HOMOGENEOUS, SEPARATING PLANE CONTINUA ARE DECOMPOSABLE

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Dedicated to my mother and father

The problem of identifying the homogeneous plane continua stems from an old question of Knaster and Kuratowski [14]. R. H. Bing [1] and Bing and Jones [3] have made notable inroads on the problem by showing that, in addition to the simple closed curve, the pseudo-arc and the circle of pseudo-arcs are homogeneous plane continua. The list of (nondegenerate) homogeneous plane continua includes only these three.

In 1955, Jones [12] classified the homogeneous plane continua into three types: (1) the ones that do not separate the plane; (2) the decomposable ones that separate the plane; and (3) the indecomposable ones that separate the plane. Continua of types (1) and (3) must be hereditarily indecomposable [9] and [11], while continua of type (2) that are not simple closed curves must admit a continuous decomposition into elements of type (1) such that the resulting quotient space is a simple closed curve [12].

The pseudo-arc is of type (1), while the simple closed curve and the circle of pseudo-arcs are of type (2). No example of a homogeneous plane continuum of type (3) is known. The pseudo-circle of Bing [2], a logical candidate, is known not to be homogeneous [6] or [15].

In this paper, we prove that there do not exist homogeneous continua of type (3).

C. E. Burgess [4, p. 77, Questions 2 and 6] has asked if there exists a homogeneous plane continuum having infinitely many complementary domains, and if there exists, for each positive integer n, a homogeneous plane continuum that separates the plane into n connected domains. It follows from the results of this paper that the answer to both questions is no.

Howard Cook [5] has described, for n a positive integer or ∞, a plane continuum that separates the plane into n complementary domains and has only pseudo-arcs as proper nondegenerate subcontinua. By the results of this paper, none of Cook's continua are homogeneous.

This paper also provides another proof that a pseudo-circle (i.e., a hereditarily indecomposable, circle-like plane continuum different from the pseudo-arc) is not homogeneous.

Received February 26, 1980.

The author was partially supported by a research grant from the Tulane University Senate Committee on Research.

Michigan Math. J. 28 (1981).

A continuum is a compact, connected, nonvoid metric space. A space X is homogeneous if, given x and y in X, there exists a homeomorphism of X onto itself taking x to y.

The following version, due to Hagopian [8], of a theorem of Effros [6] is crucial to our arguments.

THEOREM 1. *Let M be a homogeneous continuum, and let $\epsilon > 0$. There exists $\delta > 0$ such that if x and y are points of M and $d(x,y) < \delta$, then there is a homeomorphism h of M onto itself* with *$h(x) = y$ and $d(z,h(z)) < \epsilon$, for each z in M.*

The compactness of M is essential to the proof of Theorem 1; in the next section, we will prove a similar theorem for a certain noncompact, homogeneous space associated with a homogeneous plane continuum.

1. COVERING SPACES OF HOMOGENEOUS CONTINUA

Let S^1 be the unit circle in the plane with the arc-length metric. Let A be the annulus $S^1 \times I$ with the product metric. Let $S = \mathbf{R} \times I$ be the product of the real line and I with the product metric. Let $\sigma : S \to A$ be the universal covering space of A, where σ is defined by $\sigma(s,t) = (\exp(is), t)$. The map σ is a local isometry.

Let M be a homogeneous subcontinuum of A, and let $\tilde{M} = \sigma^{-1}(M)$. Even though $\tilde{M}$ is not compact and perhaps neither connected nor locally connected, it has the following important property.

THEOREM 2. *Let $0 < \epsilon < \pi$. There exists $\delta > 0$ such that if $\tilde{p}$ and $\tilde{q}$ are points of $\tilde{M}$ with $d(\tilde{p},\tilde{q}) < \delta$, then there is a homeomorphism $\tilde{h}$ of $\tilde{M}$ onto itself such that $\tilde{h}(\tilde{p}) = \tilde{q}$ and $d(z, h(z)) < \epsilon$, for each z in $\tilde{M}$.*

Proof. Let β be a positive number such that any two maps f and g of a space W into A satisfying $d(f,g) < \beta$ are ε-homotopic [10, Theorem 1.1, p. 111]. Recall that a homotopy is an ϵ-homotopy if, for each w in W, the set $\{h_t(w) : t \in I\}$ is of diameter less than ϵ. Note that $\beta \leq \epsilon$.

Let δ satisfy the conclusion of Theorem 1 for the input $\beta/3$. Then $\delta < \pi$. Let $\tilde{p}$ and $\tilde{q}$ be points of $\tilde{M}$ such that $d(\tilde{p},\tilde{q}) < \delta$. We must exhibit a homeomorphism taking $\tilde{p}$ and $\tilde{q}$ and moving no point of $\tilde{M}$ as much as ϵ.

Let $p = \sigma(\tilde{p})$ and $q = \sigma(\tilde{q})$. Since $d(p,q) < \delta$, there exists a homeomorphism $h : M \to M$ such that $h(p) = q$ and $d(h, 1_M) < \beta/3$. Extend h to a map g of a closed neighborhood $N(M)$ of M in A. Let γ be a positive number less than $\beta/3$ satisfying the inequality $d(g(y), g(y')) < \beta/3$ if $d(y,y') < \gamma$, for all y, y' in $N(M)$. Assume, by restricting $N(M)$ if necessary, that $d(y,M) < \gamma$, for all y in $N(M)$.

It now follows that, for y in $N(M)$, $d(y, g(y)) < \beta$. Hence g is ϵ-homotopic to the identity map of $N(M)$. Use the Borsuk Homotopy Extension Theorem to extend g to a map $f : A \to A$ such that f is homotopic to the identity map of A by a homotopy F that extends the ϵ-homotopy between g and the identity map of $N(M)$.

The map $G = F \circ (\sigma \times 1): S \times I \to A$ is a homotopy between σ and $f \circ \sigma$. The identity map of S lifts one end of G. Extend this lift to a lift $\tilde{G}: S \times I \to S$ of the homotopy G such that $\sigma \circ \tilde{G} = G$ and $\tilde{G}$ is a homotopy between 1_S and a map $\tilde{f}: S \to S$.

Let $\tilde{h}$ be the restriction of $\tilde{f}$ to M. The map $\tilde{h}$ is the desired homeomorphism.

To see that $\tilde{h}$ moves no point of $\tilde{M}$ as much as ϵ, observe that G restricted to $\tilde{M} \times I$ is an ϵ-homotopy. In particular, if $\tilde{x} \in \tilde{M}$, then the set $\{G_t(\tilde{x}) : t \in I\}$ has diameter less than $\epsilon < \pi$; hence, it lies in an evenly-covered set of A. Since $\sigma\{\tilde{G}_t(\tilde{x}) : t \in I\} = \{G_t(\tilde{x}) : t \in I\}$, it follows that the set $\{\tilde{G}_t(\tilde{x}) : t \in I\}$ also has diameter less than ϵ. Since $\tilde{h}(\tilde{x}) = \tilde{G}(\tilde{x}, 1)$, and $\tilde{x} = \tilde{G}(\tilde{x}, 0)$, the map $\tilde{h}$ moves no point of $\tilde{M}$ as much as ϵ.

To see that $\tilde{h}$ is a homeomorphism, follow the procedure of the above paragraphs to construct a lift $k: \tilde{M} \to \tilde{M}$ of the homeomorphism $h^{-1}: M \to M$ satisfying $d(z, k(z)) < \epsilon$, for all z in $\tilde{M}$ (we use the letter k to avoid potential notational confusion). Clearly $k \circ \tilde{h}(\tilde{x})$ and $\tilde{h} \circ k(\tilde{x})$ are points in $\sigma^{-1}(\sigma(\tilde{x}))$. Since neither $k \circ \tilde{h}$ nor $\tilde{h} \circ k$ moves a point as much as 2π, it follows that $k \circ \tilde{h}(\tilde{x}) = \tilde{x} = \tilde{h} \circ k(\tilde{x})$.

Similarly, $\tilde{h}(\tilde{p})$ is a point of $\sigma^{-1}(q)$. Since $d(\tilde{p}, \tilde{h}(\tilde{p})) < \epsilon$, it follows that $\tilde{h}(\tilde{p}) = \tilde{q}$. The proof of the theorem is complete.

THEOREM 3. *Each homogeneous, separating plane continuum M is decomposable.*

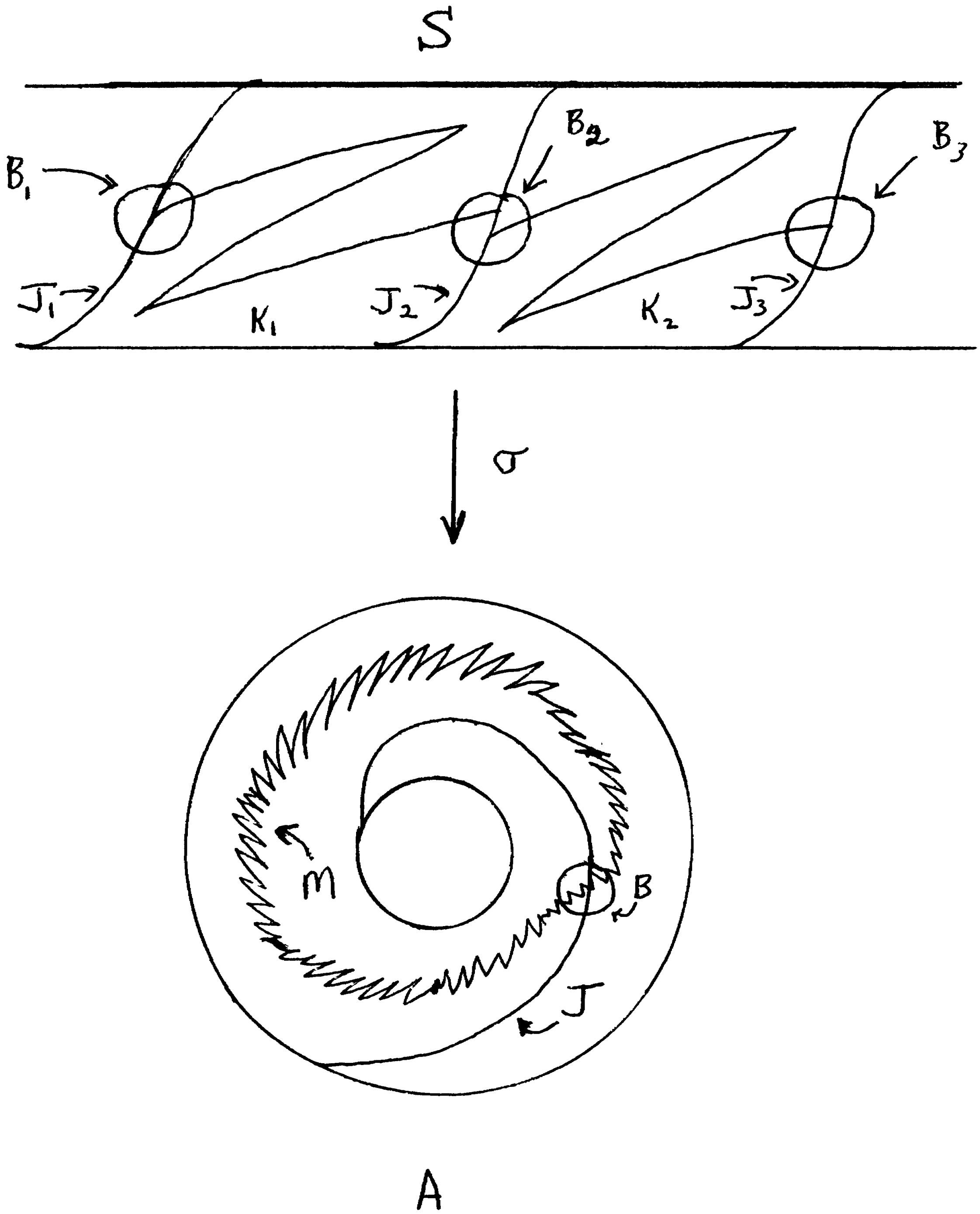
Proof. Let $T = \{z \in \mathbf{C}; 1 \leq \|z\| \leq 2\}$. Embed M in the planar annulus T so that a bounded complementary domain D of M contains the origin. If the continuum M is indecomposable, then it is hereditarily indecomposable [9] and [11]. No proper subcontinuum of M separates the plane. Hence M is the boundary of each of its complementary domains. In particular, if x is a point of M and $\gamma > 0$, then the γ-ball $B(x;\gamma)$ centered at x contains a point of D and a point of the unbounded complementary domain of M. Hence there exists an arc from D to the unbounded complementary domain of M whose intersection with M is contained in $B(x;\gamma)$.

Embed M in A, and translate the facts of the previous paragraph to this new setting. The hereditarily indecomposable, homogeneous continuum M separates the two components of the boundary of A from each other, but no proper subcontinuum of M does. If $x \in M$ and $\gamma > 0$, there is an arc in A meeting both components of the boundary of A whose intersection with M is contained in the ball $B(x;\gamma)$.

Let δ be a positive number satisfying the conclusion of Theorem 2 for $\epsilon = 1$. Let $x \in M$. Choose an arc J in A with one endpoint in $S^1 \times \{0\}$ and the other in $S^1 \times \{1\}$ whose intersection with M is contained in $B(x, \delta/2)$. Let B denote the ball $B(x, \delta/2)$.

Let B_1, B_2, and B_3 be three components of $\sigma^{-1}(B)$ such that $B_2 = B_1 + (2\pi, 0)$ and $B_3 = B_2 + (2\pi, 0)$. Let J_1, J_2, and J_3 be the corresponding lifts of J. Let $\phi: S \to S$ be the deck transformation that satisfies $\phi(B_1) = B_2$ and $\phi(B_2) = B_3$.

Let K_1 be a subcontinuum in $\tilde{M}$ irreducible between J_1 and J_2. Such a continuum must exist because M separates $S^1 \times \{0\}$ from $S^1 \times \{1\}$ in A. Let a be a point in K_1 with smallest first-coordinate.



Let $K_2 = \phi(K_1)$. Let b be a point of K_2 with largest first-coordinate. It follows that $d(a, K_2) = 2\pi = d(K_1, b)$.

Let $c \in K_1 \cap J_2$ and $d \in K_2 \cap J_2$. Then c and d are points of B_2. Use Theorem 2 to construct a homeomorphism $\tilde{h}: \tilde{M} \to \tilde{M}$ such that $\tilde{h}(c) = d$ and $d(\tilde{h}(x), x) < 1$, for all x in $\tilde{M}$. Assume the continuum $\tilde{h}(K_1) \cup K_1$ is hereditarily indecomposable; then either $\tilde{h}(K_1) \supset K_2$ or $\tilde{h}(K_1) \subset K_2$. In either case, $\tilde{h}$ moves a point of $\tilde{M}$ a distance greater than 1, which contradicts the restrictions on $\tilde{h}$.

Therefore, the proof will be complete if we show that $\tilde{h}(K_1) \cup K_2$ is hereditarily indecomposable. First we show that $K_1 \cap K_2 = \emptyset$. If z were a point of $K_1 \cap K_2$, then the component C_1 of $B_2 \cap K_1$ containing z and the component C_2 of $B_2 \cap K_2$ containing z would have the property that $\bar{C}_1 \cup \bar{C}_2$ is a decomposable continuum. Since the restriction of σ to B_2 is a homeomorphism of B_2 onto B, this contradicts the fact that M is hereditarily indecomposable.

Next we show that $\sigma(K_1 \cap J_1) \cap \sigma(K_1 \cap J_2) = \emptyset$. Since

$$\sigma(K_1 \cap J_1) = \sigma\phi(K_1 \cap J_1) = \sigma(K_2 \cap J_2),$$

and since the restriction of σ to J_2 is a homeomorphism, the claim follows from the fact that $K_1 \cap K_2 = \emptyset$.

Hence the restriction of σ to K_1 is a homeomorphism of K_1 onto $\sigma(K_1)$. Also the restriction of σ to K_2 is a homeomorphism of K_2 onto $\sigma(K_2)$.

Let $h: M \to M$ be a homeomorphism such that $h \circ \sigma = \sigma \circ \tilde{h}$. Since the restriction of $h \circ \sigma$ to K_1 is a homeomorphism, it follows that σ maps $\tilde{h}(K_1)$ homeomorphically onto $\sigma\tilde{h}(K_1)$. Since M is hereditarily indecomposable and the continua $\sigma\tilde{h}(K_1)$ and $\sigma(K_2)$ have a point in common, it follows that one contains the other. In either case, $K_2 \cup \tilde{h}(K_1)$ is homeomorphic to a subcontinuum of M and hence hereditarily indecomposable.

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