RELATION MODULES FOR EXTENSIONS OF NILPOTENT GROUPS

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1. INTRODUCTION

Let G be a group and $G = \{x_1,\ldots,x_d; r_1,\ldots,r_s\}$ a finite presentation of G, i.e., $x_1, \ldots, x_d$ generates a free group F of rank $d(F)$ and $r_1, \ldots, r_s$ are elements of F such that $G \cong F/R$, where R is the normal closure of $r_1, \ldots, r_s$ in F. In many situations, it is desirable to know the minimal number of generators, $d_F(R)$, of R as a normal subgroup of F. For example, if G is the fundamental group of a closed 3-manifold, then the maximum of the numbers $d(F) - d_F(R)$ over all finite representations must be zero [3]. Now it is notoriously difficult to determine $d_F(R)$ in general. One does not even know, if in the case G is finite, whether the number, $dF - d_F(R)$, is an invariant for G. $dF - d_F(R)$ is known not to be an invariant of G if G is infinite. Dunwoody and Pietrowski [2] have shown that the trefoil knot group $= \{a,b; a^2 = b^3\}$ has a two generator presentation needing more than one relation.

Now if one has an exact sequence of groups

$$1 \to N \to C \xrightarrow{\pi} Q \to 1,$$

then $\bar{N} = N/(N,N)$ becomes a Q-module by conjugation, $q \cdot n = \overline{cnc^{-1}}$, where $\pi c = q$. If the above sequence arises from a presentation of G, then the G-module $\bar{R}$ is called a relation module for G. Notice that any generators of N as a normal subgroup of C map to generators of $\bar{N}$ as a Q-module, i.e., $d_C(N) \geqq d_Q(\bar{N})$.

For a relation module, Gruenberg [4] has shown that if G is finite, the number $d_G(\bar{R}) - d(F)$ is an invariant for G. Moreover no examples of finite groups are known where $d_F(R) > d_G(\bar{R})$.

It is the purpose of this paper to compute the number $d_G(\bar{R})$ when G is an extension, $1 \to N \to G \to Q \to 1$, of N by Q, where N and Q are finite nilpotent groups and the orders of N and Q are relatively prime. In all that follows we shall be constantly concerned with extensions where the orders of N and Q are relatively prime. We shall refer to such an extension as a relatively prime extension. Note that such an extension is automatically split although we shall not explicitly use that fact. In the course of our investigations we shall also compute $d_G(\mathfrak{g})$, the minimal number of generators of the augmentation ideal of $\mathbf{Z}G$.

In order to state the main result we need some notation. Let $\mathbf{F}_pQ$ be semisimple and M an irreducible $\mathbf{F}_pQ$-module ($\mathbf{F}_p$ = field of p elements). Let τ_M = number of occurrences of M in $\mathbf{F}_pQ$ and if A is any $\mathbf{F}_pQ$-module, let $\tau_M(A)$ = number of

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occurrences of M in A. Let $[\![x]\!]$ be the smallest integer $\geqq x$.

If A is an F_pQ-module and s is an integer, let $\beta_s(A) = 0$ if for every irreducible module $M \neq \mathsf{F}_p$,

$$\tau_{\mathsf{F}_p}(A) \geqslant \left[\!\!\left[\frac{\tau_M(A)}{\tau_M}\right]\!\!\right] + \left(-\frac{1}{2}\right)^{s+1}.$$

Let $\beta_s(A) = (-1)^{s+1}$ otherwise. That is, if s is odd, $\beta_s(A) = 0$ if for all irreducible $M \neq \mathsf{F}_p$, $\tau_{\mathsf{F}_p}(A) > [\![\tau_M(A)/\tau_M]\!]$ and $\beta_s(A) = 1$ otherwise; if s is even, $\beta_s(A) = 0$ if for all irreducible $M \neq \mathsf{F}_p$, $\tau_{\mathsf{F}_p}(A) \geqslant [\![\tau_M(A)/\tau_M]\!]$ and $\beta_s(A) = -1$ otherwise. Define the numbers

$$\alpha_s = \max_{p,q} \{d_Q(\mathsf{F}_pH_sN) + \beta_s(\mathsf{F}_pH_sN),\ d(H_s(Q_q))\}$$

where Q_q is a Sylow q-subgroup of Q and $\mathsf{F}_pB = \mathsf{F}_p \otimes B$. $d(B)$ = the minimal number of generators of the abelian group B.

THEOREM. *If* $1 \to N \to G \to Q \to 1$ *is a relatively prime extension with* N, Q *nilpotent, then*

(i) $d_G(\mathfrak{g}) = \alpha_1$

(ii) *if* $1 \to R \to F \to G \to 1$ *is any finite presentation of* G, $d_G(\bar{R}) - d(F) = \alpha_2$.

Remarks. (a) This generalizes the corresponding result for nilpotent groups proven by Wamsley ([6], [11]).

(b) Included in the above case are all p-hyperelementary groups

$$1 \to Z/n \to G \to G_p \to 1,$$

G_p a p-group and $(p,n) = 1$. It was while wrestling with the problem of $d_G(\bar{R})$ for a particular 2-hyperelementary group that this paper came into being. For p hyperelementary groups (in fact for any N with $d_Q(\mathsf{F}_pH_s(N)) \leqq 1$ for $s = 1,2$), we easily see $\alpha_1 = d(G_p)$ and $\alpha_2 = dH_2(G_p)$. Because it comes with little extra work, we will give an independent proof of the p-hyperelementary case using particular properties of the cyclic subgroup.

I would like to thank the referee for pointing out the following two facts.

(c) For the groups considered above, $d_G(\mathfrak{g}) = d(G)$. See K. W. Gruenberg [7].

(d) A related formula for $d(G)$ if G contains a normal nilpotent subgroup appears in a paper of K. W. Gruenberg and K. W. Roggenkamp [8].

2. PRELIMINARY RESULTS

The proofs of the following results are easily accessible in Gruenberg [6, Chapter 7]. All groups will be finite.

Definition. A $\mathbf{Z}G$-lattice A is called a Swan module if

$$d_G(A) = \max_{p \in \pi G} d_G(A/pA)$$

where πG = set of primes dividing the order of G.

Using a result of Swan [10], the following result was proved by Gruenberg [4].

THEOREM 1. *All relation modules and all augmentation ideals are Swan modules.*

From this one sees that it is sufficient to compute the number of generators of a relation module or augmentation ideal locally. Let $p \in \pi G$ and let M be an irreducible $\mathsf{F}_p G$-module. Set $\rho_M = 0$ if $M = \mathsf{F}_p$ and $\rho_M = 1$ if $M \neq \mathsf{F}_p$. Recall $[\![x]\!]$ is the smallest integer greater than or equal to x.

THEOREM 2. *Let $p \in \pi G$. Then*

(i) $$d_{\mathsf{F}_p G}(\mathfrak{g}/p\mathfrak{g}) = \max\left\{\left[\!\!\left[\frac{\dim H^1(G,M)}{\dim M}\right]\!\!\right] + \rho_M : M \text{ irreducible } \mathsf{F}_p\, G\text{-module}\right\}$$

(ii) *If $1 \to R \to F \to G \to 1$ is a finite presentation of G, then*

$$d_{\mathsf{F}_p}(\bar{R}/p\bar{R}) = \max\left\{\left[\!\!\left[\frac{\dim H^2(G,M) - \dim H^1(G,M)}{\dim M}\right]\!\!\right] - \rho_M + dF : M \text{ irreducible } \mathsf{F}_p\, G\text{-module}\right\}.$$

3. SOME LEMMAS AND THE FIRST PROOF FOR THE HYPERELEMENTARY CASE

LEMMA 1. *Let $1 \to H \to G \xrightarrow{\pi} Q \to 1$ be a relatively prime extension with Q nilpotent. Let $p \in \pi(Q)$ and suppose M is an irreducible $\mathsf{F}_p G$-module. Then $H^i(G,M) = 0$ for $i \geqq 0$ unless $M = \mathsf{F}_p$. In this case $H^i(G,\mathsf{F}_p) \cong H^i(Q,\mathsf{F}_p)$.*

Proof. Let Q_p be a Sylow p-subgroup of Q. Since Q is nilpotent, there exists a projection ρ of Q onto Q_p. Let $\tilde{H}$ = kernel of $\rho \circ \pi$. Then the order of $\tilde{H}$ is relatively prime to p and since M is an elementary abelian p-group, $H^i(\tilde{H},M) = 0$ for $i > 0$. The Lyndon spectral sequence of the extension

$$1 \to \tilde{H} \to G \xrightarrow{\rho\circ\pi} Q_p \to 1$$

therefore collapses and we have $H^i(G,M) \cong H^i(Q_p, M^{\tilde{H}})$, $i \geqq 0$. Since $\tilde{H}$ is normal in $G, M^{\tilde{H}}$ is a G-invariant subspace of M and hence by the irreducibility of M must be (0) or M. If $M^{\tilde{H}} = 0$ we are done, so assume $M^{\tilde{H}} = M$, i.e., we may consider M as an irreducible $\mathsf{F}_p Q_p$-module. But every mod p representation of a p-group has a fixed point. (Construct the split extension Γ of M by Q_p and use the class formula for the action of the p-group Γ on M by conjugation.) Therefore $M^{Q_p} \neq (0)$ and so by irreducibility $M^{Q_p} = M$. Since $M^{Q_p} = M^G$, we must have $M = \mathsf{F}_p$. The last statement follows from the fact that for $i \geqq 0$, $H^i(G,\mathsf{F}_p) \cong H^i(Q_p, \mathsf{F}_p)$. But from the universal coefficient formula it follows, since Q is nilpotent and therefore the direct sum of its Sylow subgroups, that $H^i(Q_p, \mathsf{F}_p) \underset{\simeq}{\leftarrow} H^i(Q,\mathsf{F}_p)$ for $i > 0$.

Remark. For Q a p-group it is not difficult to see that

(i) $\dim H^1(Q,\mathsf{F}_p) = d(Q)$

(ii) $\dim H^2(Q,\mathsf{F}_p) - \dim H^1(Q,\mathsf{F}_p) = d(H_2(Q,\mathsf{Z}))$.

See, for example, Gruenberg [5, Chapter 7].

LEMMA 2. *Let $1 \to H \to G \to Q \to 1$ be a relatively prime extension with H nilpotent. Let $p \in \pi(H)$ and H_p the Sylow p-subgroup of H. Then if M is an irreducible* $\mathsf{F}_p G$-module,

(i) *M is trivial as H_p*-module,

(ii) *$H^i(G,M) \cong H^i(H_p,M)^{G/H_p}$ for $i \geqq 0$.*

Proof. H_p is normal in G since it is characteristic in H. Again since M is a mod p representation of H_p, $M^{H_p} \neq (0)$ and so $M^{H_p} = M$ by the irreducibility of M. As for (ii), since M is an elementary p-group, $H^q(H_p,M)$ is a F_p-vector space for all q. Now the order of G/H_p is relatively prime for p, so

$$H^s(G/H_p, H^t(H_p,M)) = 0 \qquad \text{for } s > 0,\ t \geqq 0.$$

This collapse of the Lyndon spectral sequence gives the result.

From the last lemma we see that we must investigate the action of G/H_p on $H^t(H_p,M)$. This is especially easy in the case H_p is cyclic.

LEMMA 3. *Suppose $1 \to \mathsf{Z}/n \to K \xrightarrow{\pi} K' \to 1$ is an exact sequence of groups. Let $p|n$ and suppose M is an $\mathsf{F}_p K$* module *which is trivial as an $\mathsf{F}_p(\mathsf{Z}/n)$*-module. *Then for each $s \geqq 1$, $H^{2k-1}(\mathsf{Z}/n,M) \cong H^{2k}(\mathsf{Z}/n,M)$ as K'*-modules.

Proof. If $1 \to A \to B \to C \to 1$ is an exact sequence of groups and N is a left (right) B-module, the left (right) action of C on $H^*(A,N)$, $(H_*(A,N))$ is obtained as follows: Resolve the trivial A-module Z over $\mathsf{Z}A$, $P_* \to \mathsf{Z}$. Tensor with $\mathsf{Z}B$ over $\mathsf{Z}A$ to obtain a resolution of $\mathsf{Z}B \otimes_{\mathsf{Z}A} \mathsf{Z} = \mathsf{Z}C$. "Lift" right multiplication by $c \in C$ to a chain map $f^c_* : \mathsf{Z}B \otimes_{\mathsf{Z}A} P_* \to \mathsf{Z}B \otimes_{\mathsf{Z}A} P_*$. After taking $\mathrm{Hom}_{\mathsf{Z}B}(-,N)$, $(N \otimes_{\mathsf{Z}B} -)$, the resulting induced map gives the action of $c \in C$ on $\mathrm{H}^*(A,N)$, $(H_*(A,N))$. Applying this procedure to the "standard" resolution of Z over Z/n ([1], [9]) one sees easily that the action of $q \in K'$ on $H^s(\mathsf{Z}/n,M)$ is induced from the following ladder. $M_i = \mathrm{Hom}_\Lambda(\Lambda_i,M)$, $\Lambda = \Lambda_i = \mathsf{Z}(K)$, ρ a generator of $\mathsf{Z}/n \subseteq K$, $\pi x = q$, and if $\rho^q = \rho^a$, then $\partial_\rho(\rho^q) = 1 + \rho + \dots + \rho^{a-1}$.

$$\begin{array}{ccccc}
\dots \to M_{2k-1} & \xrightarrow{1+\rho+\dots+\rho^{n-1}} & M_{2k} & \xrightarrow{\rho-1} & M_{2k+1} \to \dots \\
\downarrow {\scriptstyle x(\partial_\rho(\rho^q))^k} & & \downarrow {\scriptstyle x(\partial_\rho(\rho^q))^k} & & \downarrow {\scriptstyle x(\partial_\rho(\rho^q))^{k+1}} \\
\dots \to M_{2k-1} & \xrightarrow{1+\rho+\dots+\rho^{n-1}} & M_{2k} & \xrightarrow{\rho-1} & M_{2k+1} \to \dots
\end{array}$$

In our case Z/n acts trivially on M and since $p|n$, both the maps $1 + \rho + \dots + \rho^{n-1}$ and $\rho - 1$ are zero. Therefore $H^c(\mathsf{Z}/n,M) \cong M$ and the above action of $q \in K'$ on $H^{2k-1}(\mathsf{Z}/n,M)$ and $H^{2k}(\mathsf{Z}/n,M)$ is given by $q \cdot m = xa^k m$ with $\pi x = q$.

These lemmas together with the results of Section 2 give

THEOREM 3. *Let* $1 \to \mathbf{Z}/n \to G \to G_p \to 1$ *be a hyperelementary group.*

(i) $d_G(\mathfrak{g}) = d(G_p)$ *unless* G *is a nonabelian extension of relatively prime cyclic groups, in which case* $d_G(\mathfrak{g}) = 2$.

(ii) *If* $1 \to R \to F \to G \to 1$ *is any presentation of* G,

$$d_G(\bar{R}) = d(F) + d(H_2(G_p, Z)).$$

Proof. For the prime p we have by Lemma 1 and the remark immediately following that $d_G(\mathfrak{g}/p\mathfrak{g}) = d(G_p)$ and $d_G(\bar{R}/p\bar{R}) = dF + d(H_2(G_p))$. For the primes q dividing n, Lemmas 2 and 3 show $\dim H^2(G,M) - \dim H^1(G,M) = 0$ for all irreducible $\mathbf{F}_q G$-module M. This gives (ii). As for (i), we note that if M is an irreducible $\mathbf{F}_q G$-module, $q|n$, then by Lemma 2(i), M is trivial as a $(Z/n)_q \simeq Z/q^t$-module and $H^1(Z/q^t;M) \simeq M$. Therefore

$$\dim H^1(G;M) \simeq M^{G/Z/q^t} \leqslant \dim M.$$

It follows from Theorem 2 that $d_G(\mathfrak{g}/q\mathfrak{g}) \leqslant 2$ for $q|n$. If $d(G_p) \geqslant 2$, then

$$d_G(\mathfrak{g}) = d(G_p)$$

from Theorem 1. On the other hand if $d(G_p) = 1$, then G_p is cyclic. If G is cyclic, $d_G(\mathfrak{g}) = d(G_p)$. Otherwise G is nonabelian and $d_G(\mathfrak{g}) = 2$.

4. THE GENERAL RESULT

We have seen from Lemma 2 that if we are interested in relatively prime extensions where the subgroup is nilpotent, then we must investigate irreducible G-modules which are trivial when restricted to a subgroup. We therefore consider the following situation.

Let k be a commutative ring, $1 \to H \to G \xrightarrow{\pi} Q \to 1$, an exact sequence of groups and M a left kG-module. Q acts on the left of $H^s(H,M)$ and on the right of $H_s(H)$. Suppose M is trivial as an H-module. We can then define a left Q-module structure on $\mathrm{Hom}(H_s(H),M)$ by $(q \cdot f)(z) = qf(zq)$. Also since M is a trivial H-module, the universal coefficient theorem gives a (non-naturally) split exact sequence

$$0 \to \mathrm{Ext}(H_{s-1}(H),M) \to H^s(H,M) \xrightarrow{\sigma} \mathrm{Hom}(H_s(H),M) \to 0$$

where σ is given by $\sigma\{f\}(\{z\}) = \{f(z)\}$.

LEMMA 4. *With the above* Q-module *structures,* σ *is a homomorphism of* Q-modules. *Moreover the induced action on* $E \in \mathrm{Ext}(H_{s-1}(H),M)$ *is given by*

$$E^q = q^* E_{q_*} = \mathrm{Ext}(R_q, L_q)(E)$$

where R_q, L_q *are the actions of* q *on* $H_{s-1}(H)$ *and* M *respectively.*

Proof. Referring to Lemma 3 for the description of how to compute the action of $q \in Q$ on $H^*(H,A)$ and $H_*(H,B)$, it is easy to see that if one uses the Bar construction, $B_*(H)$, for P_*, the maps $f_n^q\colon \mathbf{Z}G \otimes_H B_n(H) \to ZG \otimes_H B_n(H)$ can be chosen to be $f_n^q(1 \otimes [h_1 | \ldots | h_n]) = x \otimes [h_1^x | \ldots | h_n^x]$ where $\pi x = q$ and $h_i^x = x h_i x^{-1}$.

Define $\bar{f}^q$ and $\tilde{f}^q$ by the following diagrams (the horizontal maps are the natural isomorphisms)

$$\begin{array}{ccc} \mathrm{Hom}_H(P_*,M) & \cong & \mathrm{Hom}_G(ZG \otimes_H P_*,M) \\ \downarrow \bar{f}^q & & \downarrow \mathrm{Hom}(f^q,1) \\ \mathrm{Hom}_H(P_*,M) & \cong & \mathrm{Hom}_G(\mathbf{Z} \otimes_H P_*,M) \end{array} \qquad \begin{array}{ccc} \mathbf{Z} \otimes_H P_* & \cong & \mathbf{Z} \otimes_G \mathbf{Z}G \otimes_H P_* \\ \downarrow \tilde{f}^q & & \downarrow 1 \otimes_G f^q \\ \mathbf{Z} \otimes_H P_* & \cong & \mathbf{Z} \otimes_G \mathbf{Z}G \otimes_H P_*. \end{array}$$

Then $\bar{f}^q$ induces the action of q in $H^*(H,M)$ and $\tilde{f}^q$ induces the action of q on $H_*(H)$. An easy calculation using the above description of f^q for the Bar construction of H shows that the following diagram is commutative (the horizontal maps are the natural isomorphisms). Recall M is trivial as an H-module. $P_H = \mathbf{Z} \otimes_H P_*$.

$$\begin{array}{ccc} \mathrm{Hom}(P_H,M) & \simeq & \mathrm{Hom}_H(P_*,M) \\ \downarrow \mathrm{Hom}(\tilde{f}^q,Id) & & \\ \mathrm{Hom}(P_H,M) & & \downarrow \bar{f}^q \\ \downarrow \mathrm{Hom}(Id,L_q) & & \\ \mathrm{Hom}(P_H,M) & \simeq & \mathrm{Hom}_H(P_*,M) \end{array}$$

That is, the action of q on $H^*(H,M)$ can be computed from the chain map $\mathrm{Hom}(\tilde{f}^q,L_q)$ on $\mathrm{Hom}(P_H,M)$.

Now the universal coefficient theorem for H with a trivial coefficient module M is obtained by using the natural homomorphism of $\mathrm{Hom}(P_H,M) \cong \mathrm{Hom}_H(P_*,M)$ and then applying the usual universal coefficient theorem to the complex, P_H, of free abelian groups. We recall this proof [9].

Let K be a chain complex of free abelian groups, C_* the cycles and B_* the boundaries. Then the middle column in the following diagram induces the universal coefficient sequence.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(H_n,M) & \longrightarrow & \mathrm{Hom}(C_n,M) & \xrightarrow{j^*} & \mathrm{Hom}(B_n,M) & & \\ & & & & \uparrow i^* & & \downarrow \partial^* & & \\ & & \mathrm{Hom}(K_{n-1},M) & \longrightarrow & \mathrm{Hom}(K_n,M) & \xrightarrow{\delta} & \mathrm{Hom}(K_{n-1},M) & & \\ & & \downarrow & & \uparrow \partial^* & & & & \\ & & \mathrm{Hom}(C_{n-1},M) & \xrightarrow{j^*} & \mathrm{Hom}(B_{n-1},M) & \longrightarrow & \mathrm{Ext}(H_{n-1},M) & \longrightarrow & 0 \\ & & \downarrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

In our case $K_* = P_H$ and it is obvious that i^*, j^*, ∂^* commute with $\operatorname{Hom}(\tilde{f}^q, L_q)$. The result now follows immediately.

In what follows we shall assume N and M are left G-modules (we interchange right and left by $g \cdot n = n \cdot g^{-1}$) and the action of G on $\operatorname{Hom}(N,M)$ and $\operatorname{Ext}(N,M)$ are as given above, i.e., by $\operatorname{Hom}(g^{-1},g)$ and $\operatorname{Ext}(g^{-1},g)$ respectively.

PROPOSITION 1. *Let* $0 \to N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \to 0$ *be an exact sequence of* G-modules *and* G-*maps. Let* M *be a* G-module. *The following is then an exact sequence of* G-modules.

$$\begin{array}{l} 0 \to \operatorname{Hom}(N_3,M) \xrightarrow{\beta^*} \operatorname{Hom}(N_2,M) \xrightarrow{\alpha^*} \operatorname{Hom}(N_1,M) \\ \xrightarrow{\Delta} \operatorname{Ext}(N_3,M) \xrightarrow{\beta^*} \operatorname{Ext}(N_2,M) \xrightarrow{\alpha^*} \operatorname{Ext}(N_1,M) \to 0. \end{array}$$

Proof. The above 6 term sequence is natural with respect to maps of short exact sequences.

LEMMA 5. *Let* N *be a finitely generated* G-module, M *an* $\mathbf{F}_p G$-module. *Suppose* $\mathbf{F}_p G$ *is semisimple, then*

$$\operatorname{Ext}(N,M)^G \simeq \operatorname{Hom}_G({}_pN, M) \qquad \textit{where} \qquad {}_pN = \{n \in N : pn = 0\}.$$

Proof. Let $f: (\mathbf{Z}G)^m \to N$ be an epimorphism of G-modules and let $\tilde{K} = \operatorname{kernel} f$ which is a free abelian group. f induces an epimorphism

$$1 \otimes f: (\mathbf{F}_p G)^m \to \mathbf{F}_p N = \mathbf{F}_p \otimes N$$

whose kernel we denote by K. Consider the following exact diagram of G-modules and maps.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \tilde{K} \cap p(ZG)^m & \to & p(ZG)^m & \xrightarrow{f} & pN & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \tilde{K} & \to & (ZG)^m & \xrightarrow{f} & N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & (\mathbf{F}_p G)^m & \to & \mathbf{F}_p N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The middle row gives by Proposition 1 a 6-term sequence of G-modules (in fact $\mathbf{F}_p G$-modules since M is a $\mathbf{F}_p$-vector space)

(1) $\quad 0 \to \operatorname{Hom}(N,M) \to \operatorname{Hom}((ZG)^m,M) \to \operatorname{Hom}(\tilde{K},M) \to \operatorname{Ext}(N,M) \to 0$

which terminates since $(ZG)^m$ is $\mathbf{Z}$-free.

Now, since M is p-elementary, if A is any abelian group, the inclusion of $\operatorname{Hom}(\mathbf{F}_pA,M)$ into $\operatorname{Hom}(A,M)$ is an isomorphism, and we will identify $\operatorname{Hom}(\mathbf{F}_pA,M)$ and $\operatorname{Hom}(A,M)$ in the following without further comment.

Since $\mathbf{F}_pG$ is semisimple, (1) splits completely and we obtain

$$\operatorname{Hom}(\mathbf{F}_pN,M) \oplus \operatorname{Hom}(\mathbf{F}_p\tilde{K},M) \simeq \operatorname{Hom}((\mathbf{F}_pG)^m,M) \oplus \operatorname{Ext}(N,M). \tag{2}$$

The map $\tilde{K} \to K$ factors though $\mathbf{F}_p\tilde{K} = \tilde{K}/p\tilde{K}$ and so we have an epimorphism $\gamma : \mathbf{F}_p\tilde{K} \to K$ with kernel isomorphic to $\tilde{K} \cap p(ZG)^m/p\tilde{K}$ as a G-module. Consider the following exact ladder of G-modules.

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K} \cap p(ZG)^m/p\tilde{K} & \longrightarrow & p(ZG)^m/p\tilde{K} & \longrightarrow & p(ZG)^m/\tilde{K} \cap p(ZG)^m & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \bar{f}} & & \\ 0 & \longrightarrow & {}_pN & \longrightarrow & N & \xrightarrow{\ p\ } & pN & \longrightarrow & 0 \end{array}$$

where κ is induced as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K} & \longrightarrow & (\mathbf{Z}G)^m & \xrightarrow{\ f\ } & N & \longrightarrow & 0 \\ & & {\scriptstyle \simeq}\downarrow{\scriptstyle p} & & {\scriptstyle \simeq}\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \kappa^{-1}} & & \\ 0 & \longrightarrow & p\tilde{K} & \longrightarrow & p(\mathbf{Z}G)^m & \longrightarrow & p(\mathbf{Z}G)^m/p\tilde{K} & \longrightarrow & 0 \end{array}$$

Since $(\mathbf{Z}G)^m$ and $\tilde{K}$ are free abelian, multiplication by p is an isomorphism onto $p\tilde{K}$ or $p(ZG)^m$. By the 5 lemma, κ is an isomorphism. Also $\bar{f}$ in $(*)$ is an isomorphism. The right-hand square in $(*)$ commutes since $p\kappa([px]) = pf(x) = \bar{f}([px])$. Therefore κ induces by the 5 lemma an isomorphism of $\tilde{K} \cap p(ZG)^m/p\tilde{K}$ with ${}_pN$. This give an exact sequence of $\mathbf{F}_pG$-modules $0 \to {}_pN \to \mathbf{F}_p\tilde{K} \to K \to 0$ which again splits and so

$$\operatorname{Hom}(\mathbf{F}_p\tilde{K},M) \simeq \operatorname{Hom}(K,M) \oplus \operatorname{Hom}({}_pN,M) \tag{3}$$

From (2) and using (3) we have

$$\operatorname{Hom}(\mathbf{F}_pN,M) \oplus \operatorname{Hom}(K,M) \oplus \operatorname{Hom}({}_pN,M) \simeq \operatorname{Hom}((\mathbf{F}_pG)^m,M) \oplus \operatorname{Ext}(N,M). \tag{4}$$

Since $(\mathbf{F}_pG)^m \simeq \mathbf{F}_pN \oplus K$, the Krull-Schmidt theorem gives

$$\operatorname{Hom}({}_pN,M) \simeq \operatorname{Ext}(N,M). \tag{5}$$

Taking fixed points gives the result.

COROLLARY. $\operatorname{Ext}(N,M)^G \simeq \operatorname{Hom}_G(\mathbf{F}_pN,M)$ *if N is finite.*

Proof. $0 \to {}_pN \to N \xrightarrow{p} N \to \mathbf{F}_pN \to 0$ is an exact sequence of abelian groups with G-action. We will show ${}_pN \simeq \mathbf{F}_pN$ as $\mathbf{F}_pG$-modules. Let $N(p) = p$-torsion of N, then $0 \to {}_pN \to N(p) \xrightarrow{p} N(p) \to \mathbf{F}_pN \to 0$ is exact. The proof will be by induction on the exponent of $N(p)$. If exponent $N(p) = 1$, then $N(p) \xrightarrow{p} N(p)$ is the zero map, so we are done. Assume exponent $N(p) = e + 1 > 1$. Now if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence of G-modules and maps, the snake lemma applied to the ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

gives a 6-term sequence of $\mathbf{F}_p G$-modules

$$0 \to {}_pA \to {}_pB \to {}_pC \to \mathbf{F}_p A \to \mathbf{F}_p B \to \mathbf{F}_p C \to 0.$$

Letting $B = N(p)$ and $A = \{n \in N : p^e n = 0\}$ and using the complete splitting of the above sequence together with the induction hypothesis and the Krull-Schmidt theorem gives ${}_pN \simeq \mathbf{F}_p N$ as $\mathbf{F}_p G$-modules.

Let now $1 \to H \to G \to Q \to 1$ be an exact sequence of groups and let N be a subgroup of H which is normal in G. Denote G/N by Q_N and consider the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q_N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \| & & \downarrow{\scriptstyle \pi} & & \\ 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

In this situation we have the following lemma.

LEMMA 6. (i) *Let A be a left* G-module *if $q \in Q_N$ and $u \in H^i(H,A)$, then* $i^*(\pi(q) \cdot u) = q \cdot i^*(u)$.

(ii) *Let B be a right* G-module, *if $q \in Q_N$ and $z \in H_i(N,B)$, then*

$$i_*(z \cdot q) = i_*(z) \cdot \pi(q).$$

Proof. Let $P_* \xrightarrow{\epsilon} \mathbf{Z}$ be a projective resolution of $\mathbf{Z}$ over $\mathbf{Z}H$ and hence over $\mathbf{Z}N$. Let $\gamma : \mathbf{Z}G \otimes_N P_* \to \mathbf{Z}G \otimes_H P_*$ be a "lift" of $\pi : \mathbf{Z}(Q_N) \to \mathbf{Z}(Q)$. For $q \in Q_N$, denote by

$$f_*^q : \mathbf{Z}G \otimes_N P_* \to \mathbf{Z}G \otimes_N P_* \qquad \text{a "lift" of} \qquad R_q : \mathbf{Z}(Q_N) \to \mathbf{Z}(Q_N)$$

and by

$$f_*^{\pi q}\, \mathbf{Z}G \otimes_H P \to \mathbf{Z}G \otimes_H P_* \qquad \text{a "lift" of} \qquad R_{\pi q} : \mathbf{Z}(Q) \to \mathbf{Z}(Q).$$

Then $f^{\pi q} \circ \gamma$ and $\gamma \circ f^q$ are chain homotopic since they are both "lifts" of

$$R_{\pi q} \circ \pi = \pi \circ R_q.$$

Since the map induced by inclusion $i: N \to H$ can be computed using γ, the results i) and ii) follow.

We are now in a position to prove the main result of this paper. Recall the following definitions. Let $\mathsf{F}_p Q$ be semisimple and M an irreducible $\mathsf{F}_p Q$-module. Let τ_M = number of occurrences of M in $\mathsf{F}_p Q$ and if A is any $\mathsf{F}_p Q$-module, let $\tau_M(A)$ = number of occurrences of M in A. Define $\beta_s(A) = 0$ if for every irreducible module $M \neq F_p$, $\tau_{F_p}(A) \geq [\![\tau_M(A)/\tau M]\!] + (-1/2)^{s+1}$ and let $\beta_s(A) = (-1)^{s+1}$ otherwise. Define

$$\alpha_s = \max_{\substack{p,q \\ \text{primes}}} \{d_Q(F_p H_s(N)) + \beta_s(F_p H_s N), d(H_s(Q_p))\}$$

where Q_q is a Sylow q-subgroup of Q.

Obviously only the primes $p \in \pi(N)$ and $q \in \pi(Q)$ make any contribution to α_s.

THEOREM 4. *If* $1 \to N \to G \to Q \to 1$ *is a relatively prime extension with* N, Q *nilpotent, then*

(i) $d_G(\mathfrak{g}) = \alpha_1$

(ii) *If* $1 \to R \to F \to G \to 1$ *is any finite presentation of* G, $d_G(\bar{R}) - d(F) = \alpha_2$.

Proof. We know from Theorems 1 and 2 that it is sufficient to evaluate the numbers

$$\left[\!\!\left[\frac{\dim H^1(G,M)}{\dim M}\right]\!\!\right] \quad \text{and} \quad \left[\!\!\left[\frac{\dim H^2(G,M) - \dim H^1(G,M)}{\dim M}\right]\!\!\right]$$

for M an irreducible $\mathsf{F}_p G$-module and $p \in \pi(G)$. Let $q \in \pi(Q)$, then from Lemma 1 and the remark following it, the only contribution will occur when $M = \mathsf{F}_q$ and this will be $d(Q_p)$ in the first case and $d(H_2(Q_q))$ in the second.

Let $p \in \pi(N)$. If M is an irreducible $\mathsf{F}_p G$-module, then Lemma 2 says M is trivial as an N_p-module ($N_p = p$ Sylow-subgroup of N) and

$$H^i(G,M) = H^i(N_p,M)^{G/N_p}.$$

From Lemma 4, the universal coefficient sequence

$$0 \to \text{Ext}(H_{s-1}(N_p),M) \to H^s(N_p,M) \xrightarrow{\sigma} \text{Hom}(H_s(N_p),M) \to 0$$

is an exact sequence of G/N_p-modules and in fact, since M is an elementary p-group, a sequence of $\mathsf{F}_p(G/N_p)$-modules. Since $p \notin \pi(G/N_p)$, $\mathsf{F}_p(G/N_p)$ is semisimple so the sequence splits. Taking fixed points we have

(1) $H^s(G,M) \cong H^s(N_p,M)^{G/N_p} \cong \text{Hom}_{G/N_p}(H_s(N_p),M) \oplus \text{Ext}(H_{s-1}(N_p),M)^{G/N_p}$.

Since $H_s(N_p)$ is finite for $s > 0$, the corollary to Lemma 5 gives for $s > 1$

(2) $$H^s(G,M) \cong \mathrm{Hom}_{G/N_p}((\mathbf{F}_p H_s(N_p),M) \otimes \mathrm{Hom}_{G/N_p}(\mathbf{F}_p H_{s-1}(N_p),M)$$

while for $s = 1$ we have

(3) $$H^s(G,M) \cong \mathrm{Hom}_{G/N_p}(\mathbf{F}_p H_1(N_p),M).$$

Because N is nilpotent, $N = \bigtimes_p N_p$ and since the homology of a p-group is annihilated by p^K in positive dimensions, we have $H_s(N) = \bigtimes_p H_s(N_p)$ for $s > 0$. Hence the inclusion $i: N_p \to N$ induces an isomorphism $i_* : \mathbf{F}_p H_s(N_p) \underset{\approx}{\to} \mathbf{F}_p H_s(N)$ for $s > 0$. I claim that $\mathbf{F}_p H_s(N_p)$ is trivial as an N/N_p-module. This follows from Lemma 6 for if $z \in \mathbf{F}_p H_s(N_p)$ and $q \in N/N_p = \ker \pi : G/N_p \to G/N$, then

$$i_*(zq) = i_*(z)\pi(q) = i_*(z).$$

Since i is an isomorphism $z \cdot q = z$ and $\mathbf{F}_p H_s(N_p)$ is trivial as an N/N_p-module. Now recall M is an irreducible $\mathbf{F}_p G$-module and that it is trivial as an N_p-module. Consider $\mathrm{Hom}_{G/N_p}(\mathbf{F}_p H_s(N_p),M)$ and suppose it is different from zero. Let

$$0 \neq f: \mathbf{F}_p H_s(N_p) \to M.$$

Since M is irreducible as $\mathbf{F}_p(G/N_p)$-module, f is an epimorphism. Let $m \in M$ and $q \in N/N_p$, $m = f(z)$, $z \in \mathbf{F}_p H_s(N_p)$ and $q \cdot m = f(z \cdot q) = f(z) = m$. That is $M = M^{N/N_p}$ or M is trivial as an N-module. Therefore the only irreducible $\mathbf{F}_p G$-modules to be considered are those which are trivial as N-modules, i.e., irreducible $\mathbf{F}_p Q$-modules. Recalling that $\mathbf{F}_p H_s(N_p) \cong \mathbf{F}_p H_s(N)$ is also an $\mathbf{F}_p Q$-module, we obtain the formulas ($\rho_M = 0$ if $M = \mathbf{F}_p$, $\rho_M = 1$ otherwise).

(i) $$d_G(\mathfrak{g}) = \max_{p,q} \left\{ \left[\!\left[\frac{\dim \mathrm{Hom}_{\mathbf{F}_p Q}(\mathbf{F}_p H_1(N),M)}{\dim M} \right]\!\right] + \rho_M, dQ_q \right\}$$

(ii) $$d_G(\bar{R}) - d(F) = \max_{p,q} \left\{ \left[\!\left[\frac{\dim \mathrm{Hom}_{\mathbf{F}_p Q}(\mathbf{F}_p H_2(N),M)}{\dim M} \right]\!\right] + \rho_M, dH_2(Q_q) \right\}$$

where M is an irreducible $\mathbf{F}_p Q$-module and Q_q is a Sylow q-subgroup of Q.

Now it is obvious that

$$\left[\!\left[\frac{\dim \mathrm{Hom}_{\mathbf{F}_p Q}(A,M)}{\dim M} \right]\!\right] = \left[\!\left[\frac{\tau_M(A)}{\tau_M} \right]\!\right]$$

for any Q module A and also that $\max_{M \text{ irr}} [\![\tau_m(A)/\tau_M]\!] = d_Q(A)$. Using these observations, it is not difficult to show

$$\max_{M \text{ irr}} \left\{ \left[\!\left[\frac{\dim \mathrm{Hom}_{\mathbf{F}_p Q}(A,M)}{\dim M} \right]\!\right] + (-1)^{s+1}\rho_M \right\} = d_Q(A) + \beta_s(A).$$

It follows that the right-hand sides of (i) and (ii) are α_1 and α_2 respectively.

Since $\beta_s(A) \leq 1$, we have the following corollary. (Compare to the hyperelementary case.)

COROLLARY. *If* $\max_q dH_s(Q_q) > \max_p d_Q(\mathsf{F}_p H_s(N))$, *then* $\alpha_s = \max_q d(H_s(Q_q))$.

We conclude this paper with the following obvious question: What is the significance of the α_s for $s \geqq 3$?

Addendum. It has been observed by the referee that the proof of the above theorem yields a calculation for $d_G(\mathfrak{g})$ and $d_G(\bar{R}) - d(F)$ also in the case Q is not necessarily nilpotent. In the definition of α_s, replace $d(H_s(Q_q)$ by $d_Q(\mathfrak{g}))$ if $s = 1$ and by $d_Q(\bar{S}) - d(E)$ if $s = 2$, where $1 \to S \to E \to Q \to 1$ is any finite presentation of Q. With this definition of α_s, Theorem 4 remains correct even if Q is not nilpotent. In order to see this one merely observes that the first half of Lemma 1 shows that if M is an irreducible $\mathsf{F}_p G$-module with $p \in \pi(Q)$, then either $H^i(G;M) = 0$ for all i or M is an irreducible Q-module and

$$H^i(G;M) = H^i(Q;M).$$

On the other hand if M is an irreducible $\mathsf{F}_p Q$-module, then M is irreducible as a G-module and again $H^i(G;M) = H^i(Q;M)$. It follows from Theorems 1 and 2 that

$$\max_{p \in \pi(Q)} \left\{ \left[\!\left[\frac{\dim H^1(G;M)}{\dim M} \right]\!\right] + \rho_M : M \text{ irreducible } \mathsf{F}_p G\text{-module} \right\} = d_Q(\mathfrak{g})$$

and

$$\max_{p \in \pi(Q)} \left\{ \left[\!\left[\frac{\dim H^2\ (G;M) - \dim H^1(G;M)}{\dim M} \right]\!\right] - \rho_M : M \text{ irr } \mathsf{F}_p G\text{-module} \right\}$$
$$= d_Q(\bar{S}) - d(E)$$

and the result follows.

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