EIGENVALUES EMBEDDED IN THE CONTINUUM FOR NEGATIVELY CURVED MANIFOLDS

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1. INTRODUCTION

Suppose that M is a complete simply connected negatively curved surface and Δ is the Laplacian of M. If M is the Poincaré upper half plane with constant curvature -1, then the spectrum of $-\Delta$ is purely continuous and consists of the half line $[1/4,\infty)$.

Denote K to be the Gauss curvature of M. McKean [10] showed that if $K \leqslant -1$ then the spectrum of $-\Delta$ is bounded below by $1/4$. By a more detailed argument, Pinsky [14] proved that if $K \leqslant -1$ then $1/4$ does not appear in the point spectrum of $-\Delta$.

Since M may have continuous spectrum starting at $1/4$, new proofs are required to prevent M from having eigenvalues greater than $1/4$. Let $ds^2 = dr^2 + g^2(r,\theta)\,d\theta^2$ be the metric in terms of geodesic polar coordinates (r,θ) about some $p \in M$. If $g = g(r)$ is independent of θ, then Pinsky [14] gave decay conditions on $K(r) + 1$, as $r \to \infty$, which insure that M has no eigenvalues greater than $1/4$. Unfortunately, his method does not generalize in a straightforward way to metrics which are not rotation invariant, at least for r suitably large.

In this paper we give decay conditions on $K(r,\theta) + 1$, K_θ, $K_{\theta\theta}$, as $r \to \infty$, which imply that M has no eigenvalues greater than $1/4$. We then easily generalize our results to dimensions $n \geqslant 2$. By adding the hypothesis $K \leqslant -1$, one obtains a criterion for a negatively curved manifold to have purely continuous spectrum consisting of the half line $[(n-1)^2/4,\infty)$.

Our method is a modification of Kato's solution [9] given the analogous problem for the Schrödinger operator on $\mathbf{R}^n$. The idea is to regard $L^2(\mathbf{R}^n) = L^2(\mathbf{R}) \times L^2(S^{n-1})$ and to systematically exploit differential inequalities for $L^2(S^{n-1})$ valued functions on $\mathbf{R}$.

The author thanks Professor Pinsky for sending us a copy of his paper [14] and for informing us [12] of the open problem which arose in that work. This provided the starting point for the present paper.

2. SURFACES HAVING ASYMPTOTICALLY CONSTANT CURVATURE

Let M be a complete simply connected negatively curved surface. Then for each $p \in M$ the exponential map exp: $T_pM \to M$ is a diffeomorphism [3, p. 184].

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Thus we may endow M with a global system of geodesic polar coordinates where the metric is given by $ds^2 = dr^2 + g^2(r,\theta)\,d\theta^2$. If K denotes the Gaussian curvature, then g satisfies the ordinary differential equation $\partial^2 g/\partial r^2 + Kg = 0$ [11, p. 278] along the geodesics emanating from p.

Using the standard formula valid for any local coordinate system [8, p. 398], the Laplacian may be computed as

$$\Delta\psi = g^{-1}\frac{\partial}{\partial r}\left(g\frac{\partial\psi}{\partial r}\right) + g^{-1}\frac{\partial}{\partial\theta}\left(g^{-1}\frac{\partial\psi}{\partial\theta}\right)$$

Define $w = g^{1/2}\psi$ and $H = g^{1/2}(-\Delta - 1/4)g^{-1/2}$. Then one has

$$Hw = \frac{-\partial^2 w}{\partial r^2} - g^{-2}\frac{\partial^2 w}{\partial\theta^2} + \Lambda\frac{\partial w}{\partial\theta} + Vw$$

where

$$\Lambda = 2g^{-3}\frac{\partial g}{\partial\theta}$$

$$V = \frac{-5}{4}g^{-4}\left(\frac{\partial g}{\partial\theta}\right)^2 + \frac{1}{2}g^{-3}\frac{\partial^2 g}{\partial\theta^2} + \frac{1}{2}g^{-1}\frac{\partial^2 g}{\partial r^2} - \frac{1}{4}g^{-2}\left(\frac{\partial g}{\partial r}\right)^2 - 1/4$$

Now suppose that $K(r,\theta) \leq 0$ satisfies the decay conditions

$$\text{(2.1)}\qquad \begin{aligned} &\text{i)} \lim_{r\to\infty} r|K+1| = 0\\ &\text{ii)} \int r|K+1|\,dr < d_1\\ &\text{iii)} \int e^{2r}|K_\theta|\,dr < d_2\\ &\text{iv)} \int e^{2r}|K_{\theta\theta}|\,dr < d_3 \end{aligned}$$

where d_1, d_2, d_3 are constants independent of θ.

One has

LEMMA 2.2. *If the decay conditions* (2.1) *are satisfied then* $\Lambda = 0(e^{-2r})$ *and* $V = o(r^{-1})$ *as* $r \to \infty$.

Proof. This follows from Propositions A.6, A.7 of the appendix.

Suppose that ψ is an eigenfunction of $-\Delta$ with eigenvalue $E + 1/4$ for $E > 0$. Then

$$\text{(2.3)}\qquad \begin{gathered} \int \|w\|^2\,dr < \infty\\ Hw = Ew \end{gathered}$$

The unique continuation theorem of Aronzajn [1, p. 235] implies that w does not vanish identically on any open set.

LEMMA 2.4. *If the decay conditions* (2.1) *are satisfied then* $d/dr\,(r\,\mathcal{G}(r)) \geqslant 0$, $r > R_0$ *where one defines* $\mathcal{G}(r) = (w', w') - (g^{-2}w_\theta, w_\theta) + E(w, w)$ *with* $w' = \partial w/\partial r$ *and where* (,) *is the standard global inner produce on* $L^2(S^1)$.

Proof. Using (2.3) and computing one obtains

$$\frac{d}{dr}(r\,\mathcal{G}(r)) = (w', w') + E(w, w) - \left(\frac{\partial}{\partial r}(rg^{-2})\,w_\theta, w_\theta\right) + 2r(Vw, w)$$

The lemma now follows from Lemma 2.2 and Proposition A.6.

Since ψ is in the domain of $-\Delta$, one has $d\psi \in L^2$. This forces $\mathcal{G}(r)$ to be integrable. As a consequence, using Lemma 2.4, we see that $\mathcal{G}(r) \leqslant 0$ for $r > R_0$.

Define $w_m = r^m w$ for any $m \geqslant 0$. Then w_m satisfies the ordinary differential equation:

$$w_m'' - 2mr^{-1}w_m' + g^{-2}w_{m,\theta\theta} + m(m+1)r^{-2}w_m - \Lambda w_{m,\theta} + (E - V)w_m = 0 \tag{2.5}$$

One has

LEMMA 2.5. *Define*

$$\mathcal{L}(m,r) = (w_m', w_m') + (E - ER_0r^{-1} + m(m+1)r^{-2})(w_m, w_m) - (g^{-2}w_{m,\theta}, w_{m,\theta}).$$

If the decay conditions (2.1) *are satisfied then* $d/dr(r^2\mathcal{L}(m,r)) > 0$ *for* $m > m_0$, $r > R_1 > R_0$.

Proof. One computes, employing (2.5):

$$\begin{aligned}\frac{d}{dr}(r^2\mathcal{L}(m,r)) &= 2r(2m+1)(w_m', w_m') + 2rE\left(1 - \frac{R_0}{2r}\right)(w_m, w_m)\\ &\quad - \left(\frac{\partial}{\partial r}(r^2g^{-2})\,w_{m,\theta}, w_{m,\theta}\right) + 2r^2\left((V - ER_0r^{-1})\,w_m', w_m\right)\end{aligned}$$

The lemma now follows from Lemma 2.1 and Proposition A.6.

By examining the formula defining $\mathcal{L}(m,r)$ we find that for some $R_2 > R_1$ there exists an $m_1 > m_0$ so that $\mathcal{L}(m_1, R_2) > 0$. By Lemma 2.6, one has $\mathcal{L}(m_1, r) > 0$ for all $r \geqslant R_2$.

Since $\int \|w\|^2 dr < \infty$, we may choose $R_3 > R_2$ so that $(w', w)(R_3) < 0$ and $-ER_0r^{-1} + m_1(2m_1 + 1)r^{-2} < 0$ if $r \geqslant R_3$.

Then

$$R_3^{-2m_1}\mathscr{L}(m_1,R_3) = \left\| w' + \frac{m}{R_3} w \right\|^2 + (E - ER_0R_3^{-1} + m(m+1)R_3^{-2})\|w\|^2 - (g^{-2}w_\theta, w_\theta)$$

and so

$$R_3^{-2m_1}\mathscr{L}(m_1,R_3) \leqslant \|w'\|^2 + E\|w\|^2 - (g^{-2}w_\theta, w_\theta)$$

$$R_3^{-2m_1}\mathscr{L}(m_1,R_3) \leqslant \mathscr{G}(R_3) \leqslant 0$$

a contradiction.

This yields,

THEOREM 2.7. *Let M be a complete simply connected negatively curved surface. Suppose that the Gauss curvature of M satisfies the decay conditions* (2.1). *Then $-\Delta$ has no eigenvalue $\lambda > 1/4$.*

3. A COUNTEREXAMPLE WITH POSITIVE CURVATURE

We observe in this section that the decay conditions (2.1) are not sufficient in themselves to guarantee the conclusion of Theorem 2.7. In fact, there are surfaces M with K identically -1 outside a compact set for which $-\Delta$ has arbitrarily large eigenvalues. Thus the hypothesis $K \leqslant 0$ for Theorem 2.7 is used there in an essential way.

Let $g(r)$ be a function satisfying $g(0) = 0$, $g'(0) = 1$, and $g(r) = e^{-r}$ for $r \geqslant 1$. Denote M to be the Riemannian manifold diffeomorphic to $\mathbf{R}^2$ with metric $ds^2 = dr^2 + g^2(r)\,d\theta^2$ in geodesic polar coordinates about some $p \in M$. Then one has $K = -g''/g = -1$ for $r \geqslant 1$.

Since g is independent of θ, the Laplacian Δ of M commutes with rotations. Each $\phi \in C_0^\infty(M)$ may be expanded as $\phi(r,w) = \sum_n \phi_n(r)e^{in\theta}$ where n is an integer. The standard local formula for the Laplacian [8, p. 398] implies that $\Delta\phi = \sum (\Delta_n\phi_n)e^{in\theta}$ where

$$\Delta_n\phi_n = \frac{d^2\phi_n}{dr^2} + \frac{g'}{g}\frac{d\phi_n}{dr} - n^{-2}g^{-2}\phi_n.$$

This decomposes the operator Δ as the direct sum of the operators Δ_n acting on $L^2(\mathbf{R}^+, g(r)\,dr)$.

Now Δ_n is unitarily equivalent to the operator $D_n = g^{1/2}\Delta_n g^{-1/2}$ acting on $L^2(\mathbf{R}^+, dr)$. A computation shows that

$$D_n\psi = \frac{d^2\psi}{dr^2} + [\omega(r) - n^2g^{-2}(r)]\,\psi$$

where $\omega(r) = -1/2h''(r) - 1/4(h'(r))^2$ and $g(r) = \exp(h(r))$. Thus if $r \geq 1$, one has

$$-D_n \psi = \frac{-d^2\psi}{dr^2} + \left[\frac{1}{4} + n^2 e^{2r}\right]\psi$$

A theorem of Titchmarsh-Weyl [16] states that operators of the form $-d^2/dr^2 + q(r)$ with $q(r) \to \infty$ as $r \to \infty$ have no essential spectrum. Thus if $n \neq 0$, $-D_n$ has pure point spectrum consisting of eigenvalues $\lambda_m(n)$ with $\lambda_m(n) \uparrow \infty$ as $m \to \infty$.

Since Δ is unitarily equivalent to the direct sum of the operators Δ_n, $\Delta = \sum_n \Delta_n$, and each Δ_n is unitarily equivalent to D_n, we see that $-\Delta$ has arbitrarily large eigenvalues.

On the other hand, $-D_0$ has no eigenvalues greater than $1/4$, since $-D_0 = -d^2/dr^2 + 1/4$ outside the compact set $0 \leq r \leq 1$. The decomposition principle [6] implies that the essential spectrum of $-D_0$ is the interval $[1/4, \infty)$. Therefore, the continuous spectrum of $-D_0$ is also the interval $[1/4, \infty)$.

In conclusion we may state that $-\Delta$ has continuous spectrum consisting of the interval $[1/4, \infty)$. Moreover, $-\Delta$ has arbitrarily large eigenvalues.

4. GENERALIZATION TO HIGHER DIMENSIONS

The above results generalize in a straightforward way to manifolds M of dimension $n \geq 2$. The additional details required are outlined below. One need only utilize the technique of Jacobi fields and suitable theorems for systems of ordinary differential equations.

Let M^n be a complete simply connected negatively curved manifold. Given $p \in M$, the exponential map $\exp: T_pM \to M$ is a diffeomorphism [15, p. 330]. For each $\omega \in S^{n-1}$, denote $\gamma(\omega, t)$ to be the unit speed geodesic starting at p with direction ω.

A vector field y along γ is called a Jacobi field [15, p. 307] if

$$y'' + R(y, V)V = 0 \tag{4.1}$$

where y'' is the second covariant derivative, R is the curvature tensor, and V is the tangent vector of γ. Suppose that y is perpendicular to γ, which is the case of interest. One may choose an orthonormal frame $e_1, \ldots, e_{n-1}, V$ of vectors parallel along γ. Setting $y = \sum \alpha_i e_i$, we find that Jacobi's equation (4.1) is equivalent to the linear system of ordinary differential equations

$$\alpha_i'' + \sum a_{ij}\alpha_j = 0 \tag{4.2}$$

where $a_{ij}(t) = R(e_j, V, V, e_i)$.

Let $\theta \in S^{n-1}$ and denote $K(\omega, \theta, t)$ to be the sectional curvature of the plane obtained by parallel translation of the (ω, θ) plane along $\gamma(\omega, t)$. Suppose that K is asymptotically constant in the sense that $\int_0^\infty t|K+1|\,dt < d_1$ where d_1 is a constant. By applying Theorem 13.1 of [7, p. 305] to the system (4.2), we deduce that any Jacobi field has the asymptotic behavior:

$$(4.3) \qquad \begin{aligned} y(t) &= e^t\,(y_\infty + o\,(t^{-1})) \\ y'(t) &= e^t\,(y_\infty + o\,(t^{-1})) \end{aligned}$$

as $t \to \infty$, for some constant vector y_∞. We will be interested in Jacobi fields satisfying the initial conditions $y(0) = 0$, $y'(0) \neq 0$. Since M has nonpositive curvature, it follows from the Rauch comparison theorem [15, p. 348] and [7, p. 305] that $y_\infty \neq 0$.

Let D denote the covariant derivative on S^{n-1}, with respect to its standard metric. Suppose that $\int_0^\infty \|D_\omega K\| e^{2t}\,dt < d_2$ and $\int_0^\infty \|D_\omega^2 K\| e^{2t}\,dt < d_3$. Applying variation of parameters, [7, p. 48] we deduce that $\|D_\omega y\| = 0(\|y\|)$ and $\|D_\omega^2 y\| = 0(\|y\|)$ as $t \to \infty$.

Suppose that $\Gamma(\omega, t) = t\omega$ is the straight line in T_pM which maps to $\gamma(\omega, t)$ under $\exp\colon T_pM \to M$. If $E_1, \ldots, E_{n-1}, \omega$ is an orthonormal frame along Γ in T_pM, let y_i be the Jacobi field satisfying $y_i(0) = 0, y_i'(0) = E_i$. It follows from the variational characterization of Jacobi fields [15, p. 314] that $y_i(t) = \exp(tE_i)$. This allows one to deduce, from (4.3), the asymptotic behavior of the components of the metric tensor in a geodesic spherical coordinate system associated to $\exp\colon T_pM \to M$.

We may now state:

THEOREM 4.4. *Let M be a simply connected complete Riemannian manifold having nonpositive curvature. Fix $p \in M$, and let r denote the geodesic distance from p. Suppose that along geodesics emanating from p, the curvature of M satisfies the decay conditions*:

i) $\int_0^\infty r\|K + 1\|\,dr < d_1$

ii) $\int_0^\infty \|D_\omega K\| e^{2r}\,dr < d_2$

iii) $\int_0^\infty \|D_\omega^2 K\| e^{2r}\,dr < d_3$

iv) $\lim_{r \to \infty} r\,\|K+1\| = 0$

for some constant d_1, d_2, d_3. Then the Laplacian $-\Delta$ of M has no eigenvalue $\lambda > (n-1)^2/4$.

Proof. Let $g = \sqrt{\det(g_{ij})}$ be the volume element for the spherical normal coordinate system given by $\exp\colon T_pM \to M$. If ψ is an eigenfunction of $-\Delta$, then set $w = g^{1/2}\psi$.

Define $\mathscr{G}(r) = (w', w') - (\langle dw, dw\rangle, 1) + E(w, w)$, where $\langle\ ,\ \rangle$ is the Riemannian metric of M and $(\ ,\ )$ is the standard global inner product on $L^2(S^{n-1})$. Now proceed by analogy with the differential inequalities arguments in Section 2. Our discussion above provides the required decay estimates on the metric.

5. LAPLACIANS HAVING PURELY CONTINUOUS SPECTRUM

By combining Theorem 4.4 with known results, one obtains conditions for a negatively curved manifold to have purely continuous spectrum. Related results were obtained in [14, p. 11] for rotation invariant metrics.

We begin by stating:

PROPOSITION 5.1 *Let M be a complete simply connected negatively curved manifold. Suppose that all sectional curvatures of M are bounded above by -1. Then the spectrum of $-\Delta$ is bounded below by $(n - 1)^2/4$. Moreover, $(n - 1)^2/4$ is not contained in the point spectrum of $-\Delta$.*

Proof. The fact that the spectrum of $-\Delta$ is bounded below by $(n - 1)^2/4$ is due to McKean [10]. Alternative proofs of this result were given in [13, p. 88] and [17, p. 498].

Let g be the volume element for geodesic spherical coordinates about some $p \in M$. According to [14, p. 4] any eigenvalue λ of $-\Delta$ must satisfy $\lambda > 1/4\mu^2$ where $\mu = \inf\limits_{r>0} g^{-1}\dfrac{\partial g}{\partial r}$. In fact, it was assumed in [14] that $n = 2$, but the same argument holds if $n \geq 2$. Since $K \leq -1$, a standard comparison theorem [3, p. 253] yields $\mu \geq n - 1$. Thus $1/4(n - 1)^2$ cannot occur in the point spectrum of $-\Delta$.

One has:

THEOREM 5.2. *Let M be a complete simply connected negatively curved manifold whose sectional curvatures are bounded above by -1. Suppose that M satisfies the decay conditions* i)-iv) *of Theorem 4.4. Then $-\Delta$ has purely continuous spectrum consisting of the half line* $[(n - 1)^2/4, \infty)$.

Proof. By Theorem 4.4 and Proposition 5.1, M has purely, continuous spectrum. It was shown in [4] that if $K \to -1$ at infinity then the essential spectrum of M is the half line $[(n - 1)^2/4, \infty)$. Since we have the stronger condition (iv) of Theorem 4.4, our conclusion follows.

APPENDIX. SOME LEMMAS FROM ORDINARY DIFFERENTIAL EQUATIONS

In this section we collect some facts concerning the second order equation:

$$g'' + Kg = 0 \tag{A.1}$$

PROPOSITION A.2. *Let $g(r)$ satisfy* (A.1) *for $0 < r < \infty$ with initial conditions $g(0) = 0$, $g'(0) = 1$. If $K \leq 0$, then $g'(r) > 0$, $g(r) > 0$ for all $r > 0$.*

Proof. Since $g(r) = \int_0^r g'(r)dr$, it is enough to show that $g'(r) > 0$. If not, let r_0 be the smallest r with $g'(r) = 0$. Then $g(r) \geq 0$ for all $r < r_0$. The formula, $g'(r_0) = 1 - \int_0^{r_0} Kgdr \leq 1$, yields a contradiction.

PROPOSITION A.3. *Suppose that $\int_0^\infty |K+1|dr < \infty$. Then the ordinary differential equation* (A.1) *has solutions g_0, g_1 which satisfy.*

(A.4)
$$g_0 \sim -g_0' \sim e^{-r}$$
$$g_1 \sim g_1' \sim e^r$$

as $r \to \infty$.

Proof. [7, p. 381]

PROPOSITION A.5. *Let $K(r) \leq 0$ and $\int_0^\infty |K+1|dr < \infty$. Denote $g(r)$ to be the unique solution of* (A.1) *satisfying $g(0) = 0$, $g'(0) = 1$. Then $g \sim g' \sim c_1 e^r$ for some $c_1 > 0$.*

Proof. By Proposition A.3 we may write $g = c_0 g_0 + c_1 g_1$ where g_0, g_1 have the asymptotic behavior (A.4). Since $K \leq 0$, Proposition A.2 shows that $c_1 > 0$.

If $K(r,\theta)$ is a family of K's parameterized by $0 \leq \theta \leq 2\pi$ then the constant $c_1(\theta)$ in Proposition A.5 has good dependence on θ. This follows by examining the estimates in [7, p. 381].

PROPOSITION A.6. *Let $K(r,\theta)$, $0 \leq \theta \leq 2\pi$ satisfy the conditions:*

i) $K(r,\theta) \leq 0$ *for all* r

ii) $\int_0^\infty |K+1|dr < d_1$

iii) $\int_0^\infty |K_\theta|e^{2r}\, dr < d_2$

iv) $\int_0^\infty |K_{\theta\theta}|e^{2r}\, dr < d_3$

where d_1, d_2, d_3 are constants independent of θ.

If $g(r,\theta)$ are the corresponding solutions of (A.1), *then as $r \to \infty$:*

i) $g(r,\theta) \sim c_1(\theta) e^r \qquad c_1(\theta) > 0$

ii) $g_\theta\, (r,\theta) = 0(g)$

iii) $g_{\theta\theta}\, (r,\theta = 0(g)$

iv) $g_r(r,\theta) \sim c_1(\theta)e^r$

Proof. Since $g'' + Kg = 0$, one has $g''_\theta + Kg_\theta + K_\theta g = 0$,

$$g''_{\theta\theta} + Kg_{\theta\theta} + 2K_\theta g_\theta + K_{\theta\theta} g = 0.$$

The result now follows from Proposition A.5 and variation of parameters [7, p. 327].

By imposing stronger decay conditions on $K(r,\theta)$, one obtains better control of the ratio $g^{-1}\, \partial g/\partial r$:

PROPOSITION A.7. *Suppose that* $K(r,\theta) \leq 0$ *satisfies the integral condition* $\int_0^\infty r|K+1|dr < \infty$, *then* $g^{-1}\,\partial g/\partial r = 1 + o(1/r)$ *as* $r \to \infty$.

Proof. [7, p. 380].

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