MEASURES ON THE TORUS WHICH ARE REAL PARTS OF HOLOMORPHIC FUNCTIONS

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We will say that a measure μ on the torus $\mathbf{T}^2$ is the real part of a holomorphic function if the p,qth Fourier coefficient

$$\hat{\mu}(p,q) = \int_{\mathbf{T}^2} x^p y^q \, d\mu(x,y) \tag{1}$$

vanishes whenever $pq < 0$. The set $\mathcal{A}$ of probability measures on $\mathbf{T}^2$ which are real parts of homomorphic functions is weak*—compact and convex. In [3] Rudin asked for a description of the extreme points of $\mathcal{A}$. Rudin's question is interesting because it concerns a phenomenon which is unique to higher dimensions; the analogous problem for the circle is trivial. In this paper we will construct some examples of extreme elements of $\mathcal{A}$.

First we establish some notation and terminology. A mapping $G: F_1 \to F_2$, where F_1 and F_2 are convex sets, will be called an *isomorphism* if it is one-to-one, onto, and preserves convex combinations. Note that isomorphisms map extreme points into extreme points. If E is a convex set and F is a convex subset of E, then F will be called a *face* of E, if $u, v \in F$, whenever $(c,u,v) \in (0,1) \times E \times E$ and $cu + (1-c)v \in F$. Note that, if F is a face of E and v is an extreme point of F, then v is an extreme point of E. A good example of a weak* closed face of $\mathcal{A}$ is $\mathcal{A}(C) = \{\mu \in \mathcal{A} \mid \mu(\mathbf{T}^2 \setminus C) = 0\}$, where C is a closed subset of $\mathbf{T}^2$. We will use B to denote the disk algebra. B can be viewed as the algebra of continuous complex valued functions on the unit circle $\mathbf{T}$ which have the property that Fourier coefficients of negative index vanish, or B can be viewed as the algebra of functions which are holomorphic on the open unit disk D and continuous on $D \cup \mathbf{T}$. In this paper we will use both viewpoints. We will assume that B is equipped with the sup-norm $\|\cdots\|$. We will indicate the Poisson kernel $\operatorname{Re}(e^{it}+w)/(e^{it}-w)$ by $P_w(e^{it})$. Finally, we let Z^k indicate the function defined by $Z^k(w) = w^k$ when $k = 0,1,2,\ldots$ and by $Z^k(w) = (\bar{w})^{-k}$ when $k = -1,-2,\ldots$. We recall that $P_w(e^{it}) = \sum_{k=-\infty}^{\infty} Z^k(w) e^{-ikt}$.

EXAMPLE 1. Consider an integer $n \geqslant 2$. Define $\pi_{n,1}: \mathbf{T} \to \mathbf{T}^2$ by $\pi_{n,1}(x) = (x^{-1}, x^{n-1})$. Let $F_{n,1} = \pi_{n,1}(\mathbf{T})$. Suppose $\mu \in \mathcal{A}(F_{n,1})$. Define the measure ν on $\mathbf{T}$ by $\nu(A) = \mu(\pi_{n,1}(A))$. It is easy to show that

$$\hat{\mu}(-p,q) = \hat{\nu}(p+(n-1)q). \tag{2}$$

It follows from (1) and (2) that $\hat{\nu}(k) = 0$ whenever $|k| \geqslant n$. Thus, there is a nonnegative trigonometric polynomial g of degree $\leqslant n-1$ such that

$$\int_{\mathbf{T}^2} f(x,y)\, d\mu(x,y) = (2\pi)^{-1} \int_0^{2\pi} f(e^{-it}, e^{i(n-1)t}) g(e^{it})\, dt. \tag{3}$$

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It is also easy to show that any measure on $\mathbf{T}^2$ satisfying (3) belongs to $\mathcal{A}(F_{n,1})$. Thus, equation (3) defines an isomorphism between $\mathcal{A}(F_{n,1})$ and the set $Q_{n,1}$ of non-negative trigonometric polynomials having degree $\leqslant n-1$ and constant term equal to 1. In [1] it is shown that the extreme elements of $Q_{n,1}$ are exactly the trigonometric polynomials of the form

$$g(e^{it}) = c\prod_{j=1}^{n-1} |e^{it} - \lambda_j|^2 \tag{4}$$

where $|\lambda_j| = 1$ for $j = 1, 2, \ldots, n-1$ and $c^{-1} = (2\pi)^{-1}\int \prod_0^{2\pi} |e^{it} - \lambda_j|^2\, dt$.

By the argument above we have established the following result: A measure μ is an extreme point of $\mathcal{A}(F_{n,1})$ if and only if it is of the form (3) where g is of the form (4).

EXAMPLE 2. Suppose that $n > 1$, that $1 \leqslant m \leqslant n$ and that n and m are relatively prime. Define $\pi_{n,m}: \mathbf{T} \to \mathbf{T}^2$ by $\pi_{n,m}(x) = (x^{-m}, x^{n-m})$. It follows from the assumption that n and m are relatively prime that $\pi_{n,m}$ is one-to-one. Let $F_{n,m} = \pi_{n,m}(\mathbf{T})$. Consider $\mu \in \mathcal{A}(F_{n,m})$. As in Example 2, we define a measure ν on $\mathbf{T}$ via $\nu(A) = \mu(\pi_{n,m}(A))$. It is easy to show that $\hat{\mu}(-p, q) = \hat{\nu}(qn + (p-q)m)$. It follows that $\nu(N) = 0$ for all integers N of the form

$$N = \pm(qn + (p-q)m) \qquad p, q > 0. \tag{5}$$

It is an exercise in elementary number theory to show that the integers which cannot be written in the form (5) are exactly those which can be written in the form

$$N = kn - jm \tag{6}$$

where $0 \leqslant k \leqslant m$, $1 \leqslant j \leqslant n-1$ and $k \leqslant j$. It follows that $d\nu(e^{it}) = g(e^{it})\, dt/2\pi$ where $g(e^{it})$ is a non-negative trigonometric polynomial of the form

$$g(e^{it}) = 1 + \sum_{s \in S} a_s e^{ist}, \tag{7}$$

where S denotes the set of integers of the form (6).

We have shown that each $\mu \in \mathcal{A}(F_{n,m})$ has the form

$$\int f(x,y)\, d\mu(x,y) = (2\pi)^{-1}\int_0^{2\pi} f(e^{-imt}, e^{i(n-m)t})\, g(e^{it})\, dt \tag{8}$$

where g belongs to the set $Q_{n,m}$ of non-negative trigonometric polynomials of the form (7). On the other hand, it is easy to show that any measure on $\mathbf{T}^2$ of the form (8) with $g \in Q_{n,m}$ belongs to $\mathcal{A}(F_{n,m})$. It follows that a measure of the form (8) is an extreme point of $\mathcal{A}(F_{n,m})$ if and only if g is an extreme point of $Q_{n,m}$. A particular example of an extreme point of $Q_{n,m}$ is the function $g(e^{it}) = 1 - \cos(m(n-1)t)$.

EXAMPLE 3. Let g be an inner function, i.e., g is analytic on D and, at almost every point of $\mathbf{T}$, g has a radial limit of absolute value 1. Suppose that there is a closed subset Q_g of $\mathbf{T}$ such that Q_g has (arc-length) measure equal to 0, and g has an analytic continuation across every open sub-arc of $\mathbf{T} \backslash Q_g$. Assume also that $g(0)$ is real. (See [0].) Motivated by Rudin's example [3], we consider the function defined on $D\times D$ by $G(z,w)=\mathrm{Re}[(1+zg(w))/(1-zg(w))]$. It is not hard to show that G has the representation $G(z,w)=\int_{\mathbf{T}^2} P_z(x)P_w(y)\,d\mu(x,y)$, where μ is the member of $\mathcal{A}$ defined by

$$\int_{\mathbf{T}^2} h(x,y)\,d\mu(x,y)=(2\pi)^{-1}\int_{\mathbf{T}} h(\overline{g(y)},y)\,|dy|$$

Note that the measure μ is supported by the closure S of the spiral

$$S_0=\{\overline{(g(y)},y)\mid y\in T\backslash Q_g\}.$$

Thus, μ belongs to $\mathcal{A}(S)$.

Consider a member ν of $\mathcal{A}(S)$. Clearly we can decompose ν as follows

$$\int_{\mathbf{T}^2} h(x,y)\,d\nu(x,y)=\int_{\mathbf{T}} h(\overline{g(y)},y)\,d\nu_0(y)+\int_{\mathbf{T}^2} h(x,y)\,d\nu_1(x,y),$$

where ν_0 is a non-negative measure on $\mathbf{T}$ with $\nu_0(Q_g)=0$ and ν_1 is a non-negative measure on $\mathbf{T}^2$ with support contained in $\mathbf{T}\times Q_g$. If we take $h(x,y)=x^{-n}y^m$, where n and m are positive integers, then

$$\int_{\mathbf{T}} (g(y))^n y^m\,d\nu_0(y)+\int_{\mathbf{T}^2} x^{-n}y^m\,d\nu_1(x,y)=0. \tag{9}$$

It follows easily from (9) that

$$\int_{\mathbf{T}} (g(y))^n f(y)\,d\nu_0(y)+\int_{\mathbf{T}^2} x^{-n}f(y)\,d\nu_1(x,y)=0 \tag{10}$$

for every $f\in B$ such that $f(0)=0$. Using a result due to Rudin (see [0, pp. 80–81]), we can find a function k in B having the properties: $k(y)=1$ for $y\in Q_g$, $|k(y)|<1$ for $y\in\mathbf{T}\backslash Q_g$, and $k(0)=0$. It follows from (10) that

$$\int_{\mathbf{T}} (g(y))^n (k(y))^r f(y)\,d\nu_0(y)+\int_{\mathbf{T}^2} x^{-n}(k(y))^r f(y)\,d\nu_1(x,y)=0 \tag{11}$$

for every $f\in B$ and for $r=1,2,\ldots$. Since ν_1 has support contained in $T\times Q_g$, it follows that (11) may be rewritten as

$$\int_{\mathbf{T}} (g(y))^n (k(y))^r f(y)\,d\nu_0(y)+\int_{\mathbf{T}^2} x^{-n}f(y)\,d\nu_1(x,y)=0. \tag{12}$$

Using the fact that $\lim_r (k(y))^r=0$ ν_0 – a.e., we have

$$\int_{\mathbf{T}^2} x^{-n}f(y)\,d\nu_1(x,y)=0 \tag{13}$$

for every $f\in B$ and for $n=1,2,\ldots$. Again using both Rudin's theorem and the fact that ν_1 is supported by $\mathbf{T}\times Q_g$ we may assert that (13) holds for every continuous complex valued function f on $\mathbf{T}$ and for $n=\pm1,\pm2,\ldots$. Let ρ be the measure defined on $\mathbf{T}$ via $\int_{\mathbf{T}} h(y)\,d\rho(y)=\int_{\mathbf{T}^2} h(y)\,d\nu_1(x,y)$. Note that $\rho(\mathbf{T}\backslash Q_g)=0$. It follows from (13) that $d\nu_1(xy) = d\rho(y)|dx|/2\pi$. Let $c = \nu_1(\mathbf{T}^2)$. Suppose ν is an extreme point of $\mathcal{A}$ and $0 < c < 1$. Then we may write

$$\int_{\mathbf{T}^2} h(x,y)\,d\nu(x,y) = (1-c)\int_{\mathbf{T}} h(\overline{g(y)},y)\,d\tilde{\nu}_0(y) + c\int_{\mathbf{T}^2} h(x,y)\,d\tilde{\rho}(y)|dx|/2\pi,$$

where $d\tilde{\nu}_0 = (1-c)^{-1}d\nu_0$ and $d\tilde{\rho} = c^{-1}d\rho$. Since the measures $d\nu$ and $d\tilde{\rho}|dx|/2\pi$ belong to $\mathcal{A}$, it follows that the measure induced on $\mathbf{T}^2$ by the functional

$$h \to \int h(g(y),y)\,d\tilde{\nu}_0(y)$$

also belongs to $\mathcal{A}$. Hence, we reach the absurd conclusion that

$$\int_{\mathbf{T}^2} h(x,y)\,d\nu(x,y) = \int h(\overline{g(y)},y)\,d\tilde{\nu}_0(y) = \int_{\mathbf{T}^2} h(x,y)\,d\tilde{\rho}(y)|dx|/2\pi.$$

The preceding argument shows that an extreme point ν of $\mathcal{A}(S)$ is either of the form $d\nu(x,y) = d\rho(y)|dx|/2\pi$, where ρ is any probability measure supported by Q_g, or ν is of the form

$$\int_{\mathbf{T}^2} h(x,y)\,d\nu(x,y) = \int_{\mathbf{T}} h(\overline{g(y)},y)\,d\nu_0(y) \tag{14}$$

where ν_0 is a probability measure on $\mathbf{T}$ which satisfies $\nu_0(Q_g) = 0$. We will examine measures of the form (14) more closely. It follows from (9) that $\int_{\mathbf{T}} g(y)y^m\,d\nu_0(y) = 0$ for $m = 1, 2, \ldots$. By the theorem of F. and M. Riesz we have $g(y)\,d\nu_0(y) = f(y)\,|dy|$ where f belongs to the Hardy space H^1. (See [0].) Thus, if ν is a member of $\mathcal{A}$ of the form (14) then ν is of the form

$$\int_{\mathbf{T}^2} h(x,y)\,d\nu(x,y) = \int_{\mathbf{T}} h(\overline{g(y)},y)\overline{g(y)}f(y)\,|dy|, \tag{15}$$

where $f \in H^1$, where $\bar{g}f \geqslant 0$ a.e., and where $\int_{\mathbf{T}} \overline{g(y)}f(y)\,|dy| = 1$. It is a trivial matter to show that any measure of the form (15) belongs to $\mathcal{A}$.

It is clear from the foregoing that the set of measures in $\mathcal{A}$ of the form (14) is isomorphic to the convex subset of H^1 given by

$$R_g = \left\{\tilde{f}\,|\,\tilde{f} \in H^1, \tilde{f}\bar{g} \geqslant 0 \text{ a.e. on } \mathbf{T} \text{ and } \int_{\mathbf{T}} \tilde{f}(y)\overline{g(y)}\,|dy| = 1\right\}.$$

Thus, the extreme points of $\mathcal{A}$ of the form (15) are exactly those for which f is an extreme point of R_g.

To complete our analysis of the face $\mathcal{A}(S)$ we will give characterization of the extreme points of R_g. We claim that a member of R_g is extreme if and only if it is an outer function. (See [0] for a discussion of outer functions.) Since R_g is a subset of the unit ball of H^1 and since the outer functions of norm 1 are the extreme points of the unit ball of H^1, it follows that any outer function in R_g is an extreme point of R_g. (See [0, p. 139].) Suppose that $f \in R_g$ is not outer. We will modify an argument due to deLeeuw and Rudin to show that f is not an extreme point of R_g. Since f is not outer, there is non-constant inner function I and a function $F \in H^1$ such that $f = IF$. Furthermore, by multiplying I by an appropriate constant if necessary, we may assume that $\int_{\mathbf{T}} |f(y)| \operatorname{Re} I(y) |dy| = 0$.

Let $h = \frac{1}{2}(1 + I^2)F$. Since I is non-constant, the function is not 0. Also since $|I(y)| = 1$ a.e. on $\mathbf{T}$, it follows that

$$h(y) = \tfrac{1}{2}(I(y) + \overline{I(y)})I(y)F(y) = f(y) \operatorname{Re} I(y)$$

a.e. on $\mathbf{T}$. Thus, we have

$$\overline{g(y)}(f(y) \pm h(y)) = \overline{g(y)} f(y)(1 \pm \operatorname{Re} I(y)) \geqslant 0$$

a.e. on $\mathbf{T}$. Furthermore, since $\overline{g(y)} f(y) = |f(y)|$, it follows that

$$\int_{\mathbf{T}} \overline{g(y)}(f(y) \pm h(y)) |dy| = \int_{\mathbf{T}} \overline{g(y)} f(y) |dy| \pm \int_{\mathbf{T}} |f(y)| \operatorname{Re} I(y) |dy| = 1.$$

Thus, we have $f \pm h \in R_g$. Hence f is not extreme.

To construct a pair of specific extreme points of R_g we observe first that it is easy to show that $(g \pm i)^2/2i$ is outer. Since $\bar{g}(g \pm i)^2/2i = 1 \pm (g - \bar{g})/2i$ on $\mathbf{T}$ and since $\int_{\mathbf{T}} g(y) |dy| = \int_{\mathbf{T}} \overline{g(y)} |dy|$, it follows that $(g \pm i)^2/2i$ is an extreme point of R_g.

EXAMPLE 4. To find our final example we will first define an isomorphism J between $\mathfrak{A}$ and a family K of linear operators on B. We then obtain our example by exhibiting extreme elements of K and applying J^{-1}.

With each $\mu \in \mathfrak{A}$ we associate an operator S_μ on B via the formula $S_\mu f(w) = \int_{\mathbf{T}^2} f(\bar{x}) P_w(\bar{x}y)\, d\mu(x, y)$, where $|w| < 1$. For $k \geqslant 0$ and $|w| < 1$ we have

$$S_\mu Z^k(w) = \sum_{l=-\infty}^{\infty} Z^l(w) \hat{\mu}(l - k, -l). \tag{16}$$

Since $\hat{\mu}(l - k, -l)$ vanishes unless $0 \leqslant l \leqslant k$, it follows that S_μ maps polynomials of degree $\leqslant n$ into polynomials of degree $\leqslant n$. Note that (16) also implies that $S_\mu 1 = 1$. If p is a polynomial, then

$$|S_\mu p(w)| \leqslant \int_{\mathbf{T}^2} |p(x)| P_w(\bar{x}y)\, d\mu(\bar{x}, y) \leqslant \|p\| S_\mu 1 = \|p\|.$$

Since the polynomials are dense in B, it follows that S_μ maps B into itself. Actually, we have proved more, namely, that S_μ belongs to the set K of operators on B which have norm 1, carry 1 into itself, and carry polynomials of degree $\leqslant n$ into polynomials of degree $\leqslant n$. Let J be a mapping from $\mathfrak{A}$ into K defined by $J(\mu) = S_\mu$. Clearly J preserves convex combinations. J is also one-to-one. For, if $J(\mu) = J(\nu)$, then it follows by (16) that

$$\hat{\mu}(k - l, -l) = \hat{\nu}(k - l, -l) \tag{17}$$

for $k \geqslant 0$ and for all l. It follows easily from (17) that $\hat{\mu}(q, r) = \hat{\nu}(q, r)$ for all pairs of integers q, r and, hence, that $\mu = \nu$. Next we will show that J maps $\mathfrak{A}$ onto K. To accomplish our task we need the following:

PROPOSITION. *Let $S \in K$. Then there exists a unique linear operator $S^{\#}$ which maps the space $C(\mathbf{T})$ of continuous complex valued functions into itself and also satisfies the following: $S^{\#}f = Sf$ for $f \in B$, $S^{\#}\bar{f} = \overline{S^{\#}f}$ for all $f \in C(\mathbf{T})$, and $S^{\#}f \geqslant 0$ whenever $f \geqslant 0$.*

Proof. It follows from the Hahn-Banach theorem that, for each $w \in \mathbf{T}$, there exists a unique probability measure α_w on $\mathbf{T}$ such that $Sf(w) = \int_{\mathbf{T}} f(x)\, d\alpha_w(x)$ for all $f \in B$. Clearly, the mapping $w \to \alpha_w$ is weak* continuous. Let $S^{\#}$ be defined by $S^{\#}g(w) = \int_{\mathbf{T}} g(x)\, d\alpha_w(x)$. It is easy to show that $S^{\#}$ has the properties asserted in the statement of the proposition.

Now we define a function R on pairs of integers by

$$R(p,q) = (2\pi)^{-1} \int_0^{2\pi} e^{iqt} S^{\#} Z^{-(p+q)}(e^{it})\, dt.$$

Using the proposition it is easy to show that R is positive definite. It follows from Bochner's theorem that there is a measure μ on $\mathbf{T}^2$ such that $\hat{\mu} = R$. We claim that $\mu \in \mathfrak{A}$. It suffices to show that $\hat{\mu}(p,q) = 0$ when $p < 0 < q$. Consider the case $0 < q < -p$. Then $S^{\#}Z^{-(p+q)} = SZ^{-(p+q)} \in B$. Thus, $\hat{\mu}(p,q) = R(p,q) = 0$ in this case. In the case $0 < -p < q$ we have

$$\mu(p,q) = (2\pi)^{-1} \overline{\int_0^{2\pi} e^{iqt} SZ^{p+q}(e^{it})\, dt}.$$

Since SZ^{p+q} is a polynomial of degree $\leqslant p+q$, and since $p+q<q$, it follows that the qth Fourier coefficient of SZ^{p+q} must vanish. Thus, $\hat{\mu}(p,q) = 0$. Finally, we show that $S = S_\mu$. For $k \geqslant 0$ we have, using (16),

$$\begin{aligned} SZ^k &= \sum_{l=0}^{k} R(l-k, -l) Z^l \\ &= \sum_{l=-\infty}^{\infty} \hat{\mu}(l-k,-l) Z^l = S_\mu Z^k. \end{aligned}$$

The preceding discussion proves the following:

THEOREM. *$J: \mathfrak{A} \to K$ is an isomorphism.*

For $n = 1, 2, \ldots$, let $\mathfrak{U}_n$ denote the set of polynomials of degree $\leqslant n$ which have sup norm $\leqslant 1$. Of course $\mathfrak{U}_n$ is the convex hull of its extreme points. Let p be an extreme element of $\mathfrak{U}_n$. We will show that there exists an extreme element of K which maps Z^n to p. Let $K(Z^n, p) = \{S \in K \mid SZ^n = p\}$. We observe that $K(Z^n,p)$ is a face of K. Thus, if we can show that $K(Z^n,p)$ is non-empty, then it will follow from the Krein–Milman theorem that $K(Z^n,p)$ contains an extreme point of K. Define operators S_1 and S_2 on B by $S_1 f(w) = n^{-1} \sum_{v^n = w} f(v)$ and $S_2 f = f \circ p$. It follows from

$$S_1 Z^{nl+h} = \begin{cases} Z^l \text{ when } h = 0, \\ 0 \quad h = 1, 2, \ldots n-1 \end{cases}$$

that the operator $S = S_2 \circ S_1$ belongs to $K(Z^n, p)$. This completes Example 4.

REMARK. The interested reader may verify that, in the notation of Examples 1, 2, and 4, we have $J(\mathfrak{A}(F_{n,m})) = K(Z^n, Z^m)$, if $m = 1$ or if n and m are relatively prime.

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