$\mathbf{Z}_2$ SURGERY THEORY

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0. NOTATION AND RESULTS

We want to give a $\mathbf{Z}_2$ surgery theory in a particularly interesting special case, namely allowing fixed point sets up to the middle dimension. This is an extension of Ted Petrie's surgery theory [20] for involutions, whose gap hypothesis would assume for a $\mathbf{Z}_2$ manifold X and each component X^s of $X^{\mathbf{Z}_2}$ that $\dim X^s < 1/2 \dim X$. We shall point out later on in which sense our obstructions differ essentially from the obstructions in Petrie's set-up and the known surgery theory.

The understanding of surgery obstruction theory is a major step in classifying G-manifolds up to diffeomorphism. Some authors [12], [14], [15] have attacked this classification problem using surgery methods in the set-up of the classical work of Kervaire and Milnor [13]. P. Löffler [15] has also obtained results in transformation groups applying obstruction theory to (much less general) surgery problems of the type studied here. Other authors have approached the classification of semilinear actions on homotopy spheres via the study of knot invariants [22], [23] and [24]. In [24], the reader can find a more complete list of references for this problem. We shall study classification problems in a much more general set-up in a later paper by constructing a long exact sequence [8]. Surgery obstruction theory as treated here is a key tool in computing the obstruction group. In this paper we use T. Petrie's G-surgery theory as developed in [18], [19].

One of the crucial problems is doing surgery while leaving a submanifold (here the fixed point set) unchanged. To show that we can do this, we apply the Atiyah-Singer signature theorem in section 2, and in section 4 we use direct computations.

The algebra we use here reflects our geometric situation. We introduce and compute new Wall groups for the group $\mathbf{Z}_2$, which, in a strong sense, are in-between the classical Wall groups [25] and the Witt groups [1]. I should point out that a less general treatment of section 2 appeared in [6].

Notation. All manifolds will be smooth oriented compact $\mathbf{Z}_2$-manifolds, thus the fact is included that the components of the fixed point set are oriented (for an orientable manifold with $\mathbf{Z}_2$ action, it does not follow that the fixed point set is orientable [3]). All maps will be equivariant. Let X and Y be $\mathbf{Z}_2$ manifolds and $f: X \to Y$. Then f is a pseudoequivalence if f is a homotopy equivalence and equivariant.

Definition 0.1. $f: X \to Y$ is an h-normal map if f is of degree 1 and we have a given $\mathbf{Z}_2$ bundle η over Y together with a given trivialization $C: \mathscr{T}X \oplus \varepsilon\, f^* \eta \oplus \varepsilon$.

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ε is a trivial $\mathbf{Z}_2$ bundle and if ∂X and ∂Y are not empty, we assume that $\partial f : \partial X \to \partial Y$ is a pseudo equivalence. In our notation, we shall usually suppress η and C. Furthermore, we assume that $\pi_1(Y_\beta)$ is abelian for every component Y_β of $Y^{\mathbf{Z}_2}$.

For h-normal maps we want to answer the $\mathbf{Z}_2$ *surgery problem:* When can we do $\mathbf{Z}_2$ surgery on (X, f) to obtain (X', f') such that f' is a pseudo equivalence? If such an (X', f') exists, we say that the *surgery problem is solvable.*

The process of G surgery is described in [19] and we shall give the details for $G = \mathbf{Z}_2$ in section 1. In our theorems we shall assume that there exists exactly one component X^s of $X^{\mathbf{Z}_2}$ such that $\dim X^s = 1/2 \dim X$. Let us assume for the introduction that $X^s = X^{\mathbf{Z}_2}$. In section 2 we give a complete answer for the $\mathbf{Z}_2$ surgery problem in the $4n$-dimensional situation. Here is a simplified version, which has been given already in [6]:

Let $f : X \to Y$ be an h normal map, $\dim Y = 4n$, $n \geq 3$ *and* $\dim Y^{\mathbf{Z}_2} = 2n$. *Let Y be simply connected and $d = \deg(f^{\mathbf{Z}_2} : X^{\mathbf{Z}_2} \to Y^{\mathbf{Z}_2})$. Then the surgery problem is solvable if and only if*

(i) $\sigma_{\mathbf{Z}_2}(f) = 0$,

(ii) $\operatorname{sign}(\mathbf{Z}_2, X) - \operatorname{sign}(\mathbf{Z}_2, Y) = 0$,

(iii) $(d^2 - 1) \operatorname{sign}(T, Y) = 0$.

Here $\sigma_{\mathbf{Z}_2}(f)$ is the obstruction to converting $f^{\mathbf{Z}_2} : X^{\mathbf{Z}_2} \to Y^{\mathbf{Z}_2}$ by surgery into a $\mathbf{Z}_2$ homology equivalence as required by Smith theory, and T is the generator of $\mathbf{Z}_2$. If we assume that $\dim Y^s < 1/2 \dim Y$ and $\mathbf{Z}_2$ acts orientation preserving, then it is known that the surgery problem is solvable if and only if obstructions (i) and (ii) vanish [20].

Assume for a moment that $\dim Y = 2m$ and $\dim Y^s \leq m$. If $Y^{\mathbf{Z}_2}$ is not connected then suppose the inequality $\dim Y^s \leq m$ holds for each component Y^s of $Y^{\mathbf{Z}_2}$. Furthermore assume:

Condition P. $f^{\mathbf{Z}_2} : X^{\mathbf{Z}_2} \to Y^{\mathbf{Z}_2}$ induces a $\mathbf{Z}_2$ homology equivalence and

$$K_i(f, \mathbf{Z}) = \operatorname{Ker}(f_* : H_i(X, \mathbf{Z}) \to H_i(Y, \mathbf{Z})) = 0 \quad \text{for} \quad i < m.$$

Denote $\mathbf{Z}[\mathbf{Z}_2]$ by Λ. Then $K = K_m(f, \mathbf{Z})$ is a free Λ-module, as [21] implies that K is a projective Λ-module and $\tilde{K}_0(\Lambda) = 0$. Assuming one component of the fixed point set has dimension m implies that $T[Y] = (-1)^m [Y]$, where [ ] denotes the fundamental class. This gives rise to a homomorphism

$$\omega : \mathbf{Z}_2 \to \{\pm 1\} \text{ by } T \mapsto (-1)^m.$$

We obtain a conjugation $\bar{\ } : \Lambda \to \Lambda$ defined by $a + bT \mapsto a + (-1)^m bT$.

Definition. $A(-1)^m$ quasi Hermitian form (K, λ, μ, T) is a free Λ-module with $\mathbf{Z}_2$ structure given by the involution T, together with a map $\lambda : K \times K \to \mathbf{Z}$ such that

(i) for $x \in K$ fixed, $y \mapsto \lambda(x, y)$ is a $\mathbf{Z}$ homomorphism,

(ii) $\lambda(x, y) = (-1)^m \lambda(y, x) = (-1)^m \lambda(Tx, Ty) (x, y \in K)$

Furthermore, we have a map $\mu : K \to \mathbf{Z}/\{1 - (-1)^m 1\}$ such that

(iii) $\lambda(x,x) = \mu(x) + (-1)^m \mu(x)$ $\quad (x \in K)$

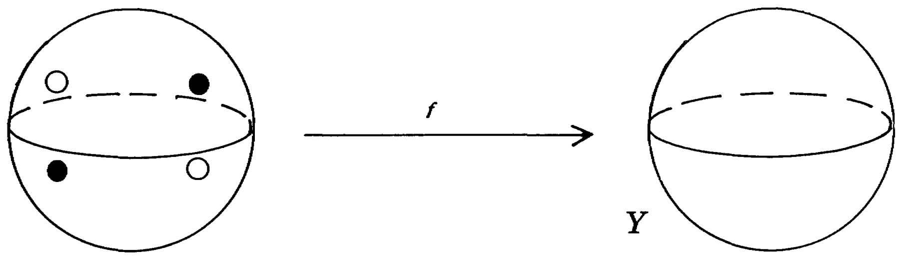
(iv) $\mu(x+y) = \mu(x) + \mu(y) + \lambda(x,y)$ $\quad (x,y) \in K$

(v) $\mu(a \cdot x) = a^2 \mu(x)$ $\quad (x \in K, a \in \mathbf{Z})$

(vi) $\mu(Tx) = \mu(x)$ $\quad (x \in K)$

If we assume condition P, then $K_m(f,\mathbf{Z})$ together with the intersection and self-intersection numbers and T induced by the involution give rise to a $(-1)^m$ quasi Hermitian form. Furthermore, $A\lambda : K \to \mathrm{Hom}_{\mathbf{Z}}(K,\mathbf{Z})$ is a Λ-module isomorphism. Here $(T\varphi)(y) = (-1)^m \varphi(Ty)$. Following Wall [27] we define furthermore $\tilde{\lambda}(x,y) = \lambda(x,y)1 + \lambda(x,Ty)T \in \Lambda$. Then $A\tilde{\lambda} : K \to \mathrm{Hom}_{\Lambda}(K,\Lambda)$ is a Λ-module isomorphism. If $\lambda(x,Tx) \equiv 0(2)$ for $x \in K$ (for example if $\dim Y^s < 1/2 \dim Y$) we also have the form $\tilde{\mu} : K \to \Lambda/\{x - (-1)x^-, x \in \Lambda\}$ and $(K,\tilde{\lambda},\tilde{\mu})$ defines a $(-1)^m$ Hermitian form which represents an element in $L_{2m}(\Lambda,\omega)$. The important—and sometimes helpful—fact in our situation is that $\lambda(x,Tx)$ need not be even. The *type* of $\lambda(\cdot,T\cdot)$ [17] is not even an invariant of the surgery problem. The type of a bilinear form $\langle\,,\rangle$ tells whether $\langle x,x\rangle$ is even for all x or not. Changing the type will be essential in the solution of the $4n+2$ dimensional case. This is done as follows:

Example 0.2.

X  Y

$X = Y = S^2$, and $T(x,y,z) = (x,y,-z)$ so the drawn equator is fixed under the action of $\mathbf{Z}_2$. Doing surgery in the above picture on two copies of $S^0 \times D^2$ which are interchanged by the involution (one is marked ●, the other ○) changes the type of $\lambda(\cdot,T\cdot) : K \times K \to \mathbf{Z}$. In the beginning, K is trivial; after the surgery step $K \cong \Lambda \oplus \Lambda$ and there exists x such that $\lambda(x,Tx) = 1$. This picture naturally generalizes to higher dimensions. As above, use Y with a fixed point component in the middle dimension and do surgery on the boundary of a fiber of the normal bundle of this fixed point component (slightly changed by an isotopy such that it does not intersect its image under the involution).

Definition 0.3. $(K,\lambda,\mu,T) \sim 0$ if there exists a Λ free submodule $N \subset K$ such that $\lambda|_{N\times N} \equiv 0$, $\mu|_N \equiv 0$, and N has a basis extending to a basis of K, defining thereby a basis of K/N, and $K/N \to \mathrm{Hom}_{\Lambda}(N,\Lambda)$ induced by $\tilde{\lambda}$ is an isomorphism. Then N is called a *subkernel.*

Direct sum, together with this equivalence relation, gives rise to new Wall groups $\tilde{W}_0(\Lambda)$ for m even and $\tilde{W}_2(\Lambda)$ for m odd. In Theorems 2.3 and 4.3 we compute that

$$\tilde{W}_0(\Lambda) \cong \mathbf{Z} \oplus \mathbf{Z} \qquad \text{and} \qquad \tilde{W}_2(\Lambda) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

Furthermore, we give invariants which determine the equivalence class of (K,λ,μ,T). It should be pointed out that our computation of $\tilde{W}_2(\Lambda)$ requires stabilization. Let us assume Condition P and $(K,\lambda,\mu,T) \sim 0$. This does not imply that $x \in N$

can be represented by a sphere which does not intersect the fixed point set, and which we can use to kill x (and Tx). In the $4n$-dimensional case, the condition $(d^2 - 1)\,\mathrm{sign}\,(T,Y) = 0$ enables us to find N such that $x \in N$ does not intersect the fixed point set. In the $4n + 2$ dimensional case we prove that we can always find N not intersecting the fixed point set, but this is an essential use of stabilization, as in example 0.2. This is of particular interest because of the following.

LEMMA 0.4. *Assume* $f: X^{2m} \to Y^{2m}$ *is an h map and satisfies Condition* P. *Furthermore,* $(K,\lambda,\mu,T) \sim 0$ *with subkernel N. For every component* $\alpha \in \pi_0(X^{Z_2})$ *with fundamental class* $[\alpha]$ *we have* $\lambda([\alpha],N) = 0$. *Then we can solve the surgery problem.*

The proof is obvious, but we shall supply the proof at the end of section 1.

Let us assume that we can do surgery on $f: X^{2m} \to Y^{2m}$ to obtain $f': X' \to Y$ and that f' satisfies Condition P. Then we define

$$r(f) = rk_\Lambda K_m(f',\mathbf{Z})(\mathrm{mod}\ 2), \qquad \text{(well-defined by 3.1);}$$

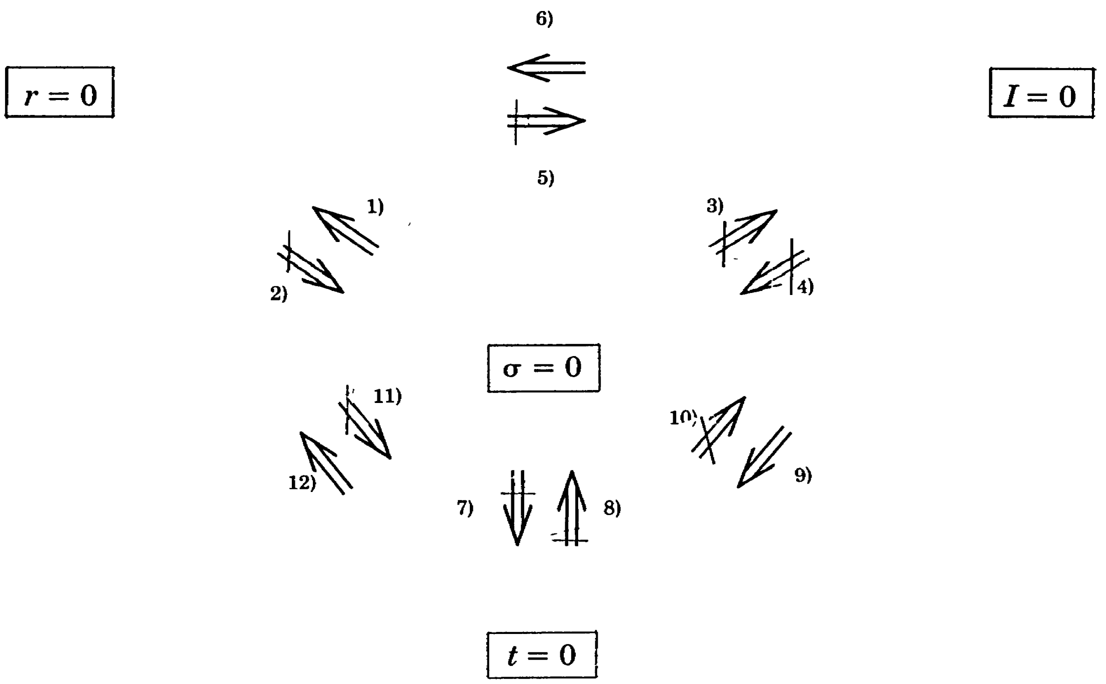
$$t(f') = \begin{cases} 0 & \text{if} \quad \lambda(x,Tx) \equiv 0 \ (\mathrm{mod}\ 2) \quad \text{for all} \quad x \in K_m(f',\mathbf{Z}) \\ 1 & \text{otherwise} \end{cases}$$

Let $i: (X')^{Z_2} \to X'$ be the inclusion and recall that we have a canonical splitting $H^m(X',\mathbf{Z}_2) \cong H^m(Y',\mathbf{Z}_2) \oplus K^m(f',\mathbf{Z}_2)$. Then

$$I(f') = \begin{cases} 0 & \text{if} \quad i^*|_{K^m(f',\mathbf{Z}_2)} \equiv 0 \\ 1 & \text{otherwise.} \end{cases}$$

Let $\sigma = 0$ denote that the surgery problem is solvable. Then the following diagram gives all the relations:

0.5



1) by 3.1
2) by 4. and 6.
3) by example 3.3. iii
4) by Lemma 3.4
5) by example 3.3. iii
6) by 9. and 12.
7) by 4. and 9.
8) by example 0.2
9) by 3.4. i) [2]
10) by 3.4. iii
11) by example 0.2
12) then (K,λ,μ) represents an element in $L_{2m}(\Lambda,\omega)$

10) follows in case we have at most one component of the fixed point set in the middle dimension.

It should be pointed out that I and t are not surgery invariants.

A simplified version of our main theorem in the $4n + 2$ dimensional case—again assuming $Y^{\mathbf{Z}_2}$ is connected, $\dim Y^{\mathbf{Z}_2} = 2n + 1$, and $n \geq 2$—is: *The surgery problem* $f: X^{4n+2} \to Y^{4n+2}$ *is solvable if and only if*

(i) $\sigma_{\mathbf{Z}_2}(f) = 0$,

(ii) $r(f) = 0$,

(iii) $c(f) = 0$.

Here $c(f)$ is the Arf invariant of f *after forgetting the* $\mathbf{Z}_2$ *action and* $r(f)$ is defined only if $\sigma_{\mathbf{Z}_2}(f) = 0$. $\sigma_{\mathbf{Z}_2}(f)$ is again the obstruction to converting $f^{\mathbf{Z}_2}$ into a mod 2 homology equivalence.

Here is an application. Assume that $f': X' \to Y^{4m+2}$ satisfies Condition P and $t(f') = 0$. Then the $\mathbf{Z}_2$ Arf invariant $c_2(f')$ is defined (this is the invariant which determines the class of $(K_{2m+1}(f',\mathbf{Z}),\tilde{\lambda},\tilde{\mu})$ in $L_{4m+2}(\Lambda,-1)$). We shall show that c_2 is not a $\mathbf{Z}_2$ surgery invariant (see 4.13). This example uses surgery steps which are not admissible in the setting of [4]. The bordism given by surgery is not isovariant. This gives an essential difference in the G-surgery theory of [4] and [19].

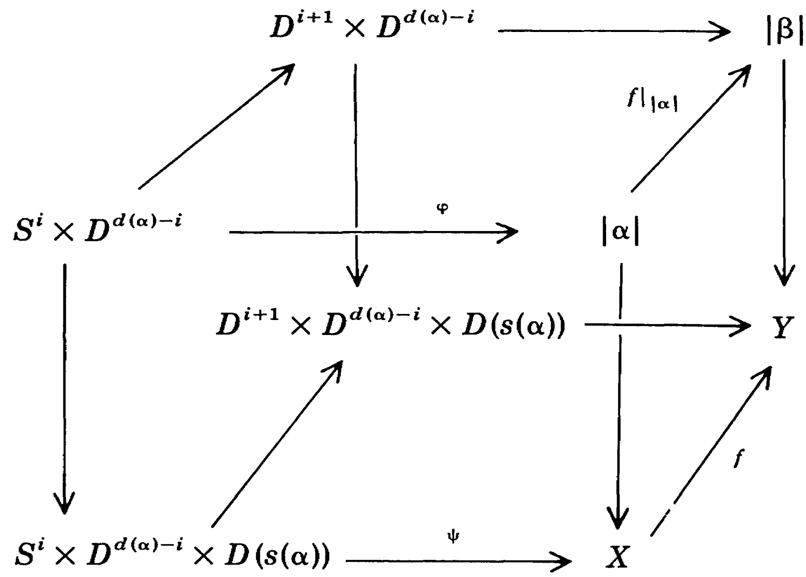
One of the purposes of this research has been to investigate the necessity of the gap hypothesis in Ted Petrie's G-surgery theory. The answer is the following. Our approach holds only for $G = \mathbf{Z}_2$ since the proofs are based on specific properties of $\mathbf{Z}_2$. Already for $\mathbf{Z}_2$ and with the least possible weakening of the gap hypothesis, there do occur new surgery obstructions. Thus I believe that it is not possible to weaken the gap hypothesis in general. Even the assumptions that $X^{\mathbf{Z}_2}$ and $Y^{\mathbf{Z}_2}$ are homotopy spheres and $\deg f^{\mathbf{Z}_2} = 1$ have not been very useful to the study of the surgery problem [14]. Only very strong assumptions such as the isovariance of f are successful. Thus I consider the gap hypothesis unavoidable if we want to investigate a wide class of normal maps.

I wish to thank Ted Petrie for suggesting this problem to me and for many long discussions which helped me solve this problem, and Valdis Vijums for some helpful suggestions.

1. $\mathbf{Z}_2$ SURGERY AND $\sigma_\alpha(f)$

Let $f: X^k \to Y^k$ be an h normal map of k-dimensional manifolds. Then we want to describe the admitted $\mathbf{Z}_2$ surgery steps. f induces a map $\tilde{f}: \pi_0(X^{\mathbf{Z}_2}) \to \pi_0(Y^{\mathbf{Z}_2})$. Let $\beta \in \pi_0(Y^{\mathbf{Z}_2})$ and $\alpha = \tilde{f}^{-1}(\beta) \subset \pi_0(X^{\mathbf{Z}_2})$.

1.1. *Surgery of Type* α: The attaching map for a handle of type α is a diagram. Here φ and ψ are imbeddings, $d(\alpha)$ = dimension of $|\alpha|$, the underlying space of α, $s(\alpha)$ is the slice representation of α, and S and D are the unit sphere and disk.



Surgery on the fixed point set means surgery of type α for some α. It is shown in [19] that any attaching map for a handle to $|\alpha|$ (this is the top square) extends to an attaching map for a handle of type α. In particular, we point out that our bundle data imply the normal data Petrie needs. In [19] it is assumed that we have a given $\mathbf{Z}_2$ bundle η over $|\beta|$ and a given isomorphism $b: \nu(|\alpha|, X) \to (f_\alpha)^*\eta$, where $f_\alpha = f|_{|\alpha|}: |\alpha| \to |\beta|$. The isomorphism is needed only after restricting the bundle to the sphere S^i on which we are doing surgery. Our candidate for η is $(\xi|_{|\beta|})_{\mathbf{Z}_2}$ which is the orthogonal complement of $(\xi|_{|\beta|})^{\mathbf{Z}_2}$ in $\xi|_{|\beta|}$. C induces a stable isomorphism $\tilde{b}: \nu(|\alpha|, X) \to (f_\alpha)^*\eta$. As $\mathbf{Z}_2$ operates trivially on $|\alpha|$ and $KO_{\mathbf{Z}_2}(|\alpha|) \cong KO(|\alpha|) \otimes R(\mathbf{Z}_2)$, KO denoting real K-theory and R denoting the real representation ring, we can neglect the $\mathbf{Z}_2$ action for η and $\nu(|\alpha|, X)$. We consider only those surgery steps where $i \leq 1/2\, d(\alpha) \leq 1/4\, k$. As

$$\pi_i(\mathrm{SO}(k - d(\alpha)) \cong \pi_i(\mathrm{SO}(k - d(\alpha) + l)),$$

every stable isomorphism $\tilde{b}: \nu(|\alpha|, X)|_{S^i} \to (f_\alpha)^*\eta|_{S^i}$ can be assumed to be of the form $b \times \mathrm{Id}: (\nu(|\alpha|, X) \oplus \varepsilon)|_{S^i} \to ((f_\alpha)^*\eta \oplus \varepsilon)|_{S^i}$. We can use this b for our surgery step.

1.2. *Surgery in the Free Part.* The attaching map for a handle is a diagram

$$\begin{array}{ccc} \mathbf{Z}_2 \times S^i \times D^{k-i} & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow f \\ \mathbf{Z}_2 \times D^{i+1} \times D^{k-i} & \to & Y \end{array}$$

where $\mathbf{Z}_2$ operates as translation on $\mathbf{Z}_2$ and φ is an imbedding.

Let $f: X \to Y$ be an h normal map. f is an *h-map* if $\tilde{f}: \pi_0(X^{\mathbf{Z}_2}) \to \pi_0(Y^{\mathbf{Z}_2})$ is a bijection. As f is of degree 1 it follows that $\tilde{f}$ is surjective [2]. If $\alpha_0 \in \pi_0(X^{\mathbf{Z}_2})$ and $\tilde{f}(\alpha_0) = \beta$, assume $d(\alpha_0) = d(\beta)$. By 0-dimensional surgery we can connect the components in $(f)^{-1}(\beta)$ for $\beta \in \pi_0(Y^{\mathbf{Z}_2})$. Thus we can convert every h-normal map into an h-map. Then we abbreviate $\pi = \pi_0(X^{\mathbf{Z}_2}) = \pi_0(Y^{\mathbf{Z}_2})$. $|\beta|$ denotes the underlying space of the component β, $d(\beta) = \dim(|\beta|)$, and $\pi_1(|\beta|)$ comes with an orientation homomorphism $\omega: \pi_1(|\beta|) \to \{\pm 1\}$ which we assume to be trivial.

So suppose $\tilde{f}: \pi_0(X^{\mathbf{Z}_2}) \to \pi_0(Y^{\mathbf{Z}_2})$ is a bijection and $f_\alpha: |\alpha| \to |\beta|$ is the map induced by f, here $\beta = \tilde{f}(\alpha)$. We assume that $\dim|\alpha| = \dim|\beta|$, so it follows from [2] that $\deg f_\alpha = d_\beta$ is odd. We are also given a bundle η^α and a stable vector bundle isomorphism $C^\alpha: \mathscr{T}|\alpha| \to f_\alpha^* \eta^\alpha$. Abbreviate the map f_α with these data by f_α.

LEMMA 1.3. *There exists an obstruction $\sigma_\beta(f)$ in a group $L(\beta)$ satisfying the following properties:*

a) *$\sigma_\beta(f)$ depends only on the normal cobordism class of f_α.*

b) *$\sigma_\beta(f) = 0$ if an only if f_α is normally cobordant to a* mod 2 *homology equivalence.*

c) *The group $L(\beta)$ depends only on $\pi_1(|\beta|)$, d_β, and* $\dim|\beta|$.

In this context we use only the weaker statement:

There is a well-defined obstruction for converting f_α by surgery into a mod *2 homology equivalence. This obstruction is denoted by $\sigma_\beta(f)$.*

It is easy to observe: If $\partial|\beta| = \emptyset$, $\pi_1(|\beta|) = 0$ and $\deg f_\alpha = 1$ we have:

$$\sigma_\beta(f) = 0 \quad \text{if} \quad \begin{cases} \dim\beta \text{ is odd} \\ \dim\beta = 4k \quad \text{and} \quad \operatorname{sign}(|\alpha|) - \operatorname{sign}(|\beta|) = 0 \\ \dim\beta = 4k+2 \quad \text{and} \quad c(f) = 0 \end{cases}$$

Here is the idea for the proof of Lemma 1.3. The geometric construction of the Wall groups in section 9 [26] defines the set $L(\beta)$. Naturally we have to use an appropriate generalization as in [19]. It is not too difficult to show that $L(\beta)$ is a group (compare [7]). Then Lemma 1.3 is an automatic consequence.

Proof of Lemma 0.4. Let $\{e_1,\dots,e_r\}$ be a Λ-basis for N. Then e_i can be represented by an imbedded sphere $S_i \subset X$ (with trivial normal bundle) and Te_i is represented by TS_i. We can assume by the usual cancellation arguments for intersection points [16] that $S_i \cap TS_j = \emptyset$ and if $i \neq j$ we can assume that $S_i \cap S_j = \emptyset$. To obtain this we first achieve that $S_i \cap |\alpha| = \emptyset$ and then we separate the spheres in the free part. By usual surgery theory, it follows that we can use S_i and TS_i to do $\mathbf{Z}_2$ surgery in the free part and kill K.

2. THE 4n-DIMENSIONAL SURGERY PROBLEM

Let X and Y be $4n$-dimensional $\mathbf{Z}_2$ manifolds and $f: X \to Y$ an h-normal map. Let Y be 1-connected. Abbreviate $\pi_0(Y^{\mathbf{Z}_2}) = \pi$. Assume there is one component $\beta_0 \in \pi$ with $d(\beta_0) = 2n$, and for $\beta \in \pi$, $\beta \neq \beta_0$, assume $6 < d(\beta) < 2n$. Let

$\alpha_0 = \tilde{f}^{-1}(\beta_0)$, and d = degree $(f|_{|\alpha_0|} : |\alpha_0| \to |\beta_0|)$. By [2], d is odd. Then we can state

THEOREM 2.1. *The surgery problem $f: X \to Y$ is solvable if and only if*

(i) $\sigma_\beta(f) = 0 \in L(\beta)$, $\beta \in \pi$,

(ii) sign $(\mathbf{Z}_2, Y)$ − sign $(\mathbf{Z}_2, X) = 0$,

(iii) $(d^2 - 1)$ sign $(T, Y) = 0$.

Remarks. $\sigma_\beta(f)$ is the obstruction to converting $f_\alpha : |\alpha| \to |\beta|$ by surgery of type α into a $\mathbf{Z}_2$ homology equivalence (section 1). $\alpha = \tilde{f}^{-1}(\beta)$ and f_α is the restriction of f.

Before we can prove this theorem we have to develop some more theory. But first let us give a relation between the invariants in the framed set-up. It is not clear to the author whether or not this relation holds in general.

LEMMA 2.2. *If $f^* \mathscr{I} Y \cong \mathscr{I} X$ then* 2.1(ii) *implies* 2.1(iii).

Proof. Let $[\alpha_0]$ and $[\beta_0]$ denote the fundamental class of α_0 and β_0, probably after making $|\alpha_0|$ connected. Then we compute in terms of Hirzebruch L classes:

$$\begin{aligned} \text{sign }(T,X) &= \langle L(\mathscr{I}|\alpha_0|, \nu(|\alpha_0|, X)), [\alpha_0] \rangle \\ &= \langle L(\mathscr{I}|\beta_0|, \nu(|\beta_0|, Y)), (f_\alpha)_* [\beta_0] \rangle \\ &= d \text{ sign }(T,Y) \end{aligned}$$

$$\begin{aligned} (d^2 - 1)\text{ sign }(T,Y) &= (d+1)(d-1)\text{ sign }(T,Y) \\ &= (d+1)(\text{sign }(T,X) - \text{sign }(T,Y)) \\ &= 0 \end{aligned}$$

Let us do the algebra first: As in [26] it is easy to see that $\tilde{W}_0(\Lambda)$ is a group.

THEOREM 2.3. $\tilde{W}_0(\Lambda) \cong \mathbf{Z} \oplus \mathbf{Z}$.

The class of (K, λ, μ, T) is given by the multisignature of $\tilde{\lambda}$. The only relations for the invariants are sign $\lambda(\cdot,\cdot) \equiv 0(8)$ and sign $\lambda(\cdot,\cdot) \equiv$ sign $\lambda(\cdot, T\cdot)$ (mod 2).

For the proof of this theorem we need two more lemmas. Define the group $WG(\mathbf{Z}_2, \mathbf{Z})$ as in [9]. Compared with the definition of $\tilde{W}_0(\Lambda)$ we consider forms over torsion free modules and $\lambda(x,x)$ need not be even. Furthermore, a form is equivalent to zero if it splits [17], [9] or is metabolic in the sense of [5]. μ is not defined in this case. There is a split exact sequence [1]

$$0 \to WG(\mathbf{Z}_2, \mathbf{Z}) \to WG(\mathbf{Z}_2, \mathbf{Z}[1/2]) \to W(\mathbf{Z}_2) \to 0.$$

As in [27] we obtain maps $\varphi_\pm : \mathbf{Z}[\mathbf{Z}_2] \to \mathbf{Z}_\pm$, $\mathbf{Z}_2$ operates on $\mathbf{Z}_\pm$ by $\pm$Id, and $a1 + bT \mapsto a \pm b$. This induces forms $\tilde{\lambda}_\pm : K_\pm \times K_\pm \to \mathbf{Z}_\pm$. After tensoring with $\mathbf{Z}[1/2]$ these forms are unimodular (the determinant of the matrix with respect to some basis is a unit) and we obtain an isomorphism

$$\varphi : WG(\mathbf{Z}_2, \mathbf{Z}[1/2]) \to WG(1, \mathbf{Z}[1/2]) \oplus WG(1, \mathbf{Z}[1/2]),$$

$$(K,\lambda,T) \mapsto ([K_+,\tilde{\lambda}_+,\mathrm{Id})],[(K_-,\tilde{\lambda}_-,-\mathrm{Id})]).$$

$$0 \to WG(1,\mathbf{Z}) \to WG(1,\mathbf{Z}[1/2]) \to W(\mathbf{Z}_2) \to 0$$

is split exact [1], $WG(1,\mathbf{Z}) \cong \mathbf{Z}$, and $W(\mathbf{Z}_2) \cong \mathbf{Z}_2$ [17]. Together this implies (already in [5]):

LEMMA 2.4. $WG(\mathbf{Z}_2,\mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_2$.

Now define $WG^{fr}(\mathbf{Z}_2,\mathbf{Z})$ to be those elements in $WG(\mathbf{Z}_2,\mathbf{Z})$ which are represented by (K,λ,T), where K is a free Λ-module.

Lemma 2.5. $WG^{fr}(\mathbf{Z}_2,\mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$, the forms are classified by the multisignature, and the only relation is sign $\tilde{\lambda}_+ \equiv$ sign $\tilde{\lambda}_-$ (mod 2).

Proof. We obtain a commutative diagram:

$$\begin{array}{ccc} 0 \longrightarrow WG^{fr}(\mathbf{Z}_2,\mathbf{Z}) & \longrightarrow & WG(\mathbf{Z}_2,\mathbf{Z}[1/2]) \\ \downarrow \psi & & \downarrow \varphi\ \cong \\ WG(1,\mathbf{Z}) \oplus WG(1,\mathbf{Z}) & \longrightarrow & WG(1,\mathbf{Z}[1/2]) \oplus WG(1,\mathbf{Z}[1/2]) \end{array}$$

ψ is defined as φ without tensoring with $\mathbf{Z}[1/2]$, but we can define it only for free objects in $WG(\mathbf{Z}_2,\mathbf{Z})$ because otherwise $(K_\pm,\tilde{\lambda}_\pm)$ are not unimodular. ψ is injective. This shows that $WG^{fr}(\mathbf{Z}_2,\mathbf{Z}) \subseteq \mathbf{Z} \oplus \mathbf{Z}$. It is well known that these forms are classified by the multisignature. It is easy to write down examples for all required signatures. sign $\tilde{\lambda}_+ \equiv$ sign $\tilde{\lambda}_-$ (mod 2) as $rk\ K_+ = rk\ K_-$.

We have a projection map $p\colon \tilde{W}_0(\Lambda) \to WG^{fr}(\mathbf{Z}_2,\mathbf{Z})$. The image are forms with $\lambda(x,x)$ even. Thus we obtain only forms with sign $\lambda(\cdot,\cdot) \equiv 0$ (mod 8).

Proof of Theorem 2.3. It is an easy exercise to give enough forms realizing all required signatures. Thus it is sufficient to show that p is injective. Consider a form (K,λ,μ,T) representing an element in $\tilde{W}_0(\Lambda)$ and $p(K,\lambda,T) \sim 0$ in $WG^{fr}(\mathbf{Z}_2,\mathbf{Z})$. Then we have to show that $(K,\lambda,\mu,T) \sim 0$ in $\tilde{W}_0(\Lambda)$.

If $\lambda(x,Tx) \equiv 0(2)$ for all $x \in K$, then $\tilde{\mu}$ is defined (as in the introduction). $(K,\tilde{\lambda},\tilde{\mu})$ defines an element in $L_4(\Lambda,1)$ and the classification by the multisignature is given in [26], [27]. This provides us with a free Λ-module N which is a subkernel.

Now assume there exists $x \in K$ such that $\lambda(x,Tx) \equiv 1$ (mod 2). We have the exact sequence

$$0 \to N \to K \to \mathrm{Hom}(N,\mathbf{Z}) \to 0,$$

where N is torsion free. The point is to show that it is split exact.

Assume $N = N_+ \oplus N_-$ where Z_2 operates by $\pm\mathrm{Id}$ on $N_\pm$. If N contains a free submodule N_0 we can split off the sequence

$$0 \to N_0 \to K_0 \to \mathrm{Hom}(N_0,\mathbf{Z}) \to 0 \qquad \text{(see 4.4 and [17])}$$

and $(K_0,\lambda|_{K_0\times K_0},\mu|_{K_0},T|_{K_0}) \sim 0$ in $\tilde{W}_0(\Lambda)$. Now write $\Lambda = \mathbf{Z}\cdot 1 + \mathbf{Z}\cdot T$. As K is a free Λ-module, write $K = \bar{K} + T\bar{K}$ in the same fashion. As Λ-modules

$$N \cong \mathrm{Hom}(N,\mathbf{Z}) \qquad \text{and} \qquad K \otimes \mathbf{Q} \cong (N \oplus N) \otimes \mathbf{Q}.$$

Thus $rk\, N_+ = rk\, N_-$. Let $\{e_1^\pm,\ldots,e_r^\pm\}$ be a basis of $N_\pm$. Then we define $\bar{e}_i^\pm \in \bar{K}$ by the equation $e_i^\pm = \bar{e}_i^\pm \pm T\bar{e}_i^\pm$. $\{\bar{e}_i^+,\bar{e}_i^-\}$ are linearly independent. Otherwise we have an $f \in \bar{K}$ and f could be chosen as the basis element $\bar{e}_1^+$ and $\bar{e}_1^-$. Then f generates a free summand N_0. This was excluded. $\{\bar{e}_i^+,\bar{e}_i^-\}$ is a $\mathbf{Z}$-basis of $\bar{K}$ and a Λ-basis for K, as otherwise $\bar{K}/\{\bar{e}_i^+,\bar{e}_i^-\}$ and K/N would contain torsion.

By assumption

$$\lambda(e_i^+,e_i^+) = 0 \Rightarrow 2(\lambda(\bar{e}_i^+,\bar{e}_i^+) + \lambda(\bar{e}_i^+,T\bar{e}_i^+)) = 0$$

As $\lambda(\bar{e}_i^+,\bar{e}_i^+) \equiv 0(2)$ we obtain $\lambda(\bar{e}_i^+,T\bar{e}_i^+) \equiv 0(2)$. In the same way, $\lambda(e_i^-,e_i^-) = 0$ implies that $\lambda(\bar{e}_i^-,T\bar{e}_i^-) \equiv 0(2)$. Thus $\lambda(x,Tx) \equiv 0(2)$ for all $x \in K$ which is a contradiction. Thus $N_+ = N_- = 0$ and N is Λ-free.

Proof of Theorem 2.1. Obviously, 2.1(i–iii) are invariants. Assume the surgery problem is solvable. Then 2.1(i) is immediate from section 1 and (ii) is folklore. To show (iii) look at the diagram (having the problem solved)

$$\begin{array}{ccc}
H_{2n}(X,\mathbf{Z}) & \xrightarrow{\quad f_* \cong \quad} & H_{2n}(Y,\mathbf{Z}) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} \\
H_{2n}(|\alpha_0|,\mathbf{Z}) & \xrightarrow{\quad (f_{\alpha_0})_* \quad} & H_{2n}(|\beta_0|,\mathbf{Z}) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\mathbf{Z} & \xrightarrow{\quad d \quad} & \mathbf{Z}
\end{array}$$

i_* is induced by the inclusion. Compute $\mathrm{sign}(T,\cdot)$ in terms of the intersection number of the fixed point set with itself [11]. Then we get

$$\begin{aligned}
\mathrm{sign}(T,X) - \mathrm{sign}(T,Y) &= \lambda([\alpha_0],[\alpha_0]) - \lambda([\beta_0],[\beta_0]) \\
&= \lambda(d[\beta_0],d[\beta_0]) - \lambda([\beta_0],[\beta_0]) \\
&= (d^2 - 1)\,\mathrm{sign}\,(T,Y).
\end{aligned}$$

By (ii), $(d^2 - 1)\,\mathrm{sign}\,(T,Y) = 0$.

Now assume 2.1(i–iii). By (i) and section 1 we can assume that f satisfies condition P. By (ii), $(K_{2n}(f,\mathbf{Z}),\lambda,\mu,T) \sim 0$ in $\tilde{W}_0(\Lambda)$. Let $\{e_1,\ldots,e_r\}$ be a Λ-basis of a subkernel. Using appropriate linear combinations we can assume that $\lambda(e_i,p_2 i_*[\alpha_0]) = 0$ for $i \geq 2$, where p_2 is the projection $H_{2n}(X,\mathbf{Z}) \to K_{2n}(f,\mathbf{Z})$. By section 1 we can kill $\{e_2,\ldots,e_r\}$ by surgery in the free part (and their duals). Now we are left with $K_{2n} = \Lambda \oplus \Lambda$ and a subkernel generated by a single element e. Choose $h \in K$ such that $\tilde{\lambda}(e,h) = (1,0)$, and let $e^* = h - 1/2(\lambda(h,h)e - \gamma Te)$, where $\lambda(h,Th) = 2\gamma + a$, $a = 0$ or 1. According to the basis $\{e,Te,e^*,Te^*\}$, λ has the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & a \\ 0 & 1 & a & 0 \end{pmatrix}$$

Consider the natural isomorphism $(f_*, p_2): H_{2n}(X,\mathbf{Z}) \to H_{2n}(Y,\mathbf{Z}) \oplus K_{2n}(f,\mathbf{Z})$. Then

$$i_*[\alpha_0] = di_*[\beta_0] + \alpha(e + Te) + \beta(e^* + Te^*), \qquad \alpha, \beta \in \mathbf{Z}.$$

Computing the signature by [11] we get

$$0 = \mathrm{sign}(T,X) - \mathrm{sign}(T,Y) = (d^2 - 1)\,\mathrm{sign}\,(T,Y) + 2\beta(2\alpha + a\beta).$$

Thus, $\beta(2\alpha + a\beta) = 0$. To show that we can solve the surgery problem we have to show by 0.4 only that we can find a subkernel N' generated by e' and $\lambda(e', p_2[\alpha_0]) = 0$. If $a = 0$ then α or β is zero. So choose e' to be e^* or e. If $a = 1$ then $\beta = 0$ or $2\alpha + \beta = 0$. In the first case, choose again $e' = e$. If $2\alpha + \beta = 0$ choose $e' = e - e^* - Te^*$.

Remark. For every pseudoequivalence $f: X \to Y$ we obtain

$$(d^2 - 1)\,\mathrm{sign}\,(T,Y) = 0,$$

if there is at most one component in the fixed point set and all other components have lower dimensions. This gives a restriction $d = \pm 1$ if $\mathrm{sign}(T,Y) \neq 0$. An example is $Y = \mathbf{C}P^{2n}$ with $Y^{\mathbf{Z}_2} = \mathbf{C}P^n \amalg \mathbf{C}P^{n-1}$. Degree theory would only tell us that d is odd [2].

3. INVARIANTS AND NONINVARIANTS FOR $\mathbf{Z}_2$ SURGERY

Let us point out some more results giving insight into the geometry of involutions. In this section we assume $\dim X = \dim Y = 2m$. Let $f: X \to Y$ by an h-normal map, and $d(\beta) \leq m$ for $\beta \in \pi$. Assume that we can do surgery on (X, f) to obtain (X', f') where f' satisfies Condition P. Then $r(f) = rk_\Lambda K_m(f',\mathbf{Z}) \bmod 2$.

LEMMA 3.1. *$r(f) \in \mathbf{Z}_2$ is well defined.*

Proof. Assume $f'': X'' \to Y$ has the same properties as f. Let $F: N \to Y \times I$ be the cobordism connecting f', and f'', which is given by equivariant surgery. $\partial N = X' \cup X''$. Do surgery in the interior of N below the middle dimension in the free part to make F highly connected. As in [26] we get the exact sequence

$$0 \to K_{m+1}(N,\partial N) \to K_m(\partial N) \to K_m(N) \to 0.$$

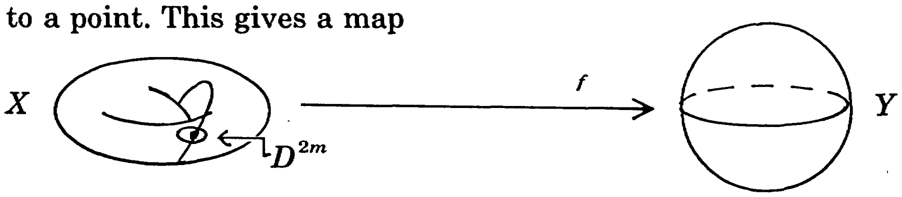
$K_m(\partial N) \cong K_m(f') \oplus K_m(f'')$ and $K_{m+1}(N,\partial N) \cong K_m(N)$ as Λ-modules. Thus

$$K_m(\partial N) \otimes \mathbf{Q} \cong (K_{m+1}(N,\partial N) \oplus K_m(N)) \otimes \mathbf{Q}.$$

Applying the Krull-Schmidt theorem, we see that $4|rk_{\mathbf{Z}}K_m(\partial N)$. Thus

$$rk_\Lambda K_m(f') \equiv rk_\Lambda K_m(f'')(\text{mod } 2)$$

Here is a simple application of this lemma. Let $X = S^m \times S^m$ with $T(x,y) = (y,x)$ and $Y = S^{2m}$ with $T(x_1,\ldots,x_{2m+1}) = (-x_1,\ldots,-x_m,x_{m+1},\ldots,x_{2m+1})$, and f given as follows. Let $p \in X^{\mathbf{Z}_2}$ and D^{2m} a small $\mathbf{Z}_2$ invariant disk with center p. Collapse $X - D^{2m}$ to a point. This gives a map



$f\colon X \to Y$ with all our assumed properties, including Condition P. But

$$H_m(X) = K_m(f) \cong \Lambda.$$

Thus $r(f) = 1$. This implies that we cannot solve the surgery problem. (If $m \equiv 0(2)$, then also $\text{sign}(T,X) = 2$). The invariant r always vanishes if $\dim X^{\mathbf{Z}_2} < 1/2 \dim X$.

One would hope to find some more invariants giving necessary conditions to solve the surgery problem. Here we want to treat two such candidates. Let $f\colon X \to Y$ and assume that f satisfies Condition P. Then we defined $t(f)$ and $I(f)$ in section 0.

LEMMA 3.2. $r(f) = 1 \Rightarrow t(f) = 1$.

Proof. If $t(f) = 0$, then (K,λ,μ,T) defines an element in $L_{2m}(\Lambda,\omega)$. This implies that $rk_\Lambda K \equiv 0(2)$.

In the introduction we gave an example where the converse does not hold.

Let $x \in K_m(f,\mathbf{Z})$, respectively, $\in K_m(f,\mathbf{Z}_2)$ and $i\colon X^{\mathbf{Z}_2} \to X$. Remember that we have a natural decomposition

$$H^m(X,\mathbf{Z}_2) \cong K^m(f,\mathbf{Z}_2) \oplus H^m(Y,\mathbf{Z}_2) \qquad [26].$$

We want to investigate the relation between $\lambda(x,Tx)$ mod 2 and $i^*\bar{x} \in H^m(X^{\mathbf{Z}_2},\mathbf{Z}_2)$ where $\bar{x}$ denotes the dual of x. The motivation is given by Lemma 1.11 [3] (see Lemma 3.3(i)). If we do not have a component $\alpha \in \pi$ such that $d(\alpha) = m$ we do not have to investigate anything.

LEMMA 3.3.

(i) $I(f) = 0 \Rightarrow t(f) = 0$.

(ii) *If there is only one component* $\alpha \in \pi$ *with* $d(\alpha) = m$, *then* $t(f) = 0 \Rightarrow I(f) = 0$.

(iii) *If there are* α, $\alpha' \in \pi$, $d(\alpha) = d(\alpha') = m$ *and* $\alpha \neq \alpha'$, *then*

$$t(f) = 0 \not\Rightarrow I(f) = 0.$$

Proof. i) is an immediate consequence of Corollary 1.11 [3]. ii). Let $\alpha \in \pi$

and $d(\alpha) = m$. Let $[\alpha] \in H_m(X^{\mathbf{Z}_2},\mathbf{Z})$ denote the fundamental class of $|\alpha|$. Then $i^*(\bar{x}) = 0 \in H^m(X^{\mathbf{Z}_2},\mathbf{Z}_2)$ if and only if $[\alpha]' = 0 \in K_m(f,\mathbf{Z}_2)$, where $[\alpha]'$ is the image of $[\alpha]$ under the composed map

$$H_m(X^{\mathbf{Z}_2},\mathbf{Z}) \to H_m(X,\mathbf{Z}) \to K_m(f,\mathbf{Z}) \to K_m(f,\mathbf{Z}_2).$$

Thus it suffices to show that $\lambda(x,i_*[\alpha]) \equiv 0 \pmod 2$ for all $x \in K_m(f,\mathbf{Z})$. Represent x by an imbedded sphere, again denoted by x, and assume $x \pitchfork Tx$ and $x \pitchfork |\beta|$. Tx denotes the imbedding of S^m we used to represent x composed with the involution T. If x and Tx intersect outside of $|\alpha|$, then they intersect in pairs of points $\{p,q\}$, $p \neq q$. Thus $\lambda(x,Tx) \equiv \lambda(x,i_*[\alpha]) \pmod 2$. Thus $t(f) = 0$ implies $I(f) = 0$.

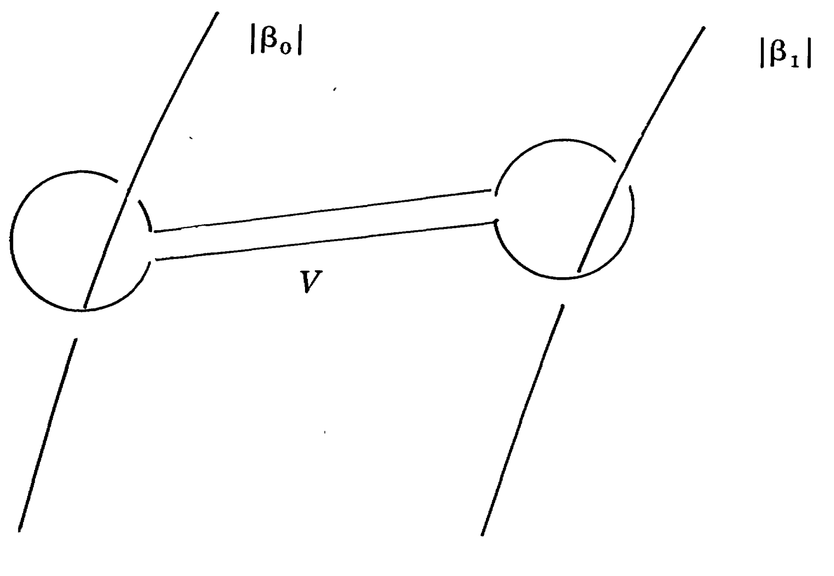
To show iii), we construct a concrete counterexample. Consider any $\mathbf{Z}_2$ manifold Y^{2m}, $m \geq 5$ and let π contain at least two components β_0 and β_1 such that

$$d(\beta_0) = d(\beta_1) = m, \qquad \text{and} \qquad i_*[\beta_i] = 0 \in H_m(Y,\mathbf{Z}).$$

Such a Y exists. Use, for example, two spheres S^{2m} with an m-dimensional sphere fixed under the involution. Then do free surgery in dimensions zero and one to obtain Y with the above properties. Consider the pseudoequivalence and h-map $\mathrm{Id}: Y \to Y$. Now do surgery as follows: Let $p_i \in |\beta_i|$, $i = 0, 1$, and

$$D_i^m = D(\nu(|\beta_i|,Y)|_{p_i}),$$

D the closed unit disk, S_i^{m-1} the boundary sphere of D_i^m. After a small isotopy we assume that $TS_i^{m-1} \cap S_i^{m-1} \neq \emptyset$. Connect S_0 and S_1 by a tube V (take $S = S_0 \# S_1$).



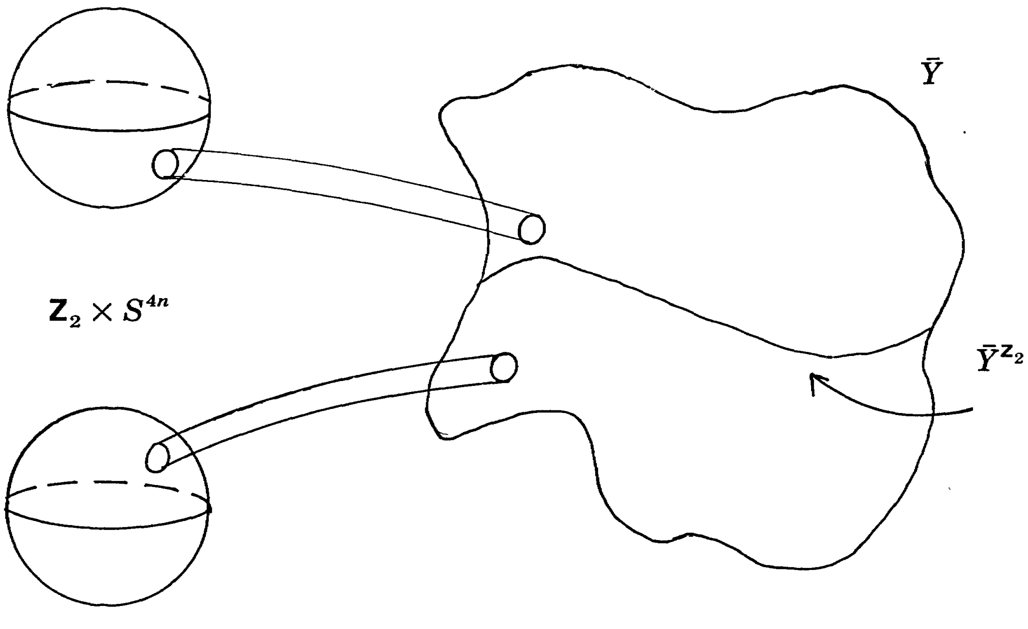
Then we obtain an imbedded $S^{m-1} \subset Y$ and $TS \cap S = \emptyset$. Now we do free surgery on $S \cup TS \subset \mathbf{Z}_2 \times S^{m-1} \times D^{m+1}$. Denote the result of this surgery by X, and we have a map $f: X \to Y$ which satisfies condition P. Then $K_m(f,\mathbf{Z}) \cong \Lambda \oplus \Lambda$, generated by e and e^* over Λ, e represented by $1 \times 0 \times S^m \subset \mathbf{Z}_2 \times D^m \times S^m$ and e^* represented by $(D_0 \cup V \cup D_1) \cup 1 \times D^m \times *$, $*$ a point in S^m. To compute intersection numbers, we assume $D_i \tilde{\pitchfork} TD_i$ and $D_0 \cap TD_0 \approx D_1 \cap TD_1$. Then we obtain $\lambda(e^*,Te^*) = a \equiv 0(2)$. (Actually, we can get $a = 0$ or $a \neq 0$.) This geometry tells us that λ has the following form for the basis $\{e,Te,e^*,Te^*\}$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 & a \\ 0 & \mp 1 & \pm a & 0 \end{pmatrix} \qquad (\pm \text{ if } m \equiv 1 \pmod 2)$$

Furthermore, $\lambda(e^*,[\beta_i]) \equiv 1(2)$ and $i^*(e^*) \neq 0 \in H^m(X^{\mathbf{Z}_2},\mathbf{Z}_2)$.

LEMMA 3.4. *There exists an example of an h-map $f: X \to Y$ such that $I(f) = 0$ but the surgery problem is not solvable.*

Proof. Consider the solved surgery problem $\mathrm{Id}: \bar{Y}^{4n} \to \bar{Y}^{4n}$ satisfying all required assumptions, as $\pi_1(\bar{Y}) = 0$, etc. Consider a further problem $h: X_0 \to S^{4n}$, with sign $X_0 \neq 0$. Let $q \in \bar{Y} - \bar{Y}^{\mathbf{Z}_2}$ and $p \in S^{4n}$. Use p and q to form the connected sum of $\mathbf{Z}_2 \times S^{4n}$ and $\bar{Y}$ and denote the result by Y. Do the same construction using $q' = q \in Y$ and $p' = h^{-1}(p) \in X_0$. Denote the result by $f: X \to Y$. There are appropriate bundle data for f and Y is 1-connected. Thus f is an h-map. $\bar{f}|_{X^{\mathbf{Z}_2}}$



is the identity. Then $\mathrm{sign}(1,X) - \mathrm{sign}(1,Y) = 2 \,\mathrm{sign}\, X_0 \neq 0$, and the surgery problem is not solvable. Assuming h to be highly connected obviously implies that $K_{2n}(f,\mathbf{Z}) = \mathbf{Z}_2 \times H_{2n}(X_0,\mathbf{Z})$ and $I(f) = 0$.

4. THE $4n + 2$ DIMENSIONAL SURGERY PROBLEM

Suppose we are given the h-normal map $f: X^{4n+2} \to Y^{4n+2}$ and $d(\beta) \leq 2n + 1$ for $\beta \in \pi$, and we have exactly one component β_0 such that $d(\beta_0) = 2n + 1$.

THEOREM 4.1. *The above surgery problem is solvable if and only if*

(i) $\sigma_\beta(f) = 0$ *for all* $\beta \in \pi$.

(ii) $r(f) = 0$,

(iii) $c(f) = 0$.

The obstructions $\sigma_\beta(f) \in L(\beta)$ again denote the obstructions to converting $f_\beta : |\tilde{f}^{-1}(\beta)| \to |\beta|$ into a mod 2 homology equivalence. $r(f)$ is defined if the obstructions in i) vanish (see section 3), and $c(f)$ is the usual Arf invariant for the surgery problem *after forgetting the* $\mathbf{Z}_2$ *action.*

Proof. It is obvious that i)–iii) are surgery invariants and that they vanish for a pseudoequivalence. Thus they are necessary. Let us show that they are sufficient. i) implies that we can do surgery on the fixed point set and obtain a $\mathbf{Z}_2$ homology equivalence for $f^{\mathbf{Z}_2}: X^{\mathbf{Z}_2} \to Y^{\mathbf{Z}_2}$. Now we do surgery in the free part below the middle dimension, which is always possible, and achieve condition P. Then $K_{2n+1}(f,\mathbf{Z})$ is a Λ-free module and it defines, together with the intersection form λ, the self-intersection form μ, and the involution T a quasi skew Hermitian form (K,λ,μ,T) representing an element in $\tilde{W}_2(\Lambda)$. Again we want to satisfy the assumptions of Lemma 0.4. We state this as a lemma (which we prove after some preparation) and then 4.1 is proven.

LEMMA 4.2. *Let* $f: X \to Y$ *be an h-map satisfying condition* P, *the assumptions of* 4.1 *and* 4.1 ii) *and* iii). *Then we can do surgery in the free part on* (X,f) *and obtain* (X',f') *such that* $(K_{2n+1}(f',\mathbf{Z}),\lambda,\mu,T)$ *satisfies the assumptions of Lemma* 0.4.

Now we proceed as follows. In Theorem 4.3 we compute $\tilde{W}_2(\Lambda)$. This shows that 4.1 ii) and iii) imply that $[(K_{2n+1}(f,\mathbf{Z}),\lambda,\mu,T)] = 0$ in $\tilde{W}_2(\Lambda)$. We have to stabilize to exhibit a subkernel, but this stabilization occurs geometrically, Lemma 4.8. After some more (if needed) stabilization we show that (K,λ,μ,T) is the direct sum of copies of $(\bar{K},\bar{\lambda},\bar{\mu},\bar{T})$ such that each $\bar{K}$ is of Λ-rank 2, has a subkernel, and there exists $x \in \bar{K}$ such that $\lambda(x,Tx) \equiv 1 (\text{mod } 2)$. This is Lemma 4.9. Then we exhibit a subkernel in $\bar{K}$ which does not intersect the fixed point set, i.e., $(\bar{K},\bar{\lambda},\bar{\mu},\bar{T})$ satisfies the assumptions of Lemma 0.4. This is Lemma 4.10 and Lemma 4.2 is a corollary of this fact.

We defined $\tilde{W}_2(\Lambda)$ in the introduction and it is a trivial check that $\tilde{W}_2(\Lambda)$ is a group. Furthermore, $(K,\lambda,\mu,T) \oplus (K,\lambda,\mu,T) \sim 0$; for if $\{e_1,\ldots,e_k\}$ is a Λ-basis of K, then we get the basis $e_1' + Te_1'',\ldots,e_k' + Te_k''$ for an appropriate $N \subset K \oplus K$. The symbols $'$ and $''$ indicate the first and second copy of K. Thus $\tilde{W}_2(\Lambda)$ contains only 2-torsion.

THEOREM 4.3. $\varphi: \tilde{W}_2(\Lambda) \to \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *given by*

$$\varphi[(K,\lambda,\mu,T)] = (rk_\Lambda K (\text{mod } 2), c(K,\lambda,\mu))$$

is an isomorphism.

Proof. φ is obviously a homomorphism. We show

a) φ is well defined,

b) φ is surjective,

c) φ is injective.

a) If $(K,\lambda,\mu,T) \sim 0$, then by definition $c(K,\lambda,\mu) = 0$. As

$$K \cong N \oplus \mathrm{Hom}(N,\mathbf{Z}) \qquad \text{and} \qquad N \cong \mathrm{Hom}(N,\mathbf{Z}),$$

it follows that $rk_\Lambda K \equiv 0(2)$.

b) Let $K = \Lambda$ with the $\mathbf{Z}$-basis $\{e,Te\}$ and λ corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then Te is dual to e and $c(K,\lambda,\mu) = (\mu(e))^2$. Setting $\mu(e) = 0$ respectively $= 1$ gives forms with invariants (1,0) respectively (1,1). We obtain forms of arbitrary invariants as linear combinations of these two forms.

c) is the difficult part and we first state an orthogonal decomposition lemma ([17, I.3.1) and do some technical preparation.

LEMMA 4.4. *Let (K,λ,μ,T) be a quasi skew Hermitian form* (q.s.H. *form) and (M,λ,μ,T) be a subform. Then*

$$(K,\lambda,\mu,T) = (M,\lambda,\mu,T) \oplus (M^\perp,\lambda,\mu,T),$$

where $M^\perp = \{x \in K \,|\, \lambda(x,M) = 0\}$.

The proof is elementary and we remark only that $M^\perp$ is again Λ-free $(\tilde{K}_0(\Lambda) = 0)$ and μ splits as λ does.

We shall need the following q.s.H forms $(K_i,\lambda_i,\mu_i,T_i)$, $i = 0,\ 1 : K_i$ is freely generated over Λ by $\{e_i,e_i^*\}$, $\mu(e_i) = \mu(e_i^*) = i$ and with respect to the $\mathbf{Z}$-basis $\{e_i,Te_i,e_i^*,Te_i^*\}$, λ corresponds to the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then by the above remark $(K_i,\lambda_i,\mu_i,T_i)$ defines a form $(K_i,\tilde{\lambda}_i,\tilde{\mu}_i)$ and $c_2(K_i,\tilde{\lambda}_i,\tilde{\mu}_i) = i$. Furthermore,

$$(K_0,\lambda_0,\mu_0,T_0) \oplus (K_0,\lambda_0,\mu_0,T_0) \cong (K_1,\lambda_1,\mu_1,T_1) \oplus (K_1,\lambda_1,\mu_1,T_1).$$

LEMMA 4.5. *Every* q.s.H. *form (K,λ,μ,T) splits stably as a direct sum of* q.s.H. *forms of Λ-rank at most 2. If a summand $(K_i,\lambda_i,\mu_i,T_i)$ is of Λ-rank 2, we can assume that $\lambda(x,Tx)$ is even for all $x \in K_i$. At most we have to stabilize with two copies of $(K_0,\lambda_0,\mu_0,T_0)$.*

Proof. Assume there exists $x \in K$ such that $\lambda(x,Tx) \equiv 1 (\mathrm{mod}\ 2)$. Then there exists a Λ-basis $\{h_1,\dots,h_r\}$ for K and $\lambda(h_1,Th_1) = 2k + 1$, $k \in \mathbf{Z}$. Then there exists $y \in K$ such that $\lambda(h_1,y) = -k$, $\lambda(Th_j,y) = \lambda(h_i,y) = 0$ for $1 \le i,j \le r$ and $i \ne 1$. Let $\bar{h} = h_1 + y + Ty$. Then $\lambda(\bar{h},T\bar{h}) = 1$, and $\bar{h}$ generates a free Λ-module H of Λ-rank 1. $(H,\lambda|_{H\times H},\mu|_H,T_H)$ is a q.s.H. form. Using 4.4 we can split this form off and obtain a form of strictly lower rank, which we denote again by (K,λ,μ,T). After a finite number of steps we can assume that $\lambda(x,Tx) \equiv 0$ (mod 2) for all K. Now

(K,λ,μ,T) defines a form $(K,\tilde{\lambda},\tilde{\mu})$. If $c_2(K,\tilde{\lambda},\tilde{\mu}) = 0$ it follows by [27] that we can decompose $(K,\tilde{\lambda},\tilde{\mu})$, respectively (K,λ,μ,T), in hyperbolic planes. If $c_2(K,\tilde{\lambda},\tilde{\mu}) = 1$ we add two copies of $(K_1,\lambda_1,\mu_1,T_1)$. Then $(K,\lambda,\mu,T) \oplus (K_1,\lambda_1,\mu_1,T_1)$ splits in hyperbolic planes. Together with a further copy of $(K_1,\lambda_1,\mu_1,T_1)$ we have a splitting of $(K,\lambda,\mu,T) \oplus$ two copies of $(K_0,\lambda_0,\mu_0,T_0)$. It is trivial to point out that $(K_0,\lambda_0,\mu_0,T_0) \sim 0$ in $\tilde{W}_2(\Lambda)$.

Remark 4.6. In particular, each of the direct summands $(\bar{K},\bar{\lambda},\bar{\mu},\bar{T})$ in our decomposition, with $rk_\Lambda \bar{K} = 2$ and $\lambda(x,Tx) \equiv 0 \pmod 2$ for all $x \in \bar{K}$, has a Λ-basis $\{e,e^*\}$ where e^* is dual to e. Computing the determinant of λ we show that we can assume that $\lambda(e,Te) = 0$. Furthermore, by the E. Schmidt orthogonalization we can assume that $\lambda(e^*,Te^*) = 0$. This shows $c(K,\lambda,\mu) = 0$.

We shall need a similar stabilization in the proof of the injectivity of φ. Let $(K_2,\lambda_2,\mu_2,T_2)$ be the following q.s.H. form. K_2 is generated by e_2 (over Λ); with respect to e_2 and Te_2, λ corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\mu(e) = \mu(Te) = 0$. This is the q.s.H form with invariants $c = 0$ and $r = 1$ in the proof of 4.3b.

LEMMA 4.7. $(K_2,\lambda_2,\mu_2,T_2) \oplus (K_2,\lambda_2,\mu_2,T_2) \cong (K_3,\lambda_3,\mu_3,T_3)$ *where* K_3 *has a basis* $\{e_3,Te_3,e_3^*,Te_3^*\}$ *such that* $\mu(e_3) = \mu(e_3^*) = 0$, *and with respect to this basis* λ_3 *has the matrix form*

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

$(K_3,\lambda_3,\mu_3,T_3) \sim 0$. (This defines $(K_3,\lambda_3,\mu_3,T_3)$.)

Proof. Let e', respectively e'', denote e in the first, respectively second, copy of K_2, and $e_3 = e' + Te''$, and $e_3^* = Te'$. e_3 generates a subkernel (trivial check).

Proof of c. (Theorem 4.3, φ is injective): Let (K,λ,μ,T) be a q.s.H. form and $rk_\Lambda K \equiv 0 \pmod 2$ and $c(K,\lambda,\mu) = 0$. Then we show that $(K,\lambda,\mu,T) \sim 0$. By Lemma 4.5, we decompose (K,λ,μ,T) (probably the stabilized form which is again denoted in the same way):

$$(K,\lambda,\mu,T) = \bigoplus_{i=1}^{r} (G_i,\lambda_i^G,\mu_i^G,T_i^G) \oplus \bigoplus_{i=1}^{s} (H_i,\lambda_i^H,\mu_i^H,T_i^H) \oplus \bigoplus_{i=1}^{t} (F_i,\lambda_i^F,\mu_i^F,T_i^F),$$

where

$rk_\Lambda G_i = 2,\ \lambda_i^G(x,T_i^G x) \equiv 0(2)$	for all $x \in G_i$	(type G)
$rk_\Lambda H_i = 1,\ c(H_i,\lambda_i^H,\mu_i^H) = 0$		(type H)
$rk_\Lambda F_i = 1,\ c(F_i,\lambda_i^F,\mu_i^F) = 1$		(type F)

Let (G,λ,μ,T) be a form of type G. Then G has a basis $\{e,Te,e^*,Te^*\}$, e^* dual to e, and $\lambda(e,Te) = \lambda(e^*,Te^*) = 0$. If $\mu(e) = 0$, then obviously $(G,\lambda,\mu,T) \sim 0$. If $\mu(e) = 1$ we consider $(G,\lambda,\mu,T) \oplus (K_2,\lambda_2,\mu_2,T_2)$, K_2 as in 4.7. Let G' be generated by $\{e + e_2 + Te_2, Te + e_2 + Te_2, e^*, Te^*\}$, giving the form (G',λ',μ',T'), which is a q.s.H. form equivalent to zero in $\tilde{W}_2$. Here $\{e + e_2 + Te, Te + e_2 + Te_2\}$ generates an appropriate subkernel. Thus stabilization with $(K_3,\lambda_3,\mu_3,T_3)$ reduces the number of copies of forms of type G by 1. The complement is a pair of forms of type H. Thus we can assume that $r = 0$, respectively, the sum of the G_i's is equivalent to zero.

As $c(K,\lambda,\mu) = 0$ it follows that s is even, and thus t is even. The following remark completes the proof. Let $(K^i,\lambda^i,\mu^i,T^i)$ be q.s.H. forms,

$$i = \{','' \},\ c(K',\lambda',\mu') = c(K'',\lambda'',\mu'') \quad \text{and} \quad rk_\Lambda K' = rk_\Lambda K'' = 1.$$

Then their direct sum is equivalent to zero. This is seen as follows. Let e^i be a basis element of K^i. If $\lambda(e',Te') = \lambda(e'',Te'')$ choose $e = e' + Te''$; if $\lambda(e',Te') = -\lambda(e'',Te'')$, choose $e = e' + e''$. Then e generates a subkernel $N \subset K' \oplus K''$. This completes the proof of Theorem 4.3.

LEMMA 4.8. *Stabilizing with* a) *one copy of* $(K_0,\lambda_0,\mu_0,T_0)$ *(see* 4.5) *or* b) *with one copy of* $(K_3,\lambda_3,\mu_3,T_3)$ (see 4.7) *arises geometrically.*

Proof of a). This is just as in ([26], Lemma 5.5) where we use connected sum with $\mathbf{Z}_2 \times S^{2k+1} \times S^{2k+1}$.

b) is exactly the stabilization given in 0.2.

Till now we stably split (K,λ,μ,T) into a direct sum of forms $(\bar{K},\bar{\lambda},\bar{\mu},\bar{T})$, either forms of type G with a basis $\{e,Te,e^*,Te^*\}$ where $\{e,Te\}$ generates a subkernel and $\lambda(e^*,Te^*) \equiv 0(2)$, or pairs of forms of type H or F. What we will need is

LEMMA 4.9. *Every* q.s.H *form* (K,λ,μ,T) *with* $rk_\Lambda K \equiv 0 \pmod 2$ *and* $c(K,\lambda,\mu) = 0$ *splits stably as a direct sum of* q.s.H. *forms of* Λ-rank 2. *For each direct summand* $(\bar{K},\bar{\lambda},\bar{\mu},\bar{T})$, *we can assume that* $\bar{K}$ *has a subkernel* N *generated by a basis element* e, *and* $\lambda(e^*,Te^*) = 1$. *We need at most to stabilize with forms* $(K_3,\lambda_3,\mu_3,T_3)$ *and* $(K_0,\lambda_0,\mu_0,T_0)$.

Proof. By the above remark, we are concerned only with the forms $(\bar{K},\bar{\lambda},\bar{\mu},\bar{T})$ of type G as above. Consider $(\bar{K},\bar{\lambda},\bar{\mu},\bar{T}) \oplus (K_2,\lambda_2,\mu_2,T_2)$ and the subform based on the module K' generated by $\{e,Te,e^* + e_2,Te^* + Te_2\}$; $\{e,Te,e^*,Te^*\}$is the basis of $\bar{K}$ and $\{e_2,Te_2\}$ the basis of K_2. $e^* + e_2$ is again dual to e and

$$\lambda(e^* + e_2,Te^* + Te_2\} = 1.$$

Then $(\bar{K},\bar{\lambda},\bar{\mu},\bar{T}) \oplus 2(K_2,\lambda_2,\mu_2,T_2)$ is isomorphic to $(K',\lambda',\mu',T') \oplus 2(K_2,\lambda_2,\mu_2,T_2)$, and each of the summands has the required properties.

LEMMA 4.10. *Let* (K,λ,μ,T) *be a* q.s.H. *form of* Λ-rank 2. *Suppose* $\{e,e^*\}$ *is a basis of* K, e^* Λ-*dual to* e, *such that* $\lambda(e,Te) = 0, \lambda(e^*,Te^*) = 1, \mu(e) = \mu(e^*) = 0$. *Let* $\bar{z} = (2m + 1)(e + Te) + 2k(e^* + Te^*)$. *Then we can find a new basis* $\{\tilde{e},\tilde{e}^*\}$ *of* K *such that* $\lambda(\tilde{e},\bar{z}) = 0$, $\lambda(\tilde{e},T\tilde{e}) = 0$ *and* $\mu(\tilde{e}) = 0$.

Proof. We can assume that $(2m+1,k)=1$. Secondly we have only to find $\tilde e$ and a dual $\tilde e^*$, where $\lambda(\tilde e,T\tilde e)=0$ and $\mu(\tilde e)=0$. Then by 4.4, $\{\tilde e,\tilde e^*\}$ is a basis for K. Choose $\tilde e=(2m+1-k)e+kTe+2ke^*$. Then it is trivial to check that $\lambda(\tilde e,T\tilde e)=0$, $\mu(\tilde e)=0$ and $\lambda(\tilde e,\tilde z)=0$. For $\tilde e$ we have to find a dual $\tilde e^*$. So let $\tilde e^*=ae+a'Te+be^*+b'Te^*$. We must show that we can find a, a', b and b' such that

α) $1=\lambda(\tilde e,\tilde e^*)=b(2m+1)+k(b'-b-2a)$,

β) $0=\lambda(T\tilde e,\tilde e^*)=b'(2m+1)+k(b-b'-2a')$.

Choose $b'=k$ and $b=k-(2m+1)+2a'$. Then α is satisfied. Then β) can be rewritten as

$$\beta) \qquad \frac{1-(2m+1)^2}{2}=a'(2m+1-k)-ak.$$

There exists a solution a, a' for β) as $(2m+1-k)$ and k are relatively prime.

Proof of Lemma 4.2. The assumptions of 4.1 and 4.1 ii) and iii) imply that $(K_{2n+1}(f,\mathbf{Z}),\lambda,\mu,T)\sim 0$ in $\tilde W_2(\Lambda)$ (4.3). By Lemmas 4.8 and 4.9, we can assume that we can do surgery in the free part on (X,f) to obtain (X',f') such that f' again satisfies the assumptions of 4.2 and $(K_{2n+1},(f',\mathbf{Z}),\lambda,\mu,T)$ splits as a sum of q.s.H. forms $(\bar K,\bar\lambda,\bar\mu,\bar T)$ each of them satisfying the assumptions of 4.10. Let z denote the image of the fundamental class of $|\alpha_0|$ in the surgery kernel. That is, $z=p_2([\alpha_0])$, where $p_2:H_{2n+1}(|\alpha_0|,\mathbf{Z})\to H_{2n+1}(X',\mathbf{Z})\to K_{2n+1}(f',\mathbf{Z})$ is the composition of the map induced by the inclusion $|\alpha_0|\to X'$ and the projection on a direct summand. Let $\bar z$ be the projection of z to $\bar K$. As in 3.3.i), $\lambda(e,Te)\equiv 0(2)$ implies $\lambda(e,z)\equiv 0(2)$ and $\lambda(e^*,Te^*)\equiv 1(2)$ implies that $\lambda(e^*,z)\equiv 1(2)$, and let $\{e,e^*\}$ be the basis of $\bar K$ as in 4.10. This easily implies that $\bar z$ has the properties in 4.10. Thus, by Lemma 4.10, we can assume that the subkernel $\bar N$ generated by e can be chosen such that $\lambda(e,\bar z)=0$. Then the collection of e for all $\bar K$ in our decomposition of $K_{2n+1}(f',\mathbf{Z})$ generates N satisfying all requirements of 4.2.

Example 4.13. We want to give an example of an h-map $f:X^{4n+2}\to Y^{4n+2}$ with the following properties.

$X^{\mathbf{Z}_2}=Y^{\mathbf{Z}_2}=S^{2n+1}$,

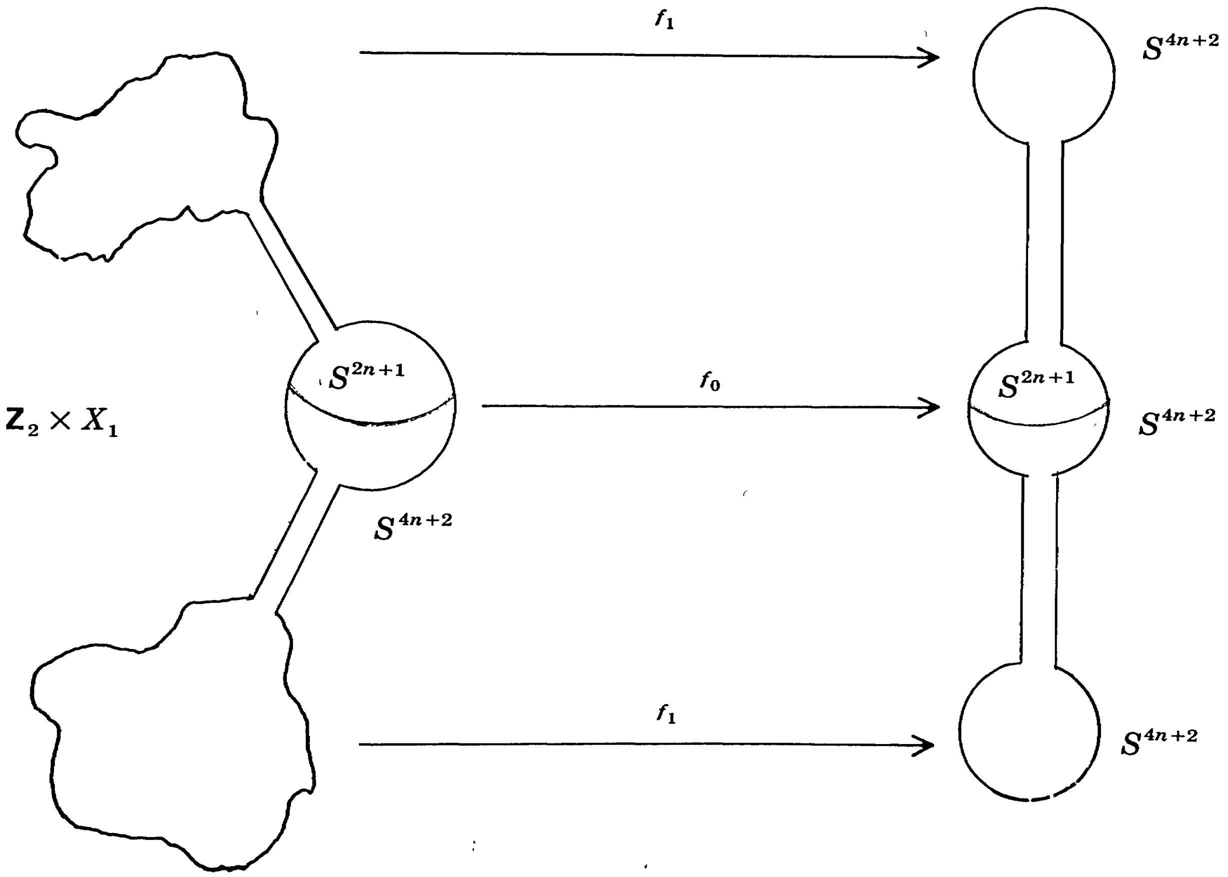
f satisfies condition P and $t(f)=0$ (i.e., $\lambda(x,Tx)\equiv 0(2)$ for all $x\in K_{2n+1}(f,\mathbf{Z})$,

Thus $(K_{2n+1}(f,\mathbf{Z}),\lambda,\mu)$ defines an element in $L_{4n+2}(\mathbf{Z},-)$ which is $c_2(K,\lambda,\mu)=1$.

But we can still do surgery to change f to a homotopy equivalence.

If we have an example with the above properties, then obviously the surgery obstruction is zero (Theorem 4.1).

Construction. Let $f_0:S^{4n+2}\to S^{4n+2}$ be the identity and suppose $S^{2n+1}\subset S^{4n+2}$ is the fixed point set. Now use a known example $f_1:X_1\to S^{4n+2}$ with Arf invariant 1 (no action). Take the connected sum of these three maps as indicated to obtain $f:X\to Y$. This map has all the required properties. In particular, it is an easy check to see that $c_2(K_{2n+1}(f,\mathbf{Z}),\tilde\lambda,\tilde\mu)=1$. Stated algebraically, this means $L_{4n+2}(\Lambda,-)\to\tilde W_2(\Lambda)$ is the trivial map. An algebraic proof of this fact is obtained from 4.3(c).



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