t)\, d\mu(u) = \sum_{-\infty}^{\infty} |\hat{\mu}(l)|^2 \gamma_l$ for finite positive measures μ on $[0, 2\pi)$. It follows that the average over all possible choices of ϵ of the numbers $\sum_{-\infty}^{\infty} |\hat{\lambda}_n^\epsilon(l)|^2 \gamma_l$ is equal to

$$\sum_1^n \int_{I_j}\int_{I_j} \phi(t-u)\, d\lambda(t)\, d\lambda(u). \tag{13}$$

Let λ_n be λ_n^ϵ for some ϵ such that $\sum_{-\infty}^{\infty} |\hat{\lambda}_n^\epsilon(l)|^2 \gamma_l$ is dominated by (13). Now $\int_0^{2\pi}\int_0^{2\pi} \phi(t-u)\, d\lambda(t)\, d\lambda(u) < \infty$ (which follows from (12) and the fact that $\lambda \in M_\alpha$) implies that the numbers in (13) go to zero as $n \to \infty$. Then (12) shows that the same is true of the norms $\|\hat{\lambda}_n\|_\alpha$. This completes the proof of Lemma 2. □

4. A remark about p-Helson sets. The following observation is relevant to the considerations of Section 6 of [1].

PROPOSITION. *Fix p_0 with $1 < p_0 < 2$. There is a compact set $K \subseteq [0, 2\pi)$ which is p-Helson if $p_0 < p < 2$, but not if $1 < p < p_0$.*

Proof. Let K be a Salem set of dimension $\alpha_0 = (2p_0 - 2)/p_0$ (see [2]). Then K has Hausdorff dimension α_0 and so is p-Helson for $p_0 < p < 2$ by the corollary in the introduction. Now fix q with $q > q_0 = p_0/(p_0 - 1) = 2/\alpha_0$. Since K is a Salem set, for any $\epsilon > 0$ K carries a nonzero measure λ with $|\hat{\lambda}(n)| = O([|n| + 1]^{\epsilon - \alpha_0/2})$. For small enough ϵ the sequence $\hat{\lambda}$ belongs to l^q. It follows from the corollary in Section 3 of [1] (analogue of our Lemma 2 (b)) that K is not p-Helson if $1 < p < p_0$. □

REFERENCES

1. M. B. Gregory, *p-Helson sets,* $1 < p < 2$, Israel J. Math. 12 (1972), 356–368.
2. J.-P. Kahane, *Some random series of functions,* D. C. Heath, Lexington, Mass., 1968.
3. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigònométriques,* Hermann, Paris, 1963.
4. D. M. Oberlin, *p-Helson sets,* Proc. Amer. Math. Soc. 54 (1976), 173–175.

Department of Mathematics
Florida State University
Tallahasee, Florida 32306