HOMOMORPHISMS OF C^* ALGEBRAS TO FINITE AW^* ALGEBRAS

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All C^* algebras and their homomorphisms are unital, and all ideals are two-sided unless otherwise qualified.

A ring R is *directly finite* if $xy = 1$ implies $yx = 1$ for all x,y in R. The ring R is *stably finite* if all rings of $n \times n$ matrices with entries from R (denoted $M_n R$) are directly finite. For C^* algebras, what is known as *finiteness* ($xx^* = 1$ implies $x^*x = 1$), is equivalent to direct finiteness [16; Theorem 27].

Stably finite rings admit a Grothendieck group (K_0) which has a natural ordering, and this in turn can lead to a great deal of structural information about the ring. For C^* algebras, the study of K_0 is becoming popular, especially for AF algebras.

I would particularly like to acknowledge the aid of Joachim Cuntz in the form of letters, helping me to understand his K_0^* and connected concepts. Conversations with Kenneth Goodearl were also of considerable value, in clarifying the proof of the existence of dimension-like functions on C^* algebras (Section 1).

Let A be a C^* algebra; following [4], [5], we define a (Cuntz's) *dimension function* as a map $D: A \to [0,1]$ satisfying:

(i) $D(1) = 1$

(ii) $D(a + b) \leq D(a) + D(b)$

(ii′) $D(a + b) = D(a) + D(b)$ if $ab = ab^* = a^*b = ba = 0$

(iii) $D(ab) \leq \text{Inf}\,\{D(a), D(b)\}$

(iv) If $\{a_n\}$ converges to a in norm, and if there exist x_n, y_n in A so that for all n, $a_n = x_n b y_n$ for some b, then $D(a) \leq D(b)$.

Consequences of these properties include the following:

(v) $D(a) = D((a^*a)^{1/2}) = D(a^*a) = D(a^*)$

(vi) $0 \leq a \leq b$ implies $D(a) \leq D(b)$

One can show that (v) and (vi) follow from (i) through (iv), essentially as in [5]; one observes (for example, for (vi)) that $0 \leq a \leq b$ implies the closure of the right ideal generated by b contains that of a. There thus exists a sequence $\{x_n\}$ in A with $\{bx_n\}$ converging to a; apply (iv).

If $D: A \to [0,1]$ satisfies (i) through (iii) (including (ii′), and is lower semicontinuous, then (iv) (and hence (v) and (vi)) follow.

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We do not require that $D(a) = 0$ imply $a = 0$; indeed, if I is a closed ideal of A, and A/I admits a dimension function, then there is one induced on A which contains I in its kernel. Observe that the kernel of a dimension function is a two-sided $*$-ideal: (ii), (iii), (v).

Our main result (2.4) asserts that every stably finite C^* algebra admits a $*$-homomorphism to a finite AW^* factor, and thus possesses a lower semicontinuous dimension function on all matrix rings. This is established by forming what amounts to an l^∞-product of the given C^* algebra, and showing with the aid of Cuntz's dimension functions that this maps onto a finite AW^* algebra.

Any trace, τ, induces a lower semicontinuous dimension function D_τ on all rings of matrices [5]. If it could be shown that a finite AW^* factor is W^* (as has been conjectured by Kaplansky), then the main result would yield:

Every stably finite C^* algebra would admit a trace.

The result, that finite AW^* factors are W^*, has been announced by Breuer-de la Harpe [3], but doubts have been cast on the proof, and there is evidence suggesting there is a counterexample.

Using the results of this article, a subsequent paper (co-authored with B. Blackadar) will contain the result, that almost every lower semicontinuous dimension function is induced by a $*$-homomorphism to a finite AW^* factor. In addition, extendibility of dimension functions to matrix rings and quotients will be discussed, along with various weakenings in the definition of lower semicontinuous dimension function.

SECTION 1

In this section, we generalize a result of Cuntz [5; 4.7]; this asserts that a simple stably finite C^* algebra possesses a dimension function on all matrix rings. Here, we establish a slight extension (by Cuntz's techniques), that the assumption of simplicity is unnecessary.

These remarks are based on [5], to which we refer the reader for many details, and the object is to discuss properties of dimension functions and prove an existence theorem, extending Cuntz's result for simple C^* algebras. Given a C^* algebra A, form $F \otimes A$, where F is the algebra of finite rank operators on a separable Hilbert space. Define an equivalence relation on the elements of $F \otimes A$ as follows:

$x \lesssim y$ if there exist x_n converging in norm to x, and corresponding a_n, b_n such that $x_n = a_n y b_n$. Declare $x \equiv y$ if $x \lesssim y$ and $y \lesssim x$. This $\equiv$ is an equivalence relation.

Let $K_0^*(A)$ be the abelian group with generators $[x]$, the equivalence classes under $\equiv$, and relations $[x] + [y] - [x+y]$ whenever x is orthogonal to y. This is easily seen to be compatible with the equivalence relation, and all elements of $K_0^*(A)$ are of the form $[x] - [y]$

As in [5], $K_0^*(A)$ admits the relation defined by $[a] - [b] \leq [c] - [d]$ if there exist x, as well as $a_1 \equiv a$, $b_1 \equiv b$, $c_1 \equiv c$, and $d \equiv d_1$ in $F \otimes A$, so that $\{a_1, d_1, x\}$

and $\{b_1, c_1, x\}$ are sets of orthogonal elements and $a_1 + d_1 + x \lesssim b_1 + c_1 + x$. Let H denote the set of elements that are $\geq [0]$ in the relation. Then

$$H \supset \{[a] : a \in F \otimes A\},$$

so $H - H = K_0^*(A)$; as in [5], $H + H \subset H$; however, it is not generally true that $H \cap -H = \{[0]\}$. Thus H defines only a preordering on the group $K_0^*(A)$.

Let e_n denote the rank n projection in F coming from the n by n identity matrix. Consider the element $e_1 \otimes 1$; since $n[e_1 \otimes 1] = [e_n \otimes 1]$, and since F consists of finite rank operators, it is clear that for all x in $K_0^*(A)$, there exists an integer n such that $[x] \leq n[e_1 \otimes 1]$. (It suffices to establish this for elements of the form $[a]$). Hence $[e_1 \otimes 1]$ is an order unit for the pre-ordered group.

We next wish to show that $n[e_1 \otimes 1] \notin -H$ for all positive integers n. Suppose not; that is, for some n, there exists a in $F \otimes A$ such that $[e_n \otimes 1] + [a] = [0]$; equivalently ([5; Section 4]) there exists $a_1 \equiv a$, as well as x with $\{e_n \otimes 1, x, a_1\}$ orthogonal so that $(e_n \otimes 1) + x + a_1 \equiv x$.

Define functions $\{s_m\}$ as in Lemma 4.1 of [5],

$$s_m(r) = \sup\{s \in \mathbf{N} \cup \{0\} : e_s \otimes 1 \lesssim e_{mn} \otimes r\}$$

(since r is already an element of $F \otimes A$, $e_{mn} \otimes r$ is to be interpreted as r repeated down the diagonal mn times. In that lemma, it was assumed that A was simple, but in fact simplicity is not required to show

(a) $d \lesssim f$ implies $s_m(d) \leq s_m(f)$;

(b) if d, f are orthogonal, then $s_m(d + f) \geq s_m(d) + s_m(f)$.

Next, we see that stable finiteness of A guarantees that all values of s_m are finite; a proof is included, as it is omitted from [5]. Say $e_m \otimes r$ fits inside a t by t matrix over A. It will be shown that $e_{t+1} \otimes 1 \lesssim e_m \otimes r$ is impossible. Let $E = e_{t+1} \otimes 1$. If $\{x_i\}$ converges to E, and $x_i \lesssim e_m \otimes r$, then for suitably large i, there exist p, q so that $px_iq = E$ (since E is a projection), whence $E = v(e_m \otimes r)u$ (for some v, u). Since $e_m \otimes r$ fits inside the top t by t square, we may suppose $E = EvE(e_m \otimes r)EuE$; by direct finiteness of $E(F \otimes A)E = M_{t+1}A$,

$$E = E(e_m \otimes r)EuEvE = (e_m \otimes r)EuEvE.$$

But $e_m \otimes r$ is annihilated on the left by a subprojection of E (as $e_m \otimes r$ is essentially a t by t matrix), a contradiction.

Applying (a), (b), and the finiteness of the s_m to the equation, we deduce for all m,

$$s_m(x) \geq s_m(x) + s_m(a) + s_m(e_1 \otimes 1);$$

thus $s_m(e_1 \otimes 1) = 0$. This is obviously impossible, and so $n[e_1 \otimes 1] \notin -H$.

By [10; 18.2], the pair $(K_0^*(A), [e_1 \otimes 1])$ admits a state, that is, there exists a group homomorphism

$$f: K_0^*(A), H, [e_1 \otimes 1] \to \mathbf{R}, \mathbf{R}^+, 1.$$

Define functions on $M_nA = M_n \otimes A \subset F \otimes A$; $D_n(x) = f([x])/n$. Then each D_n is a Cuntz's dimension function: Obvious are properties (i), (ii′), (iii), (iv), and (ii) follows from [5; Lemma 3.1].

Hence we have established:

PROPOSITION 1.1. *If A is a stably finite C^* algebra, then A possesses a Cuntz's dimension function extendible to all matrix rings.*

SECTION 2

Here we show that any stably finite C^* algebra admits a homomorphism to a finite AW^* factor.

Let $\{A_i : i \in \mathbf{N}\}$ be a countable collection of C^* algebras. Define the l^∞-product of the $\{A_i\}$,

$$l^\infty(\{A_i\}) = \{a = (a_i) \in \pi A_i : \sup_i \|a_i\| < \infty\}.$$

This is called a C^*-sum in some references (e.g. [1; Section 10]). With the supremum norm, $l^\infty(\{A_i\})$ is a C^* algebra. If $A = A_i$ for all i, we write instead, $l^\infty(A)$. Define the closed ideal of $l^\infty(\{A_i\})$,

$$c_0(A_i) = \{a = (a_i) \in l^\infty(\{A_i\}) : \limsup \|a_i\| = 0\}.$$

Set $\hat{A} = l^\infty(A)/c_0(A)$; with the quotient norm, this is a C^* algebra, and the norm is given by $\|(a_i) + c_0(A)\| = \limsup \|a_i\|$. To avoid disrupting the flow of the paper, I have incorporated some results that are either routine or well known into the Appendix.

A C^* algebra T is said to $\aleph_0$*-injective* if for all a,b in T, $a^*a \leq b^*b$ implies there exists c in T such that $a = cb$.

PROPOSITION 2.1. *If $\{T_i : i \in \mathbf{N}\}$ is a countable collection of C^* algebras, then, $\hat{T} = l^\infty(\{T_i\})/c_0(T_i)$*

(i) *is $\aleph_0$-injective and*

(ii) *satisfies, given a in $\hat{T}$, there exists c so that $\|c\| \leq 1$ and $a = c(a^*a)^{1/2}$.*

Proof. Suppose $a^*a \leq b^*b$, with a, b in $\hat{T}$. By Lemma A-1 (in the Appendix), there exists for each j in $\mathbf{N}$, an element z_j in $\hat{T}$ with $\|z_j\| \leq 1$ and $\|a - z_j b\| \leq 1/j$. Since $\hat{T}$ inherits the quotient norm from $l^\infty(\{T_i\})$, we may lift all of a,b,z_j to sequences $(a_i),(b_i),(z_{j,i})$ respectively so that

$$\sup_i \|a_i\| \leq 2\|a\|, \ \sup_i \|b_i\| \leq 2\|b\|, \qquad \text{and for all } j,\ i \qquad \|z_{ji}\| < 1 + (1/i).$$

For each j in $\mathbf{N}$, there exists an integer $K(j)$ so that for all $i \geq K(j)$,

$$\|a_i - z_{j,i} b_i\| < 2/j;$$

moreover, we may assume that $K(j) < K(j+1)$. For each positive integer $i \geq K(1)$, let $L(i)$ denote the largest integer with $K(L(i)) \geq i$. Then $L(i) \leq L(i+1)$, and $L(i)$ becomes arbitrarily large. Define

$$c_i = \begin{cases} 0 & i < K(1) \\ z_{L(i),i} & i \geq K(1). \end{cases}$$

Then for all $i \geq K(1)$, $\|a_i - c_i b_i\| < 2/L(i)$, and $\|c_i\| < 1 + L(i)^{-1}$. Hence the sequence (c_i) belongs to $l^\infty(T_i)$, and if c is the image of (c_i) in $\hat{T}$, it follows that $a = cb$ and $\|c\| \leq 1$.

Now (ii) follows from the identity $a^*a \leq (a^*a)^{1/2}(a^*a)^{1/2}$.

PROPOSITION 2.2. *Let A be a C^* algebra, I a closed ideal, and $T = A/I$ the quotient algebra.*

(i) *If A is $\aleph_0$-injective, so is T;*

(ii) *If A satisfies both conditions* (i), (ii) *of Proposition 2.1, then so does T.*

Proof. We show, to begin with, that if A is $\aleph_0$-injective (satisfies (ii)), then given x in T, there exists y in T with $(x^*x)^{1/2} = yx$ (with additionally $\|y\| \leq 1$).

Lift x to X in A. Since $((X^*X)^{1/2})^2 \leq X^*X$, there exists Y in A with $YX = (X^*X)^{1/2}$ (if (ii) holds, we may also assume $\|Y\| \leq 1$). Set $y = Y + I$.

Now given x, t in T with $t^*t \leq x^*x$, we find e in T with $t = ex$. Lift t to U in A, and lift $x^*x - t^*t$ to a *positive* element C in A. Then

$$U^*U + C + I = x^*x, \qquad \text{and} \qquad U^*U \leq U^*U + C,$$

so there exists F in A with $U = F(U^*U + C)^{1/2}$. Hence if $f = F + I$, then $t = f(x^*x)^{1/2}$. By the preceding paragraph, there exists y in T with $yx = (x^*x)^{1/2}$, so $t = fyx$.

A C^* algebra A is called an *AW^* algebra,* if every maximal commutative $*$-subalgebra is of the form $C(X)$, X extremally disconnected; equivalently, for all a in A, there exists a projection p such that $pa = a$ and $ya = 0$ implies $yp = 0$, and additionally, suprema of projections exist. If one requires only the single projection p to exist for each element a, then A is called a Rickart C^* algebra. The standard references are [16] and [1].

Two elements of a C^* algebra, x, y are called *orthogonal* if $xy = xy^* = yx = x^*y = 0$, and a set is called orthogonal if all of its elements are mutually orthogonal. We shall usually be restricting the notion of orthogonality to symmetric elements, where it reduces to $xy = 0$.

PROPOSITION 2.3. *Let A be an $\aleph_0$-injective C^* algebra that has no uncountable sets of orthogonal symmetric elements. Then A is an AW^* algebra.*

Proof. We begin by showing A is Rickart, that is, we may find for each x in A, a projection p such that $xp = x$, and that $xy = 0$ implies $py = 0$.

First suppose $x = x^*$. Let $\{y_i\}_I$ be a maximal orthogonal set of selfadjoint elements of A, such that $xy_i = 0$. By hypothesis, we may assume $I = \mathbf{N}$. Set

$$y = \sum \frac{y_i^2}{2^i \|y_i\|^2}.$$

Then $xy = yx = 0$ and $y \geq 0$. If $(x+y)z = 0$, then $(x+y)zz^* = 0$, so multiplying by x, $x^2 zz^* = 0$, whence $xzz^* = 0$, but $yzz^* = 0$ for the same reason, so $\{zz^*, y_i\}$ would be a larger orthogonal set, a contradiction. Hence $x + y$ is not a zero divisor.

Now $xx^* = x^2 \leq x^2 + y^2 = (x+y)(x+y)^*$, so there exists an element p in A with $x = (x+y)p$ ($\aleph_0$-injectivity applied on the other side). Multiplying by x on the left, $x^2 = x^2 p$, but since x is in the closure of the left ideal generated by x^2, we obtain $x = xp$, and thus $yp = 0$. Then $(x+y)(p^2 - p) = 0$, so since $x + y$ is not a zero divisor, $p = p^2$, that is, p is idempotent. Also, from $(x+y)px = x^2$, we have $(x+y)(px - x) = 0$, so $px = x$, and similarly $py = 0$. Hence $xp^* = x$, and $yp^* = 0$, whence $(x+y)(p^* - p) = 0$, and thus $p = p^*$, so p is a projection. Finally, suppose $xz = 0$; then $xpz = 0$, so $(y+x)pz = 0$, and thus $pz = 0$. So p is the desired projection for x.

Let x now be an arbitrary element of A. As x^*x is selfadjoint, there exists a projection p with $px^*x = x^*xp = x^*x$, and $x^*xz = 0$ implying $pz = 0$. Then $(px^* - x^*)(xp - x) = x^*x - x^*x + x^*x - x^*x = 0$, so $xp = x$, completing the proof that A is Rickart.

By [1; p. 45, Lemma 3], countable suprema of projections exist. Since by hypothesis (restricted to projections), the lattice of projections must thus be complete, A is AW^*.

THEOREM 2.4. *If A is a stably finite C^* algebra, there is a (unital) $*$-homomorphism from A to a finite AW^* factor.*

Proof. Form $\hat{A}$ as in Proposition 2.1. Now matrix rings over $\hat{A}$ are naturally isomorphic to $\widehat{(M_n A)}$, so $M_n\hat{A}$ is for example $\aleph_0$-injective, and all considerations of $\hat{A}$ apply to $M_n\hat{A}$. If J denotes the ideal of sequences with all but finitely many of the entries zero, then $c_0(A)$ is the closure of J, and it is a triviality to check that $l^\infty(A)/J$ is stably finite. By Lemma A-2, $\hat{A}$ is stably finite.

Let $\mathscr{C} = \{I, *\text{-ideal of } \hat{A} : \hat{A}/I \text{ is stably finite}\}$. If $K = \cup I_i$ is the union of an ascending chain of elements of $\mathscr{C}$, then $\hat{A}/K$ is stably finite (proof: if $X, Y \in M_n\hat{A}$, and $XY - 1_n \in M_n K$, then $XY - 1_n$ belongs to $M_n I_i$ for some i, whence by the stable finiteness modulo I_i, $YX - 1_n$ lies in $M_n I_i \subset M_n K$). Hence $\mathscr{C}$ possesses maximal elements. Let L be one such. If $\bar{L}$ denotes the closure of L, then $\bar{L}/L$ is the Jacobson radical of A/L; by Lemma A-2 and the maximality of L, $L = \bar{L}$, that is, L is closed.

At this point, we employ Cuntz's dimension functions, viz. section I. Since $B = \hat{A}/L$ is stably finite, B (and each of its matrix rings) has a Cuntz's dimension function (Proposition 1.1), call it D. We now show that the kernel of D is an ideal modulo which B is stably finite, hence must be zero.

Define $\operatorname{Ker} D = \{x \text{ in } B : D(x) = 0\}$. By the remarks before Proposition 1.1, $\operatorname{Ker} D$ is a $*$-ideal, and it is implicit in the definition of $K_0^*(B)$ that for all n,

$$M_n(\operatorname{Ker} D) = \operatorname{Ker} D_n,$$

where D_n is the extension to M_nB. We need only show $B/\operatorname{Ker} D$ is directly finite, since M_nB is an image of $\widehat{M_nA}$ and is thus $\aleph_0$-injective. This will follow from a lemma.

LEMMA 2.5. *Let C be a C^* algebra satisfying the condition: for all x in C, there exists u in C with $\|u\| \leq 1$ so that $x = u(x^*x)^{1/2}$. Let D be a Cuntz's dimension function on C.*

(a) *For x in C, $xx^* - 1$ belongs to* Ker D *implies $x^*x - 1$ belongs to* Ker D;

(b) *Let P be the norm closure of* Ker D *in C. For x in C, if $xx^* - 1$ lies in P, so does $x^*x - 1$.*

Proof. (a) Write

$$x = u(x^*x)^{1/2} \qquad \text{with} \quad \|u\| \leq 1. \tag{1}$$

Then $x^*x = (x^*x)^{1/2}u^*u(x^*x)^{1/2}$ whence $x^*xu^*ux^*x = (x^*x)^2$; hence

$$x^*x(1 - u^*u)x^*x = 0.$$

As $1 \geq u^*u$, $(1 - u^*u)^{1/2}x^*x = 0$; from the functional calculus, we deduce that

$$(1 - u^*u)(x^*x)^{1/2} = 0. \tag{2}$$

Multiplying on the right by u^*, we obtain

$$u^*ux^* = x^*. \tag{3}$$

From (1) and (2),

$$u^*x = u^*u(x^*x)^{1/2} = (x^*x)^{1/2}; \tag{4}$$

then premultiplying by u yields that $uu^*x = x$. Thus

$$(uu^* - 1)xx^* = 0. \tag{5}$$

Now (2) implies,

$$D(1 - u^*u + x^*x) = D(1 - u^*u) + D(x^*x); \tag{6}$$

and from (5),

$$D(1 - uu^* + xx^*) = D(1 - uu^*) + D(xx^*). \tag{7}$$

As $D(1 - xx^*) = 0$, we have $D(xx^*) + D(1 - xx^*) \geq D(1) = 1$; therefore $D(xx^*) = 1$, whence $D(x^*x) = 1$. By (6) and (7), it follows that

$$D(1 - uu^*) = D(1 - u^*u) = 0. \tag{8}$$

Inasmuch as $D(xx^* - 1) = 0$ and $u^*xx^*u = x^*x$ (from (4)),

$$D(x^*x - u^*u) = D(u^*(xx^* - 1)u) \leq D(xx^* - 1) = 0,$$

and thus

$$D(x^*x - u^*u) = 0. \tag{9}$$

By (8), (9),

$$D(x^*x - 1) \leq D(x^*x - u^*u) + D(u^*u - 1) = 0,$$

whence $D(x^*x - 1) = 0$. This concludes the proof of (a). (b). Suppose $xx^* - 1$ belongs to P. Writing $(xx^*)^{1/2} - 1 = (xx^* - 1)\cdot((xx^*)^{1/2} + 1)^{-1}$, we see that $(xx^*)^{1/2} - 1$ lies in P. Let J be the ideal $P/\operatorname{Ker} D$ in $C/\operatorname{Ker} D$; J is the Jacobson radical. Let Y denote the image of $(xx^*)^{1/2}$ in $C/\operatorname{Ker} D$. As $Y + J$ is invertible in C/P, Y is invertible in $C/\operatorname{Ker} D$. Hence there exists z in C with

$$(xx^*)^{1/2}z \equiv 1 \equiv z(xx^*)^{1/2} \qquad \operatorname{mod}(\operatorname{Ker} D).$$

As $\operatorname{Ker} D = (\operatorname{Ker} D)^*$, we may assume that $z = z^*$. Thus $zxx^*z - 1$ lies in $\operatorname{Ker} D$; by part (a), $x^*z^2x - 1$ does as well. Now

$$\begin{aligned}(1 - x^*x)x^*x(x^*z^2)(z^2x) &= (1 - x^*x)x^*(xx^*)^{1/2}(xx^*)^{1/2}zz(z^2x)\\ &\equiv (1 - x^*x)x^*z^2x \qquad \operatorname{mod} P\\ &\equiv 1 - x^*x \qquad\qquad \operatorname{mod} P.\end{aligned}$$

But $(1 - x^*x)x^*x = x^*(1 - xx^*)x$, so the former lies in P, whence $1 - x^*x$ does as well.

Conclusion of Proof of Theorem 2.4. With $C = M_nB$, 2.5 applies; hence with P the closure of $\operatorname{Ker} D$, B/P is stably finite.

Since B was constructed so that it had no proper stably finite images, we must have $P = \{0\}$, and thus $\operatorname{Ker} D = \{0\}$. Then B can have no uncountable sets of orthogonal elements (if $\{y_i\}_I$ were an uncountable orthogonal set of nonzero symmetric elements, then for some n, the subset of I defined by

$$K_n = \left\{ j \in I : D(y_j) > \frac{1}{n} \right\}$$

would be infinite, so if y were a sum of $n + 1$ distinct elements of K_n, $D(y) > 1$, a contradiction). By Propositions 2.1, 2.2, 2.3, B is an AW^* algebra, and by construction, B is finite. Hence $A \to \hat{A} \to B$ yields a unital $*$-homomorphism to a finite AW^* algebra. This completes the proof of the Theorem.

SECTION 3

Propositions 2.1, 2.2, 2.3 admit a surprising consequence. Let $\{A_i : i \in \mathbf{N}\}$ be a countable collection of C^* algebras, and form

$$l^\infty(\{A_i\}) = \{(a_i) : a_i \in A_i, \sup_i \|a_i\| < \infty\}$$

with the sup norm, and let M be its ideal of null sequences. Suppose it is known that all (simple) images of $l^\infty(\{A_i\})$ with M in the kernel are stably finite, and let P be a maximal ideal containing M. Then $B = l^\infty(\{A_i\})/P$ has a faithful dimension function, whence from Proposition 1 (and its remark), and Propositions 2.2 and 2.3, B must be an AW^* algebra and a factor.

One condition on the A_i that will guarantee that all images of $l^\infty(\{A_i\})$ are stably finite, is *unitary 1-stable range* [14]:

A C^* algebra A satisfies *unitary 1-stable range* if for all a,b in A, if $aA + bA = A$ implies that there exists a unitary u such that $a + bu$ is invertible.

The condition $aA + bA = A$ is better expressed: $aa^* + bb^*$ is invertible. Now C^* images of such C^* algebras retain this property [14; 8(c)], as do the l^∞-products, $l^\infty(\{A_i\})$, if each of the A_i has it. Unitary 1-stable range trivially implies stable finiteness (for it implies the usual 1-stable range of algebraic K-theory, and this goes up to matrix rings and implies direct finiteness), so all images are going to be stably finite. Included in the class of C^* algebras with unitary 1-stable range are AF algebras [14; 12] and finite AW^* algebras [14; 3]. Robertson [18] has characterized C^* algebras with unitary 1-stable range as those whose unit group is dense.

If A is a UHF algebra, let $\bar{A}$ denote the II_1 hyperfinite factor generated by the tracial representation of A. Define an ideal I of $l^\infty(\bar{A})$,

$$I = \{c = (c_i) \in l^\infty(\bar{A}) : \limsup \operatorname{tr}(c_i^* c_i)^{1/2} = 0\}.$$

Then an easy consequence of Kaplansky's density theorem yields that the natural embedding, $l^\infty(A)/(I \cap l^\infty(A)) \to l^\infty(\bar{A})/I$ is actually onto. Since all maximal ideals of $l^\infty(A)$ that contain M also contain $I \cap l^\infty(A)$, we deduce that all nontrivial simple images of $l^\infty(A)$ (that is, not arising from a point of $\mathbf{N}$) is equal to a W^* factor constructed as an ultraproduct of W^* algebras.

More is true; viz. 3.1.

The definition, $\aleph_0$-injective, stems from the following considerations. The condition $a^*a \leq b^*b$ implies $a = cb$ is equivalent to:

If $f(b) = a$ extends to a continuous A-module homomorphism $bA \to aA$, then there exists c in A with f given by left multiplication by c.

This equivalence was found by William Paschke around ten years ago, but I misplaced the reference. Of course the closure of a countably generated right ideal is the closure of a principal right ideal, so such C^* algebras are precisely those which satisfy:

all continuous module homomorphisms to A from a countably generated right ideal, are given by left multiplication by an element of A.

For general rings, drop the word, "continuous," and the definition of (right) $\aleph_0$-injective for rings arises. This paper was motivated by the simple observation that if R is any von Neumann regular ring then $\left(\overset{\aleph_0}{\pi} R\right)/(\oplus R)$ is $\aleph_0$-injective, so that any simple stably finite image will be a continuous self-injective regular ring. For regular rings, a condition that will guarantee that all images of πR are stably finite is unit regularity ([11; Section 2] or, for a comprehensive treatment see [10]). Not surprisingly, in view of the earlier remarks, for regular rings, unit-regularity is equivalent to 1-stable range.

The "product by sum" result, in turn was suggested by a result (apparently well known) mentioned by C. U. Jensen of Copenhagen in a lecture series, that if $\{A_i\}$ is a countable collection of abelian groups, then $(\pi A_i)/(\oplus A_i)$ is $\aleph_0$-algebraically compact.

PROPOSITION 3.1. *Let A be an AF algebra. Then any simple image of $l^\infty(A)$ whose kernel contains $c_0(A)$, is a finite W^* factor.*

Proof. Let $S = \mathbf{N}^{\mathbf{N}}$, equipped with the pointwise ordering. Write A as the C^* limit of finite dimensional algebras $\{C_m\}_{m\in\mathbf{N}}$. To each sequence $s = (n(1), n(2), \dots)$ in S assign a W^* algebra of finite type, $B_s = l^\infty(C_{n(1)}, C_{n(2)}, \dots)$. With the obvious maps, $l^\infty(A)$ is the C^* direct limit, over the directed set S, of finite W^* algebras $\{B_s\}_{s\in S}$.

Let M be a maximal two-sided ideal of $l^\infty(A)$ containing $c_0(A)$. Then $R = l^\infty(A)/M$ is a finite AW^* factor, as follows from the discussion immediately preceding. On the other hand, R is the C^* limit of quotients of finite W^* algebras,

$$\left\{\frac{B_s}{(B_s \cap M)}\right\}.$$

Any image of a finite W^* algebra admits a homomorphism to a finite W^* factor (e.g. [1; p. 208, Example 2]), hence possesses a (not necessarily faithful) trace. As the trace space of R is the inverse limit over a directed set of nonempty compact sets (the trace spaces of the $B_s/(B_s \cap M)$), it is nonempty, so R has a trace. A finite AW^* algebra factor with a trace is W^* [9], completing the proof.

Minor modifications of the proof can be made in case either A is nonseparable, or the l^∞-product is taken over a larger index set than $\mathbf{N}$.

APPENDIX: TECHNICALITIES

LEMMA A-1. *Let A be any C^* algebra, and let a, b be elements of A such that $a^*a \le b^*b$. Then there exist z_j in A with $\|z_j\| \le 1$ and $\lim z_j b = a$ (in the norm).*

Proof. (Essentially due to Joachim Cuntz). This is divided into three steps. Set $t = (a^*a)^{1/2}$, $s = (b^*b)^{1/2}$. We may obviously assume 0 is in Spec b^*b.

(i) There exist w_j in A with $\|w_j\| \le 1$ and $a = \lim w_j t$.

Regard A as a subalgebra of $B(H)$ for some Hilbert space H. If $a = ut$ is the polar decomposition of a (in $B(H)$), and f is a continuous real-valued function on $\mathbf{R}$ with $f(0) = 0$, then $uf(t)$ lies in A ([4; 1.3]). In particular, if f_n is of the form,

$$f_n(x) = \begin{cases} 0 & x \leq 0 \\ nx & 0 \leq x \leq 1/n \\ 1 & x \geq 1/n \end{cases}$$

then $f_n(t)t$ converges to t (as 0 lies in its spectrum); thus $uf_n(t)t$ converges to a; set $w_j = uf_j(t)$.

(ii) There exist x_j in A with $\|x_j\| \leq 1$ and $t = \lim x_j s$.

This follows the idea in [6; I.7.2]. Set $Y_n = s^2(1/n + s^2)^{-1}$; then

$$\|Y_n - 1\| = \left\| \left(\frac{1}{n}\right)\left(\frac{1}{n} + s^2\right)^{-1} \right\| \leq 1.$$

Now

$$\begin{aligned}[(Y_n - 1)t]\,[(Y_n - 1)t]^* &= (Y_n - 1)t^2(Y_n - 1) \leq (Y_n - 1)s^2(Y_n - 1) \\ &= \left(\frac{s^2}{n^2}\right)\left(\frac{1}{n} + s^2\right)^{-2}.\end{aligned}$$

Since the real valued function $f(v) = v/(1/n + v)^2$ assumes a maximum for positive v of $n/4$, we deduce that $\|(Y_n - 1)t\|^2 \leq 4/n$; set $x_j = Y_j - 1$.

(iii) There exist y_j with $\|y_j\| \leq 1$ and $s = \lim y_j b$.

If we write $b = ws$ in its polar decomposition in $B(H)$ (as in (i)), as is well known, $w^*b = s$. As in (i), we find continuous functions $\{f_n\}$ with

$$f_n(0) = 0, \qquad 0 \leq f_n \leq 1, \qquad \text{and} \qquad wf_n(s)s$$

converging to b. Now $f_n(s)w^* = (wf_n(s))^*$ belongs to A ([4; 1.3]), and $f_n(s)w^*b = f_n(s)s$ which converges to s. Set $y_j = f_j(s)w^*$.

Finally, set $z_j = w_j x_j y_j$.

A-2. If T is any ring, and J is its Jacobson radical, then direct (stable) finiteness of T implies that of T/J.

Proof. Since $J(M_n T) = M_n J$, we need only show direct finiteness holds, for $n = 1$. If $xy - 1$ belongs to J, then xy belongs to $1 + J$ whence xy is invertible in T. There thus exists z in T with $xyz = 1 = zxy$. By direct finiteness of T, y is invertible, so is invertible mod J, and thus $yx - 1$ belongs to J.

REFERENCES

1. S. K. Berberian, *Baer *-rings*, Springer, New York, 1972.

2. M. Breuer, *A homotopy-theoretic proof of the additivity of the trace.* Rocky Mountain J. Math. 10 (1980), 185–198.

3. M. Breuer and P. de la Harpe, *Charactérisation des AW^*-algèbres semi-finies qui sont des algèbres de von Neumann.* C.R. Acad. Sci. Paris Sér. A-B 289 (1979), no. 1, A21–A23.

4. J. Cuntz, *The structure of multiplication and addition in simple C^* algebras.* Math. Scand. 40 (1977), no. 2, 215–233.

5. ———, *Dimension functions on simple C^* algebras.* Math. Ann. 233 (1978), no. 2, 145–153.

6. J. Dixmier, *Les C^*-algèbres et leurs répresentations,* 2ème edition, Gauthier, Paris, 1969.

7. G. Elliott and D. Handelman, *Extending traces to factorial traces,* preprint.

8. E. G. Effros and J. Rosenberg, *C^*-algebras with approximately inner flip.* Pacific J. Math. 77 (1978), no. 2, 417–443.

9. J. Feldman, *Embedding of AW^* algebras.* Duke Math. J. 23 (1956), 303–307.

10. K. R. Goodearl, *Von Neumann regular rings,* Pitman, London, 1979.

11. K. R. Goodearl and D. E. Handelman, *Rank functions and K_0 of regular rings.* J. Pure Appl. Algebra 7 (1976), no. 2, 195–216.

12. D. Handelman, *Finite Rickart C^* algebras and their properties.* Studies in Analysis, pp. 171–196, Academic Press, New York, 1979.

13. ———, *Rings with involution as partially ordered abelian groups.* Rocky Mountain J. Math., to appear.

14. ———, *Stable range in AW^* algebras.* Proc. Amer. Math. Soc. 76 (1979), no. 2, 241–249.

15. I. Kaplansky, *Any orthocomplemented complete modular lattice is a continuous geometry.* Ann. of Math. (2) 61 (1955), 524–541.

16. ———, *Rings of operators,* Benjamin, New York, 1965.

17. N. Prijatelj and I. Vidav, *On special $*$-regular rings.* Michigan Math. J. 18 (1971), 213–221.

18. A. Guyan Robertson, *Stable range in C^* algebras,* preprint.

19. I. Vidav, *On some $*$-regular rings.* Acad. Serbe Sci. Publ. Inst. Math. 13 (1959), 73–80.

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